TÉCNICO LISBOA

Quantum Field Theory

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Contents

Preface to the 2021 Edition

This is a complete revision of the old text [1]. Chapters 2 to 9 were revised and translated into English. As for Chapter 1 and Appendix A, a complete rewrite is planned but not in time for the 2021-2021 course.

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Chapter 1

A Equação Relativista para o Eletrão

1.1 Introdução

Pretendemos neste capítulo juntar as ideias da mecânica quântica com as de relatividade restrita tornando-as compatíveis. Isso vai levar-nos à substituição da equação de Schrödinger pelas equações relativistas de Klein-Gordon e Dirac. Como veremos, esta tentativa de descrever a Física ao nível quântico através duma equação para uma partícula terá que ser abandonada e substituída por uma descrição em termos dum número *variável* de partículas permitindo a sua criação e aniquilação. Esse será o objetivo de chamada *segunda quantificação* que explicaremos mais à frente. Contudo existem muitos problemas onde a interpretação em termos das equações para uma partícula é adequada e conduz a bons resultados. Isto passa-se para distâncias não muito pequenas, como se compreenderá melhor no seguimento. Além disso, o formalismo desenvolvido para tratar das equações de Klein-Gordan e Dirac irá ser o suporte dos desenvolvimentos futuros. Isto justifica que estudemos em algum detalhe estas equações e as suas soluções.

Como dissemos anteriormente queremos encontrar equações que sejam compatíveis com a mecânica quântica e a relatividade restrita. Vamos aqui rever brevemente os princípios básicos destas duas teorias. A mecânica quântica [2, 3] baseia-se nos seguintes princípios:

- Para o estado físico existe uma função de estado $|\Phi\rangle$ que contém toda a informação possível sobre o sistema. Na maior parte dos casos tratemos com uma representação do estado $|\Phi\rangle$ em termos das coordenadas, a chamada função de onda $\Psi(q_i, s, t)$ onde s designa outros números quânticos para além dos possíveis de descrever a partir das coordenadas (por exemplo o spin). $|\Psi(q_i, s_i, t)|^2 \geq 0$ tem a interpretação duma densidade de probabilidade de encontrar o sistema num estado com coordenadas q_i, números quânticos internos s_i, no instante t.

- As observáveis físicas são representadas por operadores hermíticos lineares. Por exemplo

$$p_i \to -i\hbar \frac{\partial}{\partial q_i} \tag{1.1}$$

$$E \to i\hbar \frac{\partial}{\partial t} \tag{1.2}$$

- Um estado $|\Phi\rangle$ do sistema é um estado próprio de operador Ω se

$$\Omega \, |\Phi_n\rangle = \omega_n \, |\Phi_n\rangle \tag{1.3}$$

onde $|\Phi_n\rangle$ é o estado próprio a que corresponde o valor próprio ω_n. Se Ω for hermítico então os ω_n são reais. Na representação das *coordenadas* temos

$$\Omega(q, s, t)\Psi(q, s, t) = \omega_n \Psi(q, s, t) \tag{1.4}$$

- Existe um conjunto completo e ortonormal de funções próprias, Ψ_n, dum conjunto completo de operadores que comutam $\{\Omega_1, \Omega_2, \ldots\}$. Uma função de onda arbitrária pode ser expandida em termos desse conjunto completo

$$\Psi = \sum_n a_n \Psi_n \tag{1.5}$$

- O resultado duma medição é qualquer um dos valores próprios. Se $\Psi = \sum_n a_n \Psi_n$ com $\Omega\Psi_n = \omega_n \Psi_n$ então o resultado da medição será o valor ω_n com probabilidade $|a_n|^2$. O valor médio duma observável é dado por

$$< \Omega >_\Psi = \sum_s \int dq_1 ... \Psi^*(q_i, s_i, t)\Omega\Psi(q_i, s_i, t) = \sum_n |a_n|^2 \omega_n \tag{1.6}$$

Depois da medição o estado fica projetado no vetor próprio (ou combinações de vetores próprios) correspondentes ao valor próprio.

- A evolução no tempo dum sistema físico é dada pela equação

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi \tag{1.7}$$

- onde o Hamiltoniano H é um operador linear e hermítico. A linearidade implica o princípio de sobreposição e a hermiticidade conduz à *conservação* de probabilidade

$$\frac{d}{dt} \sum_s \int dq_1 \cdots \Psi^*\Psi = \frac{i}{\hbar} \sum_s \int dq_1 \cdots [(H\Psi)^*\Psi - \Psi^*(H\Psi)] = 0 \tag{1.8}$$

Estes são os princípios básicos de mecânica quântica que procuraremos conservar. Por outro lado a relatividade restrita baseia-se nos princípios da relatividade e da constância da velocidade da luz. Para as nossas aplicações basta recordar [4] que as coordenadas de dois referenciais de inércia estão relacionadas pela relação

$$x'^{\mu} = a^{\mu}{}_{\nu}\, x^{\nu} \tag{1.9}$$

A invariância do intervalo

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = dx^{\mu} dx_{\mu} \tag{1.10}$$

onde a métrica $g_{\mu\nu}$ é diagonal e, com as nossas convenções, dada por $g_{\mu\nu} = \mathrm{diag}(+---)$, restringe os coeficientes $a^{\mu}{}_{\nu}$ de transformação, Eq. (1.9), a obedecerem a

$$g_{\mu\nu} a^{\mu}{}_{\alpha} a^{\nu}{}_{\beta} dx^{\alpha} dx^{\beta} = g_{\alpha\beta} dx^{\alpha} dx^{\beta} \tag{1.11}$$

ou ainda

$$a^{\mu}{}_{\alpha} g_{\mu\nu} a^{\nu}{}_{\beta} = g_{\alpha\beta} \tag{1.12}$$

que pode ser escrita matricialmente na forma

$$a^T g\, a = g \tag{1.13}$$

As matrizes que obedecem à Eq. (1.13) constituem o grupo de Lorentz, designado por $O(3,1)$. Para ver as principais propriedades do que é um grupo ver o Complemento 1.1. É fácil verificar que

$$\det\ a = \pm 1 \tag{1.14}$$

As transformações que têm $\det\ a = +1$ constituem o grupo de Lorentz *próprio* e podem ser construídas a partir de transformações infinitesimais. Exemplos são as rotações no espaço a três dimensões e as transformações de Lorentz. Uma rotação dum ângulo θ em torno do eixo dos zz será descrita pela matriz

$$a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.15}$$

enquanto que uma transformação de Lorentz segundo o eixo dos xx será dada pela matriz

$$a = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.16}$$

onde

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad ; \quad \beta = \frac{V}{c} \ . \tag{1.17}$$

e V é a velocidade do referencial S' em relação a S. Exemplos de transformações com det $a = -1$ são as inversões no tempo ou no espaço. Por exemplo $t' \to -t$ corresponde à matriz

$$a = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.18}$$

No seguimento vamos admitir que os princípios básicos de mecânica quântica e da relatividade restrita são conhecidos. Nos problemas no final do capítulo são dados exemplos que servem para ilustrar os conceitos de que vamos necessitar. No Complemento 1.2, a notação invariante é usada para descrever as equações de Maxwell, o que virá a ser útil em capítulos posteriores.

1.2 A equação de Klein-Gordon.

Comecemos pela partícula livre. Em mecânica quântica não relativista a equação de Schrödinger é obtida da equação fundamental

$$i\hbar \frac{\partial}{\partial t}\psi = H\psi \tag{1.19}$$

usando o Hamiltoniano livre não relativista que é

$$H = \frac{p^2}{2\ m} \tag{1.20}$$

e fazendo a substituição $\vec{p} \to -i\hbar\vec{\nabla}$. Obtemos então

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi \tag{1.21}$$

A primeira ideia que surgiu para generalizar esta equação para uma partícula relativista foi usar em vez da Eq. (1.20) o Hamiltoniano relativista. Para uma partícula livre o Hamiltoniano é a sua energia e devemos ter

$$H = E \tag{1.22}$$

A energia está relacionada com o momento linear através da relação

$$p_\mu p^\mu = m^2c^2 \tag{1.23}$$

onde

$$p^\mu \equiv \left(\frac{E}{c}, \vec{p}\right) \tag{1.24}$$

Temos então

$$\frac{E^2}{c^2} - \vec{p}\cdot\vec{p} = m^2c^2 \tag{1.25}$$

ou seja

$$E^2 = p^2c^2 + m^2c^4 \tag{1.26}$$

Classicamente exige-se que as energias sejam positivas por isso deveríamos ter no caso relativista

$$H = \sqrt{p^2c^2 + m^2c^4} \tag{1.27}$$

Somos imediatamente confrontados com o problema de interpretar a raiz quadrada dum operador. Para evitar este problema vamos encontrar uma equação para H^2. Isto obtém-se facilmente iterando a Eq. (1.19) e observando que $\left[i\hbar\frac{\partial}{\partial t}, H\right] = 0$. Obtém-se então

$$-\hbar^2\frac{\partial^2}{\partial t^2}\psi = (-\hbar^2c^2\vec{\nabla}^2 + m^2c^4)\psi \tag{1.28}$$

ou ainda

$$\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right]\psi = 0 \tag{1.29}$$

onde $\Box = \partial_\mu\partial^\mu$. Agora não temos dificuldades em interpretar os operadores mas introduzimos no problema as soluções de energia negativa que também são soluções da Eq. (1.29). Como veremos as soluções de energia negativa não podem deixar de existir em mecânica quântica relativista e a sua interpretação está relacionada com as antipartículas. A observação experimental de antipartículas veio a confirmar esta interpretação.

Mas não foi a existência de soluções com energia negativa que levou ao abandono da Eq. (1.29), chamada equação de Klein-Gordon [5–7], como equação relativista para o eletrão mas antes outro problema relacionado com a *densidade* de probabilidade. Partindo da Eq. (1.29) e da equação complexa conjugada obtemos

$$\psi^*\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right]\psi - \psi\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right]\psi^* = 0 \tag{1.30}$$

ou

$$0 = \psi^*\Box\psi - \psi\Box\psi^* = \partial_\mu(\psi^*\overleftrightarrow{\partial}^\mu\psi) \tag{1.31}$$

onde $\psi^*\overleftrightarrow{\partial}^\mu\psi \equiv \psi^*\overrightarrow{\partial}^\mu\psi - \psi^*\overleftarrow{\partial}^\mu\psi$. Temos então

$$\partial_\mu J^\mu = 0 \qquad ; \qquad J^\mu = \psi^* \overleftrightarrow{\partial}^\mu \psi \tag{1.32}$$

Na identificação usual $J^\mu = (\rho c, \vec{J})$ pelo que a densidade será

$$\rho = \frac{1}{c^2}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) \tag{1.33}$$

Esta equação mostra que ρ não pode ser interpretado como uma densidade de probabilidade por não ser definida positiva. Finalmente uma terceira razão fez abandonar a equação da Klein-Gordon. De facto ela não conduz aos níveis de energia do átomo de *hidrogénio* (ver Problema 1.39).

Se excetuarmos esta última razão, a Eq. (1.29) foi abandonada pelas razões erradas. De facto pode-se mostrar que ela é a boa equação relativista para partículas de *spin zero*, razão pela qual não pode explicar os níveis do átomo de hidrogénio onde os efeitos do spin são importantes. As soluções de energia negativa serão compreendidas e a densidade ρ será re-interpretada não como uma densidade de *probabilidade* mas antes como uma *densidade de carga*.

1.3 A equação de Dirac

Confrontado com os problemas anteriores Dirac propôs uma outra equação relativista para o eletrão [8, 9]. Como na equação fundamental, Eq. (1.19), a derivada em ordem ao tempo aparece linearmente é natural admitir num contexto relativista que o Hamiltoniano seja também linear nas derivadas em ordem às coordenadas e portanto escrevemos

$$i\hbar \frac{\partial \psi}{\partial t} = \left(-i\hbar c \vec{\alpha} \cdot \vec{\nabla} + \beta m c^2\right) \psi \equiv H\psi \tag{1.34}$$

É fácil de ver que α^i e β não podem ser números pois então a relação entre energia e momento duma partícula relativista não seria verificada. Também ψ não pode ser um escalar se $\rho = \psi^*\psi$ é para ser interpretada como a componente temporal dum 4-vetor corrente. Assim Dirac propôs que $\vec{\alpha}$ e β sejam matrizes hermíticas $N \times N$ (para que H seja hermítico) e que ψ seja uma matriz coluna com N elementos.

$$\psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_N \end{bmatrix} \tag{1.35}$$

A Eq. (1.34) é então interpretada como uma equação matricial. Para que ela faça sentido devemos satisfazer as condições:

- Deve dar a relação correta entre a energia e o momento isto é $E^2 = p^2c^2 + m^2c^4$, para uma partícula livre.

- Deve fornecer uma probabilidade definida positiva.
- Deve ser covariante para transformações de Lorentz.

Vejamos os dois primeiros requisitos. Para que se obtenha a relação energia-momento correta basta que cada componente satisfaça à equação de Klein Gordon. Para isso iteramos a Eq. (1.34)

$$\begin{aligned} -\hbar^2\frac{\partial^2\psi}{\partial t^2} &= \left(-i\hbar c\alpha^i\nabla_i + \beta mc^2\right) i\hbar\frac{\partial\psi}{\partial t} \\ &= \left[-\hbar^2c^2\frac{\alpha^i\alpha^j+\alpha^j\alpha^i}{2}\nabla_i\nabla_j - i\hbar mc^2(\alpha^i\beta+\beta\alpha^i)\nabla_i + \beta^2m^2c^4\right]\psi \end{aligned} \tag{1.36}$$

onde se usaram as propriedades de simetria e anti-simetria dos tensores. No Complemento 1.3 faz-se uma revisão destas propriedades. Para que cada componente satisfaça a equação de Klein- Gordon devemos ter

$$\begin{cases} \alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij} \\ \alpha^i\beta + \beta\alpha^i = 0 \\ (\alpha^i)^2 = \beta^2 = 1 \end{cases} \tag{1.37}$$

Temos portanto que construir 4 matrizes que *anticomutem*, sejam *hermíticas* e cujo *quadrado* seja a *unidade*. É desde logo claro que não podem ser 2×2 pois só há 3 matrizes 2×2 que anticomutam, as matrizes de Pauli. Para ver a dimensão mínima em que é possível realizá-las, observemos que sendo hermíticas os seus valores próprios são reais e iguais a ± 1 pois $\alpha^{i2} = \beta^2 = 1$. Das relações de anticomutação pode-se concluir que têm traço nulo. Por exemplo

$$\alpha^i = -\beta\alpha^i\beta \tag{1.38}$$

ou seja

$$Tr(\alpha^i) = Tr(-\beta\alpha^i\beta) = -Tr(\alpha^i) = 0 \tag{1.39}$$

Isto tem como consequência que N deve ser par para que o número de valores próprios $+1$ e -1 seja igual. Como $N = 2$ está excluído devemos ter $N = 4$ como a dimensão mais baixa onde se realiza a Eq. (1.37). Uma representação explicita, a chamada *representação de Dirac* é

$$\alpha^i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} \qquad ; \qquad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1.40}$$

onde σ_i são as matrizes de Pauli:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad ; \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad ; \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1.41}$$

É um exercício trivial verificar que a Eq. (1.40) satisfaz as condições da Eq. (1.37). Claro que a escolha não é *única*, mas voltaremos a este assunto mais tarde.

Vamos agora ver a questão da corrente de probabilidade. Para isso escrevemos a equação conjugada hermítica da Eq. (1.34). Atendendo a que α^i e β são hermíticas, obtemos

$$-i\hbar\frac{\partial\psi^\dagger}{\partial t} = \psi^\dagger(i\hbar c\alpha^i\overleftarrow{\partial}_i + \beta mc^2) \tag{1.42}$$

Multiplicando a Eq. (1.34) à esquerda por $\psi^\dagger$ e a Eq. (1.42) à direita por ψ e subtraindo obtemos

$$i\hbar\frac{\partial}{\partial t}(\psi^\dagger\psi) = -i\hbar c\nabla_i(\psi^\dagger\alpha^i\psi) \tag{1.43}$$

ou ainda

$$\frac{\partial}{\partial t}(\psi^\dagger\psi) + \vec{\nabla}\cdot(\psi^\dagger c\vec{\alpha}\psi) = 0 \tag{1.44}$$

o que permite identificar uma densidade de probabilidade e uma corrente de probabilidade:

$$\rho = \psi^\dagger\psi \tag{1.45}$$

$$\vec{j} = \psi^\dagger c\vec{\alpha}\psi \tag{1.46}$$

Integrando a Eq. (1.44) em todo o espaço obtemos

$$\frac{d}{dt}\int d^3x\psi^\dagger\psi = 0 \tag{1.47}$$

o que está de acordo com identificarmos $\psi^\dagger\psi$ como uma densidade de probabilidade conservada no tempo.

A notação das Eqs. (1.44) e (1.46) antecipa o facto de $\vec{j}$ ser um 3-vetor. De facto temos de mostrar isso e muito mais. Na secção seguinte demonstraremos que $j^\mu = (c\rho, \vec{j})$ é um 4-vetor conservado, $\partial_\mu j^\mu = 0$ e que a equação de Dirac é *covariante*, isto é, que mantém a mesma forma em todos os referenciais de inércia.

1.4 Covariância da equação de Dirac

Antes de mostrar-mos a covariância da equação de Dirac vamos introduzir uma conveniente notação 4-dimensional. Multiplicamos a Eq. (1.34) por $\frac{1}{c}\beta$ à esquerda e introduzimos as matrizes

$$\gamma^0 \equiv \beta \qquad ; \qquad \gamma^i \equiv \beta\alpha^i \qquad i = 1, 2, 3 \tag{1.48}$$

Então a equação de Dirac escreve-se

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0 \tag{1.49}$$

ou ainda

$$(i\hbar\not{\partial} - mc)\psi = 0 \tag{1.50}$$

onde se introduziu a notação, devida a Feynman

$$\not{\partial} \equiv \gamma^\mu\partial_\mu \tag{1.51}$$

As matrizes γ^μ, na representação de Dirac, são[1,2]

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad ; \qquad \gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix} \tag{1.53}$$

As matrizes γ^μ não são hermíticas mas obedecem à relação importante,

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \,. \tag{1.54}$$

É fácil de ver que as relações da Eq. (1.37) se escrevem duma forma compacta em termos das matrizes γ, isto é

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \tag{1.55}$$

onde introduzimos a notação para anticomutador $\{A, B\} \equiv AB + BA$.

Devemos notar que apesar da sugestiva notação da Eq. (1.49) ainda *não* demonstrámos a covariância da equação. Antes de o fazermos vejamos a relação entre diferentes representações das matrizes γ.

1.4.1 Transformações de equivalência

Consideremos duas representações das matrizes γ, γ^μ e $\tilde{\gamma}^\mu$. Isto quer dizer que tanto γ^μ como $\tilde{\gamma}^\mu$ satisfazem a Eq. (1.55). A equação de Dirac nestas representações será

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0 \tag{1.56}$$

[1]Na nossa convenção não subimos ou descemos índices nas matrizes de Pauli. Elas são sempre definidas com o índice em baixo como na Eq. (1.41).

[2]As matrizes estão representadas em blocos 2×2, por isso a matriz 1 é a matriz identidade em dimensão 2×2, isto é

$$1_{2\times2} \equiv \mathbb{1} \equiv 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.52}$$

Nós vamos usar a notação simplificada de 1 e dimensionalidade deve ser entendida pelo contexto.

e

$$(i\hbar\tilde{\gamma}^\mu\partial_\mu - mc)\tilde{\psi} = 0 \tag{1.57}$$

Se ambas as equações descrevem a mesma Física, deve haver uma relação entre ψ e $\tilde{\psi}$. Seja

$$\psi = U\tilde{\psi} \tag{1.58}$$

onde U é uma matriz constante que admite inversa. Então substituindo na Eq. (1.56) e multiplicando à esquerda por U^{-1} obtemos

$$\tilde{\gamma}^\mu = U^{-1}\gamma^\mu U \tag{1.59}$$

Uma transformação deste tipo é chamada *transformação de equivalência* e embora mude a função de onda não altera a Física (ver Problema 1.19 para a definição das representações de *Majorana* e *Quiral*).

1.4.2 Demonstração da covariância

Consideremos então a equação de Dirac em dois referenciais de inércia O e O'

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi(x) = 0 \tag{1.60}$$

e

$$(i\hbar\gamma'^\mu\partial'_\mu - mc)\psi'(x') = 0 \tag{1.61}$$

A matriz γ'^μ satisfaz as mesmas relações de anticomutação que γ^μ e além disso $\gamma'^{0\dagger} = \gamma'^0$ e $\gamma'^{i\dagger} = -\gamma'^i$. Pode-se então demonstrar que γ'^μ e γ^μ estão relacionados por uma transformação de equivalência

$$\gamma'^\mu = U^{-1}\gamma^\mu U \tag{1.62}$$

onde U é uma matriz unitária (ver Problema 1.18). Assim podemos passar toda a transformação para a função de onda e usar a *mesma representação* em *todos* os referenciais de inércia. As funções de onda $\psi'(x')$ e $\psi(x)$ devem então estar relacionados por

$$\psi'(x') = \psi'(ax) = S(a)\psi(x) = S(a)\psi(a^{-1}x') \tag{1.63}$$

com

$$x'^\mu = a^\mu{}_\nu x^\nu \tag{1.64}$$

e a matriz $S(a)$ deverá depender apenas de velocidade relativa e/ou rotação entre os dois referenciais O e O'. Substituindo a Eq. (1.63) na Eq. (1.61) obtemos

$$(i\hbar\gamma^\mu \frac{\partial}{\partial x'^\mu} - mc)S(a)\psi(x) = 0 \tag{1.65}$$

Sabendo que

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = (a^{-1})^\nu{}_\mu \partial_\nu \tag{1.66}$$

obtemos

$$\left[i\hbar S^{-1}(a)\gamma^\mu S(a)(a^{-1})^\nu{}_\mu \partial_\nu - mc\right]\psi(x) = 0 \tag{1.67}$$

o que comparando com a Eq. (1.60) dá

$$S^{-1}(a)\gamma^\mu S(a)(a^{-1})^\nu{}_\mu = \gamma^\nu \tag{1.68}$$

ou ainda

$$S(a)\gamma^\mu S^{-1}(a)a^\nu{}_\mu = \gamma^\nu \tag{1.69}$$

As Eqs. (1.69) são as relações fundamentais que permitem obter S. Para se obter a matriz S começamos por considerar transformações infinitesimais

$$a^\nu{}_\mu = g^\nu{}_\mu + \omega^\nu{}_\mu + \cdots \tag{1.70}$$

com

$$\omega^{\mu\nu} = -\omega^{\nu\mu} \tag{1.71}$$

o que resulta da aplicação da Eq. (1.70) na Eq. (1.12) conservando apenas termos de ordem ω. A Eq. (1.71) quer dizer que há somente seis parâmetros independentes. Veremos que eles podem ser identificados com os três graus de liberdade duma rotação mais os três graus de liberdade duma transformação de Lorentz numa direção arbitrária. Então se definirmos

$$S = 1 - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu} + \cdots \tag{1.72}$$

$$S^{-1} = 1 + \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu} + \cdots \tag{1.73}$$

onde as matrizes $\sigma_{\mu\nu}$ são antissimétricas

$$\sigma_{\mu\nu} = -\sigma_{\nu\mu} \tag{1.74}$$

obtemos a partir das relações da Eq. (1.69),

$$[\gamma^\mu, \sigma_{\alpha\beta}] = 2i(g^\mu{}_\alpha\gamma_\beta - g^\mu{}_\beta\gamma_\alpha) \tag{1.75}$$

Usando as relações de anticomutação dos $\gamma's$ é fácil de verificar que

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \tag{1.76}$$

satisfaz a condição da Eq. (1.75). Isto determina S e S^{-1} infinitesimalmente. De facto a forma Eq. (1.72) exponencia[3] pelo que a expressão para uma transformação finita é

$$S = e^{-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}} \tag{1.77}$$

Para encontrarmos a forma explícita da matriz S vamos distinguir a caso das rotações do das transformações de Lorentz propriamente ditas (conhecidas por "boosts"). Para as rotações definimos

$$(\theta^1, \theta^2, \theta^3) \equiv (\omega^2{}_3, \omega^3{}_1, \omega^1{}_2) \tag{1.78}$$

e

$$(\Sigma^1, \Sigma^2, \Sigma^3) \equiv (\sigma^{23}, \sigma^{31}, \sigma^{12}) \tag{1.79}$$

Então

$$S_R = e^{\frac{i}{2}\vec{\theta}\cdot\vec{\Sigma}} \tag{1.80}$$

Na representação de Dirac

$$\vec{\Sigma} \equiv \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \tag{1.81}$$

pelo que a Eq. (1.80) representa a generalização para spinores de 4 componentes da maneira como spinores de 2 componentes se transformam para rotações. O fator $\frac{1}{2}$ na Eq. (1.80) tem a ver com o facto de somente depois duma rotação de 4π a função de onda do eletrão retomar o mesmo valor. Usando

$$(\vec{\theta}\cdot\vec{\Sigma})(\vec{\theta}\cdot\vec{\Sigma}) = \vec{\theta}\cdot\vec{\theta} \tag{1.82}$$

podemos escrever, desenvolvendo a Eq. (1.80) em série

$$S_R = \cos\frac{\theta}{2} + i\hat{\theta}\cdot\vec{\Sigma}\sin\frac{\theta}{2} \tag{1.83}$$

onde $\hat{\theta}$ é o versor na direção da rotação. Esta relação pode ser usada para verificar a Eq. (1.69) para o caso das rotações finitas (ver Problema 1.25).

Para as transformações de Lorentz propriamente ditas (*boosts*), definimos o 3-vetor $\vec{\omega}$ tal que ($\omega^i \equiv \omega^{0i}$)

[3]Isto é verdade para todos os grupos de transformações contínuas, os chamados grupos de Lie. Para estas transformações é suficiente conhecer o que se passa para transformações infinitesimais (álgebra de Lie) para saber o que acontece para transformações finitas (grupos de Lie).

$$\begin{cases} \hat{\omega} \equiv \hat{V} \\ \tanh\omega = \frac{V}{c} \end{cases} \tag{1.84}$$

onde V é a velocidade relativa dos dois referenciais. Então usando

$$\sigma^{0i} = \frac{i}{2}\left[\gamma^0, \gamma^i\right] = i\gamma^0\gamma^i = i\alpha^i \tag{1.85}$$

temos

$$S_L = e^{-\frac{1}{2}\vec{\omega}\cdot\vec{\alpha}} \tag{1.86}$$

com $\vec{\alpha}$ dada pela Eq. (1.40). Pode-se também mostrar que

$$(\vec{\omega}\cdot\vec{\alpha})^2 = \vec{\omega}\cdot\vec{\omega} \tag{1.87}$$

pelo que obtemos

$$S_L = \cosh\frac{\omega}{2} - \hat{\omega}\cdot\vec{\alpha}\sinh\frac{\omega}{2} \tag{1.88}$$

Esta expressão pode ser usada para verificar a Eq. (1.69) para o caso das transformações de Lorentz finitas (ver Problema 1.25). Isto demonstra que a expressão da Eq. (1.77) é correta para transformações finitas. No Complemento 1.4, as propriedades de transformação dos spinores são usadas para calcular a precessão de Thomas.

É fácil verificar que S_R é unitária enquanto S_L não o é. É contudo possível demonstrar[4] que

$$S^{-1} = \gamma^0 S^\dagger \gamma^0 \tag{1.89}$$

tanto para S_R como para S_L. Esta relação é importante pois permite mostrar que a corrente é um 4-vetor. Na notação 4-dimensional a Eq. (1.46) escreve-se

$$j^\mu(x) = c\psi^\dagger(x)\gamma^0\gamma^\mu\psi(x) \tag{1.90}$$

Vejamos então como j^μ se transforma:

$$\begin{aligned} j'^\mu &= c\psi'^\dagger(x')\gamma^0\gamma^\mu\psi'(x') \\ &= c\psi^\dagger(x)S^\dagger\gamma^0\gamma^\mu S\psi(x) \\ &= c\psi^\dagger(x)\gamma^0\gamma^0 S^\dagger\gamma^0\gamma^\mu S\psi(x) \\ &= c\psi^\dagger(x)\gamma^0 S^{-1}\gamma^\mu S\psi(x) \end{aligned} \tag{1.91}$$

[4]Basta recordar que $[\gamma^0, \vec{\Sigma}] = 0$ e $\{\gamma^0, \vec{\alpha}\} = 0$.

Se usarmos a Eq. (1.69) obtemos então $S^{-1}\gamma^\mu S = a^\mu{}_\nu \gamma^\nu$ e portanto

$$j'^\mu = a^\mu{}_\nu j^\nu \tag{1.92}$$

como seria de esperar para um 4-vetor. Na Eq. (1.90) aparece a combinação $\psi^\dagger\gamma^0$. Como veremos no seguimento, esta expressão aparece tantas vezes que é conveniente definir um símbolo para ela

$$\overline{\psi} \equiv \psi^\dagger\gamma^0 \tag{1.93}$$

que se designa por *adjunto de Dirac*. Uma propriedade importante do adjunto de Dirac é a modo como se transforma numa mudança de referencial. Obtemos

$$\overline{\psi'}(x') = \psi'^\dagger(x')\gamma^0 = \psi^\dagger(x)S^\dagger\gamma^0 = \psi^\dagger\gamma^0\gamma^0 S^\dagger\gamma^0 = \overline{\psi}(x)S^{-1}\,, \tag{1.94}$$

onde se usou a Eq. (1.89) e $\gamma^0\gamma^0 = 1$.

1.4.3 Inversão no espaço

Embora mais tarde voltemos ao caso das simetrias discretas, (P,C e T) é útil introduzir aqui a inversão no espaço ou Paridade. A inversão no espaço é uma transformação de Lorentz com det $a = -1$ dada pela matriz

$$a^\mu{}_\nu \equiv \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{1.95}$$

Queremos encontrar a matriz S_P que transforma os spinores e que deve satisfazer a Eq. (1.69), isto é,

$$S_P^{-1}\gamma^\mu S_P = a^\mu{}_\nu \gamma^\nu \tag{1.96}$$

Vemos facilmente que esta relação é satisfeita para

$$\mathcal{P} \equiv S_P = e^{i\varphi}\gamma^0 \tag{1.97}$$

onde $e^{i\varphi}$ é uma fase arbitrária.

1.4.4 Covariantes bilineares

Tal como qualquer matriz complexa 2×2 se pode exprimir em termos de 4 matrizes linearmente independentes (por exemplo a matriz identidade mais as matrizes de Pauli) assim qualquer matriz 4×4 se pode exprimir em termos de 16 matrizes 4×4 linearmente independentes. Para introduzir estas matrizes é conveniente definir a seguinte matriz

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{1.98}$$

que na representação de Dirac tem a forma

$$\gamma_5 = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix} \tag{1.99}$$

Da definição resultam as propriedades importantes

$$\{\gamma_5, \gamma^\mu\} = 0 \tag{1.100}$$

$$(\gamma_5)^2 = 1, \quad \gamma_5^\dagger = \gamma_5\,. \tag{1.101}$$

Estamos agora em posição de definir as 16 matrizes 4×4

$$\Gamma^S = 1 \tag{1.102}$$

$$\Gamma^V_\mu = \gamma_\mu \tag{1.103}$$

$$\Gamma^T_{\mu\nu} = \sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \tag{1.104}$$

$$\Gamma^A_\mu \equiv \gamma_5\gamma_\mu \tag{1.105}$$

$$\Gamma^P = \gamma_5 \tag{1.106}$$

onde os símbolos S, V, T, A e P designam respetivamente: escalar, vetor, tensor, pseudo vetor e pseudo-escalar e têm a ver com a maneira como os bilineares

$$\overline{\psi}\ \Gamma^a\psi \qquad a = S, V, T, A \text{ e } P \tag{1.107}$$

se transformam para transformações de Lorentz. Por exemplo

$$\begin{aligned} \overline{\psi'}(x')\ \Gamma^A\psi'(x') &= \overline{\psi'}(x')\gamma_5\gamma^\mu\psi'(x') \\ &= \overline{\psi}(x)S^{-1}\gamma_5\gamma^\mu S\psi(x) \\ &= \det\ a\ a^\mu{}_\nu\overline{\psi}(x)\gamma_5\gamma^\nu\psi(x) \end{aligned} \tag{1.108}$$

onde se usou o facto de $[S, \gamma_5] = 0$ para transformações de Lorentz próprias e $\{\mathcal{P}, \gamma_5\} = 0$ para a inversão no espaço. Isto mostra que $\overline{\psi}(x)\gamma_5\gamma_\mu\psi(x)$ se transforma como um vetor axial ou pseudo-vetor. De forma semelhante se podiam demonstrar as propriedades de transformação dos outros bilineares.

É fácil de mostrar (ver Problema 1.16) que as matrizes Γ^a satisfazem as propriedades

- $(\Gamma^a)^2 = \pm 1$
- $\mathrm{Tr}(\Gamma^a) = 0 \qquad \forall a \neq S$
- $\gamma^\mu\gamma_\mu = 4 \quad ; \quad \gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu \quad ; \quad \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}$
- $\gamma^\mu\gamma^\nu\gamma^\rho = g^{\mu\nu}\gamma^\rho - g^{\mu\rho}\gamma^\nu + g^{\nu\rho}\gamma^\mu + i\varepsilon^{\mu\nu\rho\alpha}\gamma_\alpha\gamma_5$ (1.109)

1.4.5 Sistema de unidades naturais

Em física de partículas tratamos de grandezas à escala sub-atómica, para as quais o sistema internacional (SI) não é bem adaptado. Assim faz sentido escolher um sistema de unidades mais adaptado a estas escalas, o chamado sistema de unidades naturais. Neste sistema as unidades [Kg,m,s] são substituídos por $[\hbar, c,\ \mathrm{GeV}]$, onde $1\ \mathrm{GeV} = 10^9\ \mathrm{eV} = 1.602 \times 10^{-10}$ J, é uma unidade de energia.

No sistema de unidades naturais é usual fazer uma simplificação adicional, escolhendo $\hbar = c = 1$, complementado com $\epsilon_0 = \mu_0 = 1$ (notar que $c = 1$ implica $\epsilon_0\mu_0 = 1$). Assim só há uma unidade independente, a energia. Por vezes, em vez da energia usa-se também a distância ou o tempo, sendo a conversão feita usando as relações:

$$1 = c = 2.999792 \times 10^8\ \mathrm{ms}^{-1} \quad \rightarrow \quad 1\ \mathrm{s} = 2.999792 \times 10^8\ \mathrm{m} \tag{1.110}$$

$$1 = \hbar c = 197.327\ \mathrm{MeV.fermi} \quad \rightarrow \quad 1\ \mathrm{MeV}^{-1} = 197.327 \times 10^{-15}\ \mathrm{m} \tag{1.111}$$

$$1 = \hbar = 1.054571 \times 10^{-34}\ \mathrm{Js} \quad \rightarrow \quad 1\ \mathrm{J.s} = 9.482529 \times 10^{33} \tag{1.112}$$

Como exemplo, vamos escrever as várias unidades em termos da energia. Temos sucessivamente

$$\begin{aligned} 1\ \mathrm{m} &= 5.067730 \times 10^{12}\ \mathrm{MeV}^{-1} \\ 1\ \mathrm{s} &= 1.520214 \times 10^{21}\ \mathrm{MeV}^{-1} \\ 1\ \mathrm{Kg} &= \frac{1\ \mathrm{J.s}}{1\ \mathrm{m}^2 \times 1\ \mathrm{s}^{-1}} = \frac{1\ \mathrm{J.s} \times 1\ \mathrm{s}}{1\ \mathrm{m}^2} = 5.613088 \times 10^{29}\ \mathrm{MeV}\ . \end{aligned} \tag{1.113}$$

Particularmente úteis são as relações:

$$\begin{aligned} 1\ \mathrm{s}^{-1} &= 6.578023 \times 10^{-22}\ \mathrm{MeV} \\ 1\ \mathrm{barn} &= 10^{-24}\ \mathrm{cm}^2 = 2.568189 \times 10^{-3}\ \mathrm{MeV}^{-2} \\ 1\ \mathrm{pb} &= = 2.568189 \times 10^{-15}\ \mathrm{MeV}^{-2} \\ 1\ \mathrm{MeV}^{-2} &= 3.893794 \times 10^{14}\ \mathrm{pb} \end{aligned} \tag{1.114}$$

$$\begin{aligned} 1\ \mathrm{GeV}^{-2} &= 3.893794 \times 10^8\ \mathrm{pb} \\ 1\ \mathrm{eV}^{-2} &= 1.5202 \times 10^{15}\ \mathrm{Hz} \end{aligned}$$

Poderia parecer que ao fazer $\hbar = c = 1$ se perde informação. No entanto é sempre possível voltar atrás e re-introduzir estas constantes. Tomemos como exemplo a secção eficaz $e^- + e^+ \to \mu^- + \mu^+$ em QED (isto é a baixas energias). No limite em que se desprezam as massas o resultado é

$$\sigma = \frac{4\pi\alpha^2}{s}\ \mathrm{GeV}^{-2} \tag{1.115}$$

onde s é o quadrado da energia no centro de massa e $\alpha = 1/137.032\cdots$, é a constante de estrutura fina. Se quisermos voltar para o sistema SI, usamos o facto de que uma secção eficaz tem as dimensões duma área. Então

$$\begin{aligned} L^2 =& \left(ML^2T^{-2}\right)^{-2} \hbar^\beta c^\gamma \\ =& M^{-2}L^{-4}T^4 \left(ML^2T^{-1}\right)^\beta \left(LT^{-1}\right)^\gamma \\ =& M^{-2+\beta}\, L^{-4+2\beta+\gamma}\, T^{4-\beta-\gamma}\,, \end{aligned} \tag{1.116}$$

que tem como solução, $\beta = 2, \gamma = 2$ e portanto a expressão correta, do ponto de vista dimensional, seria

$$\sigma = \frac{4\pi\hbar^2\, c^2\, \alpha^2}{s}\ . \tag{1.117}$$

1.5 Spin e a equação de Dirac

1.5.1 O operador de spin na equação de Dirac

Em mecânica quântica uma observável é conservada se comutar com o Hamiltoniano do sistema. Por exemplo, em mecânica não relativista o Hamiltoniano para a partícula livre (equação de Schrödinger),

$$H_{\mathrm{S}} = \frac{p^2}{2m} \tag{1.118}$$

comuta com o operador momento angular $\vec{L} = \vec{r} \times \vec{p}$ e portanto o momento angular é conservado. A questão que se põe agora é saber o que acontece em mecânica quântica relativista para o Hamiltoniano de Dirac,

$$H_{\mathrm{D}} = \vec{\alpha} \cdot \vec{p} + \beta m\ . \tag{1.119}$$

Vamos calcular este comutador. Isto faz-se mais facilmente se usarmos as expressões com índices em vez de vetores. Como se trata de índices do espaço vamos usar os índices $i, j, k, \ldots$. Obtemos

$$\left[H_{\mathrm{D}}, L^i\right] = \left[\alpha^j p^j, L^i\right] \tag{1.120}$$

porque no espaço de Dirac, L^i é proporcional à matriz identidade que comuta com a matriz constante β. Usando agora $L^i = \epsilon^{ikm} x^k p^m$, obtemos sucessivamente,

$$\begin{aligned}\left[H_{\rm D}, L^i\right] =& \epsilon^{ikm}\left[\alpha^j p^j, x^k p^m\right] \\ =& \epsilon^{ikm}\alpha^j\left[p^j, x^k\right] p^m \\ =& -i\epsilon^{ikm}\alpha^k p^m = -i\left(\vec{\alpha}\times\vec{p}\right)^i \end{aligned} \tag{1.121}$$

isto é, o momento angular não comuta com o Hamiltoniano de Dirac,

$$\left[H_{\rm D}, \vec{L}\right] = -i\vec{\alpha}\times\vec{p} \tag{1.122}$$

e não é portanto uma quantidade conservada, mesmo para a partícula livre.

Se pensarmos um pouco isto não devia ser uma surpresa, pois do estudo do átomo de hidrogénio em mecânica quântica não relativista sabemos que o eletrão tem spin e é o momento angular total que é conservado. Em mecânica quântica não relativista o operador de spin é dado por ($\hbar = 1$),

$$\vec{S} = \frac{1}{2}\vec{\sigma}\,. \tag{1.123}$$

Como os spinores de Dirac têm quatro componentes, vamos generalizar este operador para

$$\vec{S} = \frac{1}{2}\vec{\Sigma}, \quad \vec{\Sigma} \equiv \begin{bmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{bmatrix}, \tag{1.124}$$

e vamos ver quais as relações de comutação deste operador com $H_{\rm D}$. Como $\vec{\Sigma}$ é diagonal comuta com a matriz também diagonal[5] β, portanto temos só de ver as relações de comutação com as matrizes α^i. Obtemos

$$\begin{aligned}\left[\alpha^i, \Sigma^j\right] =& \begin{bmatrix}0 & \sigma^i\\ \sigma^i & 0\end{bmatrix}\begin{bmatrix}\sigma^j & 0\\ 0 & \sigma^j\end{bmatrix} - \begin{bmatrix}\sigma^j & 0\\ 0 & \sigma^j\end{bmatrix}\begin{bmatrix}0 & \sigma^i\\ \sigma^i & 0\end{bmatrix} \\ =& \begin{bmatrix}0 & \left[\sigma^i, \sigma^j\right]\\ \left[\sigma^i, \sigma^j\right] & 0\end{bmatrix} \\ =& 2i\epsilon^{ijk}\begin{bmatrix}0 & \sigma^k\\ \sigma^k & 0\end{bmatrix} = 2i\epsilon^{ijk}\alpha^k \end{aligned} \tag{1.125}$$

e portanto

$$\left[\vec{\alpha}\cdot\vec{p}, \vec{\Sigma}\right] = 2i\vec{\alpha}\times\vec{p} \tag{1.126}$$

onde usámos $[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$. Usando os resultados das Eqs. (1.122) e (1.126) podemos definir o momento angular total,

$$\vec{J} = \vec{L} + \vec{S} = \vec{r}\times\vec{p} + \frac{1}{2}\vec{\Sigma} \tag{1.127}$$

[5]Estamos a considerar a representação de Dirac, claro.

que satisfaz,

$$\left[H_{\rm D}, \vec{J}\right] = 0 \tag{1.128}$$

e portanto o momento angular total é conservado. Usando a Eq. (1.124) e as propriedades das matrizes de Pauli podemos facilmente mostrar que

$$S^2 = \frac{1}{4}\Sigma^2 = \frac{3}{4} \tag{1.129}$$

o que mostra que o eletrão tem $s = 1/2$.

1.5.2 O spin e o operador de Pauli-Lubanski

Introduzimos o spin na secção anterior duma forma muito intuitiva, procurando uma extensão do conceito em mecânica quântica não relativista. Vamos agora ver como o spin aparece numa forma mais formal, em particular como se deve generalizar a Eq. (1.127) no formalismo da relatividade restrita.

Comecemos com o caso dum campo escalar. Então numa transformação de Lorentz $x'^{\mu} = a^{\mu}{}_{\nu}\, x^{\nu}$ um campo escalar é invariante, isto é

$$\phi'(x') = \phi(x) \tag{1.130}$$

que pode ainda ser escrita como

$$\phi'(x) = \phi(a^{-1}\, x)\ . \tag{1.131}$$

Consideremos agora uma rotação em torno do eixo dos z. Usando uma notação matricial temos

$$\begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\epsilon & \sin\epsilon & 0 \\ 0 & -\sin\epsilon & \cos\epsilon & 0 \\ 0 & 0 & 0 & \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \epsilon & 0 \\ 0 & -\epsilon & 1 & 0 \\ 0 & 0 & 0 & \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{1.132}$$

onde a segunda forma é para rotações infinitesimais. Definindo $\vec{\epsilon} = \epsilon\, \vec{e}_z$ obtemos para rotações infinitesimais

$$x' = (x^0, \vec{x} - \vec{\epsilon} \times \vec{x}) \tag{1.133}$$

ou ainda

$$a^{-1}\, x = (x^0, \vec{x} + \vec{\epsilon} \times \vec{x}) \tag{1.134}$$

Obtemos portanto da Eq. (1.131)

$$\begin{aligned} \phi'(x) =& \phi(x^0, \vec{x} + \vec{\epsilon} \times \vec{x}) \simeq \phi(x) + \vec{\epsilon} \cdot (\vec{x} \times \vec{\nabla})\phi(x) \\ =& (1 + i\vec{\epsilon} \cdot \vec{L})\, \phi(x) \end{aligned} \tag{1.135}$$

mostrando que $\vec{L}$ é o gerador das rotações no espaço tridimensional. Agora definimos para as transformações de Lorentz infinitesimais uma relação semelhante[6], usando o caso de spinores (seria semelhante para qualquer campo)

$$\psi'(x) \equiv \left(1 - \frac{i}{2} J_{\mu\nu}\omega^{\mu\nu}\right)\psi(x) \tag{1.136}$$

onde os operadores $J_{\mu\nu}$ são os geradores do grupo de Lorentz (ver Problema 1.26 para uma descrição dos grupos de Lorentz e Poincaré).
Mas nós vimos que numa transformação de coordenadas os spinores se transformam de acordo com

$$\psi'(x') = \left(1 - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right)\psi(x) \tag{1.137}$$

ou ainda

$$\begin{aligned}\psi'(x) &= \left(1 - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right)\psi(x^\rho - \omega^\rho{}_\nu x^\nu) \\ &= \left(1 - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu} + x_\mu\omega^{\mu\nu}\partial_\nu\right)\psi(x)\ .\end{aligned} \tag{1.138}$$

Comparando a Eq. (1.138) com a Eq. (1.136) e usando a antisimetria do tensor $\omega^{\nu\nu}$, obtemos então,

$$J_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \frac{1}{2}\sigma_{\mu\nu}\ . \tag{1.139}$$

Esta relação é a generalização da Eq. (1.127) como se pode verificar tomando o caso das rotações.

Para voltar ao problema de descrever o spin no formalismo quadrimensional da relatividade restrita recordemos que o grupo de Poincaré tem dois *invariantes*, P^2 e W^2 (ver Problema 1.26), onde $P^2 = P_\mu P^\mu$ e $W^2 = W_\mu W^\mu$, com P_μ o operador do *momento linear* e W_μ o chamado 4-vetor de Pauli-Lubanski, definido por

$$W_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^\sigma \tag{1.140}$$

Pode-se mostrar em geral que se P^2 tem valores próprios m^2 então W^2 tem valores próprios [10] (ver também o Complemento 1.5),

$$W^2 = -m^2 s(s+1) \tag{1.141}$$

onde s é o spin (inteiro ou semi inteiro). No Complemento 1.5 faz-se uma explicação mais aprofundada do significado de W_μ e da razão da Eq. (1.141).

[6]Veja o Problema 1.29 para mostrar a compatibilidade das definições entre as Eqs. (1.136) e (1.135).

Vejamos a forma de W_μ para a equação de Dirac. Consideremos transformações de Lorentz infinitesimais. Usando a Eq. (1.139) na definição de W_μ obtemos

$$W_\mu = -\frac{i}{4}\varepsilon_{\mu\nu\rho\sigma}\sigma^{\nu\rho}\partial^\sigma \tag{1.142}$$

Calculando W^2 é fácil de ver (Problema 1.28) que os valores próprios para a equação de Dirac são

$$W^2 = -\frac{3}{4}m^2 \tag{1.143}$$

o que confirma que $s = \frac{1}{2}$. Voltaremos a este assunto depois de ter estudado as soluções de onda plana.

Notemos que da definição de W_μ só a parte que tem que ver com o spin contribui, já que a parte que é a generalização do momento angular orbital, $i(x_\mu\partial_\nu - x_\nu\partial_\mu)$, se anula devido à antisimetria do tensor $\varepsilon_{\mu\nu\rho\sigma}$. Assim, para um campo escalar como não há a parte do spin, isto é $J_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$ obtemos $W_\mu = 0$, implicando então da Eq. (1.141) que um campo escalar tem spin zero.

1.6 Soluções para a partícula livre

1.6.1 Ondas planas

Tomemos a equação de Dirac para a partícula livre

$$(i\partial\!\!\!/ - m)\psi(x) = 0 \tag{1.144}$$

A Eq. (1.144) admite como soluções ondas planas

$$\psi(x) = w(\vec{p})e^{-ip_\mu x^\mu} \tag{1.145}$$

desde que $p_\mu p^\mu = m^2$. Isto implica que $(p^0)^2 = E^2 = \vec{p}\cdot\vec{p} + m^2$ e portanto temos soluções com energia positiva e negativa. Nas nossas convenções fazemos $p^0 = E = \sqrt{|\vec{p}|^2 + m^2} > 0$ sempre, pelo que devemos ter

$$\psi^r(x) = w^r(\vec{p})e^{-i\varepsilon_r p_\mu x^\mu} \tag{1.146}$$

onde $\varepsilon_r = \pm 1$ para soluções de energia positiva e negativa, respetivamente, e o índice r explicita as diferentes soluções independentes, como veremos de seguida.

Para determinar $w^r(\vec{p})$ vamos considerar primeiro o caso da partícula em repouso e depois efetuaremos uma transformação de Lorentz para obter $w^r(\vec{p})$. No referencial próprio a equação de Dirac reduz-se a

$$\left(i\gamma^0\frac{\partial}{\partial t} - m\right)\psi = 0 \tag{1.147}$$

Usando a representação de Dirac, Eq. (1.53), é fácil de ver que a equação se escreve

$$m\left(\varepsilon_r\gamma^0 - 1\right)\psi^r = 0 \tag{1.148}$$

onde

$$\psi^r = w^r(0)e^{-i\varepsilon_r mt} \tag{1.149}$$

com

$$\varepsilon_r = \begin{cases} +1 & r = 1,2 \\ -1 & r = 3,4 \end{cases} \tag{1.150}$$

e

$$w^1(0) = \sqrt{2m}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad ; \quad w^2(0) = \sqrt{2m}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{1.151}$$

$$w^3(0) = \sqrt{2m}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad ; \quad w^4(0) = \sqrt{2m}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{1.152}$$

Vemos portanto que $r = 1,2$ são soluções da energia *positiva* e $r = 3,4$ da energia *negativa*. O fator $\sqrt{2m}$ da normalização foi introduzido por conveniência como será claro mais tarde (esta normalização é a nossa *única* diferença em relação às convenções de Bjorken e Drell [11]). Se usarmos o operador $\Sigma^3 = \sigma^{12}$ vemos ainda que $w^{(r)}(0)$ são funções próprias de Σ^3 com valores próprios ± 1. Assim as soluções $r = 1,2$ descrevem o eletrão de *Schrödinger-Pauli* e as soluções de energia negativa, $r = 3,4$ serão interpretadas mais tarde.

Para obtermos as soluções $w^r(\vec{p})$ efetuamos então uma transformação de Lorentz para um sistema que se mova com velocidade $-\vec{V}$. Usando a Eq. (1.86) e a Eq. (1.88) obtemos

$$\begin{aligned} w^r(\vec{p}) &= e^{-\frac{1}{2}\vec{\omega}\cdot\vec{\alpha}}w^r(0) \\ &= \left[\cosh\frac{\omega}{2}1 - \hat{\omega}\cdot\vec{\alpha}\sinh\frac{\omega}{2}\right]w^r(0) \\ &= \cosh\frac{\omega}{2}\left[1 + \frac{\vec{p}\cdot\vec{\alpha}}{E+m}\right]w^r(0) \end{aligned} \tag{1.153}$$

onde se usou (notar que $\cosh\omega = \gamma$, $\sinh\omega = \gamma\beta$),

$$\tanh\omega = |\vec{V}| = \beta \to \tanh\frac{\omega}{2} = \frac{|\vec{p}|}{E+m}\ . \tag{1.154}$$

Se notarmos que

$$\vec{\alpha}\ w^r(0) = -\vec{\gamma}\gamma^0 w^r(0) = -\varepsilon_r \vec{\gamma}\ w^{(r)}(0) \tag{1.155}$$

$$w^r(0) = \gamma^0\gamma^0 w^r(0) = \varepsilon_r \gamma^0 w^r(0) \tag{1.156}$$

podemos finalmente escrever

$$w^r(\vec{p}) = \frac{\cosh \omega/2}{E+m}(\varepsilon_r \not{p} + m) w^r(0) \tag{1.157}$$

onde

$$\cosh \frac{\omega}{2} = \sqrt{\frac{E+m}{2m}} \tag{1.158}$$

Notar que o fator $\sqrt{\frac{1}{2m}}$ na Eq. (1.158) cancela com o $\sqrt{2m}$ em $w^r(0)$.

A forma explicita da Eq. (1.157) permite mostrar as seguintes relações importantes (ver Problema 1.17)

$$(\not{p} - \varepsilon_r m) w^r(\vec{p}) = 0 \qquad \overline{w}^r(\vec{p})(\not{p} - \varepsilon_r m) = 0 \tag{1.159}$$

$$\overline{w}^r(\vec{p}) w^r(\vec{p}) = 2m\ \delta_{rr'}\varepsilon_r \tag{1.160}$$

$$\sum_{r=1}^{4} \varepsilon_r w^r_\alpha(\vec{p})\overline{w}^r_\beta(\vec{p}) = 2m\ \delta_{\alpha\beta} \tag{1.161}$$

$$w^{r\dagger}(\varepsilon_r \vec{p}) w^{r'}(\varepsilon_{r'}\vec{p}) = 2E\ \delta_{rr'} \tag{1.162}$$

Para mostrar estas relações é conveniente ter uma forma explicita para $w^\dagger(\vec{p})$ que pode ser obtida a partir da Eq. (1.157) e da relação

$$\gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu \tag{1.163}$$

que resulta da própria definição e da hermiticidade de $\vec{\alpha}$ e β. Obtemos

$$w^{r\dagger}(\vec{p}) = w^{r\dagger}(0)(\not{p}\gamma^0 + m)\frac{1}{\sqrt{2m}\sqrt{E+m}} \tag{1.164}$$

ou para $\overline{w}^r(\vec{p})$

$$\overline{w}^r(\vec{p}) = \overline{w}^r(0)(\varepsilon_r \not{p} + m)\frac{1}{\sqrt{2m}\sqrt{E+m}} \tag{1.165}$$

Convém notar que $\overline{w}^r(\vec{p}) w^r(\vec{p})$ é um escalar que na nossa *normalização* vale $2m$ enquanto que $w^{\dagger r}(\vec{p}) w^r(\vec{p}) = 2E$ se transforma como a componente temporal dum 4-vetor o que está de acordo com a interpretação de $\rho = \psi^\dagger\psi$ como a *densidade* de *probabilidade*. O facto que é $\overline{w}$ e não $w^\dagger$ que intervém na relação de fecho, Eq. (1.161), deve-se à não unitariedade das transformações de Lorentz.

Exemplo 1 *Mostrar a relação Eq. (1.161).*

Comecemos por notar que, usando as Eqs. (1.157) e (Eq. 1.165) podemos escrever

$$w^r_\alpha(\vec{p})\overline{w}^r_\beta(\vec{p}) = \frac{1}{2m}\frac{1}{E+m}\left(\varepsilon_r \not{p} + m\right)_{\alpha\alpha'}\left(\varepsilon_r \not{p} + m\right)_{\beta'\beta} w^r_{\alpha'}(0)\overline{w}^r_{\beta'}(0). \quad (1.166)$$

Calculando agora separadamente, para $r = 1, 2$,

$$\sum_{r=1}^{2} w^r_{\alpha'}(0)\overline{w}^r_{\beta'}(0) = 2m \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{\alpha'\beta'} = 2m \left(\frac{1+\gamma^0}{2}\right)_{\alpha'\beta'} \quad (1.167)$$

e portanto

$$\begin{aligned} \sum_{r=1}^{2} w^r_\alpha(\vec{p})\overline{w}^r_\beta(\vec{p}) &= \frac{1}{E+m}\left[\left(\not{p}+m\right)\left(\frac{1+\gamma^0}{2}\right)\left(\not{p}+m\right)\right]_{\alpha\beta} \\ &= \left(\not{p}+m\right)_{\alpha\beta} \end{aligned} \quad (1.168)$$

e para $r = 3, 4$,

$$\sum_{r=3}^{4} w^r_{\alpha'}(0)\overline{w}^r_{\beta'}(0) = 2m \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}_{\alpha'\beta'} = -2m \left(\frac{1-\gamma^0}{2}\right)_{\alpha'\beta'} \quad (1.169)$$

e portanto

$$\begin{aligned} \sum_{r=3}^{4} w^r_\alpha(\vec{p})\overline{w}^r_\beta(\vec{p}) &= -\frac{1}{E+m}\left[\left(-\not{p}+m\right)\left(\frac{1-\gamma^0}{2}\right)\left(-\not{p}+m\right)\right]_{\alpha\beta} \\ &= -\left(-\not{p}+m\right)_{\alpha\beta} \\ &= \left(\not{p}-m\right)_{\alpha\beta} \end{aligned} \quad (1.170)$$

Combinando Eq. (1.168) com a Eq. (1.170) obtemos então a Eq. (1.161).

1.6.2 O spin das soluções de onda plana

Consideremos agora as soluções de onda plana. Então

$$W_\mu = -\frac{1}{4}\varepsilon_r \; \varepsilon_{\mu\nu\rho\sigma}\sigma^{\nu\rho}p^\sigma = -\frac{1}{4}\gamma_5[\gamma_\mu, \not{p}] \; \varepsilon_r \quad (1.171)$$

onde se usou a relação

$$\varepsilon_{\mu\nu\alpha\beta}\,\sigma^{\alpha\beta} = -2i\,\sigma_{\mu\nu}\,\gamma_5 = \gamma_5\left[\gamma_\mu, \gamma_\nu\right] \quad (1.172)$$

No referencial próprio $p^\mu = (m, 0, 0, 0)$ e portanto,

$$W^0 = 0 \qquad , \qquad \frac{\vec{W}}{m} = \frac{1}{2}\vec{\Sigma}\ \varepsilon_r \tag{1.173}$$

onde $\vec{\Sigma}$ coincide com a definições das Eqs. (1.81) e (1.124). Calculando W^2 é fácil de ver que os valores próprios para a equação de Dirac são

$$W^2 = -\frac{3}{4}m^2 \tag{1.174}$$

o que confirma que $s = \frac{1}{2}$, tendo em conta a Eq. (1.141). Se introduzirmos um 4-vetor para descrever o spin s^μ que verifica[7] $s^\mu s_\mu = -1$ e $p_\mu s^\mu = 0$ o operador de spin numa direção arbitrária será (ver Problema 1.27)

$$-\frac{W \cdot s}{m} = \frac{1}{2m}\gamma_5\ \not{s}\ \not{p}\ \varepsilon_r \tag{1.175}$$

Usando este operador e escolhendo $s^\mu = (0, 0, 0, 1)$ no referencial próprio é fácil de ver que $w^r(0)$ são os estados próprios com valor $\pm 1/2$ segundo o eixo dos zz ($+1/2$ para $r = 1, 4$ e $-1/2$ para $r = 2, 3$). De facto, no referencial próprio temos

$$\begin{aligned} \frac{1}{2m}\gamma_5 \not{s} \not{p} \varepsilon_r &= -\frac{1}{2}\gamma_5\gamma^3 = -\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}\sigma_3 & 0 \\ 0 & -\frac{1}{2}\sigma_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & & & \\ & -\frac{1}{2} & & \\ & & -\frac{1}{2} & \\ & & & \frac{1}{2} \end{bmatrix} \end{aligned} \tag{1.176}$$

É convencional introduzir aqui a seguinte notação. Designamos por $u(p, s)$ uma solução de energia positiva de momento p_μ e spin s^μ. Satisfaz às equações

$$(\not{p} - m)u(p, s) = 0 \tag{1.177}$$

e

$$\vec{\Sigma} \cdot \vec{s}\ u(p, s) = u(p, s) \tag{1.178}$$

onde a Eq. (1.178) é no referencial próprio. De modo semelhante designamos por $v(p, s)$ uma solução de energia negativa que satisfaz

$$(\not{p} + m)v(p, s) = 0 \tag{1.179}$$

[7] Basta ver que no referencial próprio $s^\mu = (0, \vec{s})$ e $p^\mu = (m, \vec{0})$.

e que no referencial próprio tem spin $-\vec{s}$, isto é

$$\vec{\Sigma} \cdot \vec{s}\, v(p,s) = -v(p,s) \tag{1.180}$$

Com estas definições temos

$$w^1(\vec{p}) = u(p, s_z) \tag{1.181}$$

$$w^2(\vec{p}) = u(p, -s_z) \tag{1.182}$$

$$w^3(\vec{p}) = v(p, -s_z) \tag{1.183}$$

$$w^4(\vec{p}) = v(p, s_z) \tag{1.184}$$

onde s_z^μ é um 4-vetor que no referencial próprio toma a forma

$$s_z^\mu = (0, 0, 0, 1) \tag{1.185}$$

É fácil de verificar que as expressões explícitas para $u(p,s)$ e $v(p,s)$ são

$$u(p,s) = \sqrt{E+m} \begin{bmatrix} \chi(s) \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi(s) \end{bmatrix} \tag{1.186}$$

$$v(p,s) = \sqrt{E+m} \begin{bmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi(-s) \\ \chi(-s) \end{bmatrix} \tag{1.187}$$

onde $\chi(s)$ é um spinor de Pauli. Por exemplo

$$v(p,\uparrow) = \sqrt{E+m} \begin{bmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi(\downarrow) \\ \chi(\downarrow) \end{bmatrix} = \sqrt{E+m} \begin{bmatrix} \frac{p_-}{E+m} \\ -\frac{p_z}{E+m} \\ 0 \\ 1 \end{bmatrix} = w^4(\vec{p}) \tag{1.188}$$

onde $p_- = p_x - ip_y$.

1.6.3 Projetores de energia-momento e spin

A partir da equação de Dirac

$$(\not{p} - m)\, u(p,s) = 0, \quad (\not{p} + m)\, v(p,s) = 0 \tag{1.189}$$

é fácil de ver que

$$\Lambda \pm (p) = \frac{\pm \not{p} + m}{2\ m} \tag{1.190}$$

são projetores para as soluções de energia positiva e negativa, respetivamente. Satisfazem as relações

$$\begin{cases} \Lambda_\pm^2 = \Lambda_\pm \\ \Lambda_+\Lambda_- = \Lambda_-\Lambda_+ = 0 \\ \Lambda_+ + \ \Lambda_- = 1 \end{cases} \tag{1.191}$$

Para o spin apliquemos a Eq. (1.175) aos spinor $u(p,p)$ e $v(p,s)$. Obtemos

$$\begin{aligned} \frac{-W\cdot s}{m}\ u(p) &= \frac{1}{2}\frac{1}{m}\gamma_5 \not{s}\not{p}\ u(p.s) = \frac{1}{2}\gamma_5 \not{s}\ u(p,s) \\ \frac{-W\cdot s}{m}\ v(p) &= -\frac{1}{2}\frac{1}{m}\gamma_5 \not{s}\not{p}\ v(p.s) = \frac{1}{2}\gamma_5 \not{s}\ v(p,s) \end{aligned} \tag{1.192}$$

onde se usou a Eq. (1.189). Atendendo a que $(\gamma_5 \not{s})(\gamma_5 \not{s}) = 1$ é fácil de ver que o projetor de spin deverá ser

$$P(s) \equiv \frac{1+\gamma_5 \not{s}}{2} \tag{1.193}$$

Podemos verificar que $P^2(s) = P(s)$, $P(s)P(-s) = 0$ e $P(s) + P(-s) = 1$. É ainda fácil de ver que no referencial em que a partícula está em repouso temos

$$\begin{aligned} P(-s_z) &= \frac{1-\gamma_5 \not{s}_z}{2} = \frac{1+\gamma_5\gamma^3}{2} \\ &= \frac{1-\Sigma^3\gamma^0}{2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{1.194}$$

pelo que

$$P(-s_z)w^3(0) = \frac{1-\Sigma^3\gamma^0}{2}w^3(0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} w^3(0) \tag{1.195}$$

$$= w^3(0) \tag{1.196}$$

Isto justifica a identificação de $w^3(\vec{p})$ com $v(p,-s_z)$.

Os projetores $\Lambda_\pm(p)$ e $P(\pm s)$ desempenham um papel muito importante em desenvolver meios de cálculo eficazes sem recurso às formas explicitas dos spinores.

1.6.4 Grupos de onda

Como a equação de Dirac é linear, soluções localizadas da equação podem ser obtidas como sobreposição das soluções de onda plana. Vamos estudar estas sobreposições.

Comecemos por formar um grupo de onda com soluções de energia positiva, somente. Então

$$\psi^{(+)}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E}\sum_{\pm s} b(p,s)u(p,s)e^{-ip\cdot x} \tag{1.197}$$

onde os fatores foram escolhidos para tornarem a normalização simples. Obtemos

$$\begin{aligned}\int d^3x\psi^\dagger(x)\psi(x) &= \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{2E}\right)^2\sum_{(s,s')} b^*(p,s')b(p,s)u^\dagger(p,s')u(p,s)\\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E}\sum_s |b(p,s)|^2 = 1\end{aligned} \tag{1.198}$$

onde se usaram as condições de normalização, Eq. (1.162). Notar que com a nossa escolha o fator $\frac{d^3p}{E}$ é invariante de Lorentz.

Podemos agora calcular a densidade de corrente associada a este grupo de onda

$$\begin{aligned}\vec{J}^{(+)} &= \int d^3x\psi^{(+)\dagger}\vec{\alpha}\psi^{(+)} = \int d^3x\overline{\psi}^{(+)}\vec{\gamma}\psi^{(+)}\\ &= \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{2E}\right)^2\sum_{s,s'} b^*(p,s')b(p,s)\overline{u}(p,s')\vec{\gamma}u(p,s)\end{aligned} \tag{1.199}$$

Para prosseguir convém introduzir a decomposição de Gordon (ver Problema 1.20)

$$\overline{u_1}(p_1,s_1)\gamma^\mu u_2(p_2,s_2) = \frac{1}{2\ m}\overline{u}_1(p_1)\Big[(p_1+p_2)^\mu + i\sigma^{\mu\nu}(p_1-p_2)_\nu\Big]u_2(p_2) \tag{1.200}$$

Então

$$\overline{u}(p,s')\vec{\gamma}u(p,s) = 2\vec{p}\ \delta_{ss'} \tag{1.201}$$

e portanto

$$\begin{aligned}\vec{J}^{(+)} &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E}\frac{\vec{p}}{E}\sum_s |b(p,s)|^2\\ &= <\frac{\vec{p}}{E}>\end{aligned} \tag{1.202}$$

onde a Eq. (1.202) resulta da Eq. (1.198). Mas $< \frac{\vec{p}}{E} >$ é a velocidade de grupo pelo que obtemos o resultado familiar em mecânica quântica não relativista. Há contudo uma inconsistência em considerar unicamente as soluções de energia positiva. Por exemplo, se localizarmos um eletrão em $t = 0$, então com o decorrer do tempo são necessárias as soluções de energia negativa para o descrever (ver Problema 1.33). O conjunto completo de soluções inclui as soluções de energia positiva e negativa.

Seja então

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E} \sum_s \left[b(p,s)u(p,s)e^{-ip\cdot x} + d^*(p,s)v(p,s)e^{ipx} \right] \tag{1.203}$$

Um cálculo simples dá para a probabilidade

$$\int d^3x \psi^\dagger \psi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E} \sum_s \left[|b(p,s)|^2 + |d(p,s)|^2 \right] = 1 \tag{1.204}$$

e para a corrente

$$\begin{aligned} J^k = \int d^3x \overline{\psi} \gamma^k \psi \quad &= \quad \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E} \left\{ \sum_s \left[|b(p,s)|^2 + |d(p,s)|^2 \right] \frac{p^k}{E} \right. \\ &\quad + i \sum_{s,s'} b^*(\tilde{p},s') d^*(p,s) e^{2iEt} \overline{u}(\tilde{p},s') \sigma^{k0} v(p,s) \\ &\quad \left. - i \sum_{s,s'} b(\tilde{p},s') d(p,s) e^{-2iEt} \overline{v}(p,s) \sigma^{k0} u(\tilde{p},s) \right\} \end{aligned} \tag{1.205}$$

onde $\tilde{p} \equiv (p^0, -\vec{p})$. Vemos que para além do termo da velocidade de grupo há termos cruzados entre as soluções de energia positiva e negativa que oscilam rapidamente com frequências $> 2 \times 10^{21}$ Hz[8]. Estas oscilações são proporcionais às amplitudes das soluções de energia negativa no grupo de ondas. Serão importantes se estas amplitudes forem grandes. Do Problema 1.33 pode-se ver que isso é verdade se quisermos ter eletrões localizados em dimensões da ordem do seu comprimento de Compton $\lambda_c = \frac{1}{m} \simeq 4 \times 10^{-11}\ cm$. Isto quer dizer que a interpretação em termos de funções de onda começa a ter problemas quando queremos descrever fenómenos a esta escala (ver Problema 1.34).

1.7 Antipartículas

Apesar de todos os sucessos da equação de Dirac descritas anteriormente o problema das soluções com energia negativa continua por resolver. Este problema não é um

[8]Temos $\omega = 2E > 2m \simeq 1$ MeV $= 1.5 \times 10^{21}$ s^{-1}.

problema académico, pois é preciso explicar porque é que os eletrões nos átomos não efetuam transição para estados de energia negativa. Por exemplo um cálculo simples dá para o eletrão, no estado fundamental do hidrogénio, uma taxa de transição de 10^8 s^{-1} para decair no intervalo $[-mc^2, -2mc^2]$

1.7.1 A teoria dos buracos de Dirac.

Foi Dirac quem primeiro forneceu um tratamento consistente das soluções de energia negativa. O argumento de Dirac só funciona para fermiões pois faz uso do *Princípio de Exclusão de Pauli*. Assim para Dirac o *vácuo* da teoria é constituído por todos os estados de energia negativa preenchidas. Devido ao princípio de exclusão de Pauli um eletrão com energia $E > 0$ não pode então efetuar uma transição para um estado de energia negativa, explicando a estabilidade dos átomos. Claro que o vácuo tem energia e momento infinitos mas fisicamente só medimos diferenças em relação ao vácuo e essas serão finitas.

A principal consequência desta interpretação é a existência de antipartículas, neste caso o positrão. Consideremos que o vácuo tem uma lacuna ou buraco. Isto quer dizer a *ausência* dum eletrão de energia $-E$ e carga $-|e|$. Mas isto pode ser igualmente interpretado como *presença* duma partícula de carga $+|e|$ a energia positiva $+E$, isto é, o positrão. Assim a produção dum par eletrão-positrão é explicada esquematicamente na Figura 1.1

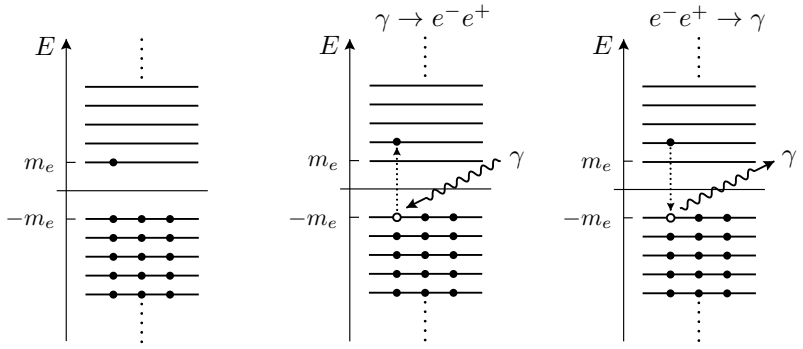


Figure 1.1: Esquema do mar de Dirac. Produção e aniquilação de pares.

Isto é, um eletrão é excitado dum estado de energia negativa deixando atrás de si uma lacuna no mar de Dirac. Como esta lacuna corresponde a um positrão ficou criado um par e^+e^-. Igualmente a aniquilação eletrão-positrão pode ser interpretada como um eletrão com $E > 0$ que faz uma transição para um estado com $E < 0$ que estava livre (positrão) desaparecendo portanto o eletrão e o positrão, conforme indicado na Figura 1.1

Com a teoria dos buracos abandonamos a interpretação em termos de funções de onda de uma partícula para passar a ser uma explicação em termo de muitas partículas. Só o formalismo da segunda quantificação, com os seus operadores de criação e destruição permitirá fazer uma descrição consistente desta teoria de muitas

partículas. Essa explicação, como veremos, também se aplicará aos bosões, o que a este nível não é possível de explicar por não satisfazerem ao princípio de exclusão de Pauli. Contudo a interpretação de Dirac teve um papel determinante no desenvolvimento da teoria e a descoberta experimental das antipartículas foi um grande sucesso.

1.7.2 A interpretação de Feynman-Stückelberg

A interpretação moderna das soluções de energia negativa foi desenvolvida por Stückelberg e Feynman no contexto de teoria quântica dos campos. As partículas de energia negativa ($E < 0$) são interpretadas como partículas de energia negativa que se propagam para trás no tempo. Estas partículas de energia negativa correspondem a antipartículas de energia positiva que se propagam para o futuro. A dependência no tempo das funções de onda não virá alterada por esta dupla transformação, $E \to -E$ e $t \to -t$, isto é

$$e^{-iEt} = e^{-i(-E)(-t)} \tag{1.206}$$

Para ilustrar esta ideia consideremos os diagrama da Fig.1.2. No diagrama da es-

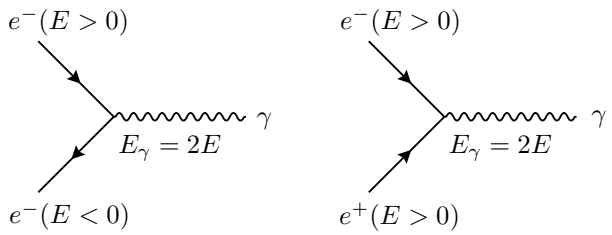


Figure 1.2: Equivalência entre eletrões de energia negativa e positrões de energia positiva.

querda um eletrão de energia E emite um fotão de energia $2E$ e para conservar energia um eletrão de energia $-E$. Sendo uma solução de energia negativa propaga-se para trás no tempo. Na interpretação de Feynman-Stückelberg, no diagrama da direita, um positrão de energia $E > 0$ aniquila-se com um eletrão de energia $E > 0$ para produzir um fotão de energia $2E$. Nesta interpretação tanto a partícula como a antipartícula se propagam para o futuro. Notar no entanto que nos diagramas de Feynman as antipartículas são desenhadas com a seta para trás no tempo, como no diagrama do lado esquerdo. Voltaremos a esta questão no próximo capítulo.

1.7.3 Operadores e os spinores das antipartículas

Há um detalhe subtil, mas importante, quando descrevemos as antipartículas por spinores $v(p)$ em termos dos momentos físicos, isto é,

$$\psi = v(E, \vec{p}) e^{i(Et - \vec{p}\cdot\vec{x})} \tag{1.207}$$

onde E e $\vec{p}$ são a energia e momento reais do positrão (antipartícula). A aplicação dos operadores de energia e momento dão

$$H\psi = i\frac{\partial\psi}{\partial t} = -E\psi, \quad \vec{p}_{\rm op}\psi = -i\vec{\nabla}\psi = -\vec{p}\psi \tag{1.208}$$

O sinal menos provém do facto que os spinores v não deixam de ser os estados de energia negativa das soluções da equação de Dirac. Isto quer dizer que os operadores que dão a energia e momento *físicos* nos spinores v são

$$H^{(v)} = -i\frac{\partial}{\partial t}, \quad \vec{p}^{(v)}_{\rm op} = i\vec{\nabla} \tag{1.209}$$

Uma consequência desta substituição $(E, \vec{p}) \to (-E, -\vec{p})$ é que o momento angular também muda de sinal,

$$\vec{L} = \vec{r} \times \vec{p} \to -\vec{L} \tag{1.210}$$

Para que o comutador $[H_D, \vec{L} + \vec{S}]$ seja nulo mantendo-se a conservação do momento angular total, então o operador de spin nos spinores v também tem de inverter o sinal,

$$\vec{S}^{(v)} = -\vec{S} \tag{1.211}$$

Em termos da explicação de Dirac, isto significa que a ausência dum eletrão de energia negativa e spin *up* é equivalente a um positrão com energia positiva e spin *down*. Este mesmo resultado levou à identificação da Eq. (1.180) (ver também as Eqs. (1.173) e (1.188)).

1.7.4 Conjugação de carga

Da teoria dos buracos emerge assim numa nova simetria de natureza: para cada partícula existe uma antipartícula. Esta simetria designa-se por *conjugação de carga*. Vejamos como a podemos definir. Para isso temos de definir a interação entre eletrões, positrões e fotões. Como veremos na secção 1.10, a interação é definida pela chamada prescrição mínima em que,

$$p^\mu \to p^\mu - q_e A^\mu \quad \Longrightarrow \quad i\partial^\mu \to i\partial^\mu - q_e A^\mu \tag{1.212}$$

onde usámos q_e para o eletrão (ver convenções na Eq. (1.324)). De acordo com a teoria dos buracos devemos ter uma correspondência unívoca entre as soluções de energia negativa da equação de Dirac para os eletrões

$$(i\not{\partial} - q_e\not{A} - m)\psi = 0 \tag{1.213}$$

e as soluções de energia positiva da equação de Dirac para os positrões,

$$(i\not{\partial} + q_e\not{A} - m)\psi_c = 0 \tag{1.214}$$

onde ψ_c é a função de onda para o positrão. Para encontrar a relação observemos que o sinal relativo entre $i\partial\!\!\!/$ e $q_e A\!\!\!/$ é o contrário nas duas equações. Isso leva-nos a considerar o complexo conjugado da Eq. (1.213). Obtemos

$$(-i\gamma^{\mu^*}\partial_\mu - q_e\gamma^{\mu^*}A_\mu - m)\psi^* = 0 \tag{1.215}$$

Usando agora $\gamma^{0T}\psi^* = \overline{\psi}^T$ e $\gamma^{0T}\gamma^{\mu^*}\gamma^{0T} = \gamma^{\mu T}$ obtemos

$$\left[-\gamma^{\mu T}(+i\partial_\mu + q_e A_\mu) - m\right]\overline{\psi}^T = 0 \tag{1.216}$$

Se encontrarmos uma matriz C, não singular, tal que

$$C\gamma^{\mu T}C^{-1} = -\gamma^\mu \tag{1.217}$$

podemos então identificar (a menos duma fase que tomamos igual a 1)

$$\psi_c \equiv C\overline{\psi}^T \tag{1.218}$$

Que existe uma matriz C verificando a Eq. (1.217) pode ser demonstrado construindo um exemplo específico. Na representação de Dirac é

$$C = i\gamma^2\gamma^0 = -C^{-1} = -C^\dagger = -C^T \tag{1.219}$$

ou mais explicitamente

$$C = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{1.220}$$

É instrutivo ver como é que a Eq. (1.218) relaciona as soluções de energia negativa com as funções de onda do positrão. Consideremos um eletrão de energia negativa em repouso com spin para baixo. Então

$$\psi = N \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{imt} \tag{1.221}$$

onde N é uma renormalização. Aplicando a Eq. (1.218) obtemos

$$\psi_c = N \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{-imt} \tag{1.222}$$

isto é, um positrão de energia positiva e spin para cima. Portanto a *ausência* dum eletrão de spin $\downarrow$ e energia negativa corresponde à *presença* dum positrão de energia

positiva e spin $\uparrow$. Foi este facto que nos levou a identificar $v(p,\uparrow)$ com $w^4(\vec{p})$ e $v(p,\downarrow)$ com $w^3(\vec{p})$.

Consideremos agora uma função de onda com spin e momento arbitrários, ψ'. Então (recordar que $\epsilon = \pm 1$ para os estados de energia positiva (negativa), respetivamente),

$$\psi = \left(\frac{\varepsilon \not{p} + m}{2m}\right)\left(\frac{1+\gamma_5 \not{s}}{2}\right)\psi' \tag{1.223}$$

e

$$\begin{aligned}
\psi_c &= C\overline{\psi}^T = C\gamma^0\psi^* \\
&= C\gamma^0\left(\frac{\varepsilon \not{p} + m}{2m}\right)^*\left(\frac{1+\gamma_5 \not{s}}{2}\right)^*\psi'^* \\
&= C\left(\frac{\varepsilon \not{p}^T + m}{2m}\right)\left(\frac{1-\gamma_5 \not{s}^T}{2}\right)\gamma^0\psi'^* \\
&= \left(\frac{-\varepsilon \not{p} + m}{2m}\right)\left(\frac{1+\gamma_5 \not{s}}{2}\right)\psi'_c
\end{aligned} \tag{1.224}$$

onde se usou $[C,\gamma_5]=0$ e $\gamma_5^T = \gamma_5 = \gamma_5^*$. Vemos que ψ_c é descrito pelos *mesmos* p^μ e s^μ mas o sinal da energia mudou. Notar que embora s^μ seja o mesmo, o spin é invertido como vimos na Eq. (1.222). Isto deve-se ao facto de o projetor de spin no referencial próprio ter a forma $\frac{1+\gamma^0\vec{\Sigma}\cdot\vec{s}}{2}$ e a mudança de sinal vem da matriz γ^0. Em termos de spinores para a partícula livre temos

$$\begin{aligned}
v(p,s) &= e^{i\phi(p,s)}u^c(p,s) \\
u(p,s) &= e^{i\phi(p,s)}v^c(p,s)
\end{aligned} \tag{1.225}$$

o que mostra que, à parte duma fase, $u(p,s)$ e $v(p,s)$ são spinores conjugadas de carga.

A conjugação de carga, forma conjuntamente com a paridade e a inversão no tempo, um conjunto de *simetrias discretas* muito importantes para a caracterização das partículas e suas interações. Para um estudo mais aprofundado em teoria quântica dos campos ver [12].

1.8 Spin e helicidade

Para partículas no referencial próprio, os spinores $u((E,\vec{0}),s)$ e $v((E,\vec{0}),s)$, são estados próprios do operador S_z,

$$S_z = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} . \tag{1.226}$$

Isto deixa de ser verdade quando $\vec{p} \neq 0$. No entanto, para o caso particular do momento linear ser segundo o eixo dos z, essa situação ainda se mantém. De facto se $\vec{p} = \pm|\vec{p}|\vec{e}_z$, obtemos das Eqs. (1.186) e (1.187),

$$u_\uparrow = N \begin{bmatrix} 1 \\ 0 \\ \frac{\pm|\vec{p}|}{E+m} \\ 0 \end{bmatrix} , \; u_\downarrow = N \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{\pm|\vec{p}|}{E+m} \end{bmatrix} , \; v_\uparrow = N \begin{bmatrix} 0 \\ \frac{\mp|\vec{p}|}{E+m} \\ 0 \\ 1 \end{bmatrix} , \; v_\downarrow = N \begin{bmatrix} \frac{\pm|\vec{p}|}{E+m} \\ 0 \\ 1 \\ 0 \end{bmatrix} , \tag{1.227}$$

e obtemos

$$\begin{aligned} S_z u_\uparrow(E,\pm|\vec{p}|\vec{e}_z) &= +\frac{1}{2} u_\uparrow(E,\pm|\vec{p}|\vec{e}_z) \\ S_z u_\downarrow(E,\pm|\vec{p}|\vec{e}_z) &= -\frac{1}{2} u_\downarrow(E,\pm|\vec{p}|\vec{e}_z) \\ S_z^{(v)} v_\uparrow(E,\pm|\vec{p}|\vec{e}_z) &= -S_z v_\uparrow(E,\pm|\vec{p}|\vec{e}_z) = +\frac{1}{2} v_\uparrow(E,\pm|\vec{p}|\vec{e}_z) \\ S_z^{(v)} v_\downarrow(E,\pm|\vec{p}|\vec{e}_z) &= -S_z v_\downarrow(E,\pm|\vec{p}|\vec{e}_z) = -\frac{1}{2} v_\downarrow(E,\pm|\vec{p}|\vec{e}_z) . \end{aligned} \tag{1.228}$$

Portanto para uma partícula com momento $\vec{p} = (0,0,\pm|\vec{p}|)$ os spinores $u_\uparrow, v_\uparrow$ correspondem a spin *up* e os spinores $u_\downarrow, v_\downarrow$ a spin down, conforme indicado na Fig. 1.3.

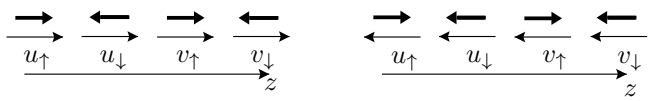


Figure 1.3: Spinores e spins para movimento segundo $\pm\vec{e}_z$.

1.8.1 Helicidade

As propriedades dos spinores para movimento segundo o eixo dos z descritas acima não são particularmente úteis nas aplicações, pois nem as partículas resultantes das colisões vão segundo o eixo dos z, nem as soluções anteriores fornecem uma base em que expandir os estados pois $[H_D, S_z] \neq 0$, e portanto não é possível definir uma base simultânea de H_D e S_z.

A base mais conveniente leva-nos ao conceito de helicidade. A helicidade é definida como a projeção do spin na direção do movimento, isto é

$$h = \frac{\vec{S}\cdot\vec{p}}{|\vec{p}|} = \frac{1}{2}\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}\,. \tag{1.229}$$

É fácil de mostrar que $\left[H_D, \vec{\Sigma}\cdot\vec{p}\right] = 0$ (ver Problema 1.23), e que portanto h comuta com o Hamiltoniano livre de Dirac. Como o spin medido segundo qualquer eixo está quantizado e só pode tomar os valores $\pm\frac{1}{2}$, os valores próprios da helicidade são também $\pm\frac{1}{2}$. Designamos estes estados por $\uparrow$ ou RH para $h = +\frac{1}{2}$ e $\downarrow$ ou LR para $h = -\frac{1}{2}$, conforme indicado na Fig. 1.4. Notar que o conceito de helicidade não é

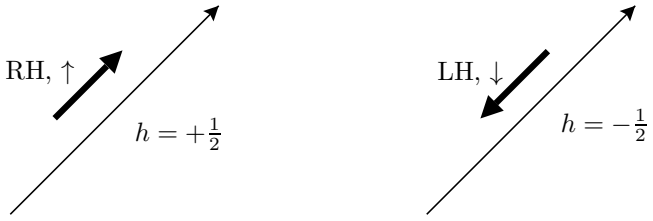


Figure 1.4: Estados próprios da helicidade para spin 1/2.

invariante de Lorentz pois, para partículas com massa, é sempre possível ir para um referencial onde se muda o sentido do momento. Já o conceito de quiralidade que, como veremos, está relacionado é invariante de Lorentz. Preferimos a notação $\uparrow,\downarrow$, para não confundir com os estados próprios da quiralidade que veremos depois.

1.8.2 Spinores de helicidade

Para as aplicações é útil ter uma representação explícita dos spinores de helicidade. Comecemos pelos spinores u para as soluções de energia positiva. Queremos resolver a equação aos valores próprios,

$$h\,u = \lambda u\,. \tag{1.230}$$

Podemos escrever esta equação na forma

$$\frac{1}{2|\vec{p}|}\begin{bmatrix}\vec{\sigma}\cdot\vec{p} & 0\\ 0 & \vec{\sigma}\cdot\vec{p}\end{bmatrix}\begin{bmatrix}u_A\\ u_B\end{bmatrix} = \lambda\begin{bmatrix}u_A\\ u_B\end{bmatrix}\,, \tag{1.231}$$

donde resulta

$$(\vec{\sigma}\cdot\vec{p})u_A = 2|\vec{p}|\lambda u_A, \quad (\vec{\sigma}\cdot\vec{p})u_B = 2|\vec{p}|\lambda u_B\,. \tag{1.232}$$

Usando agora $(\vec{\sigma}\cdot\vec{p})^2 = |\vec{p}|^2$, obtemos,

$$|\vec{p}|^2 u_A = 2|\vec{p}|\lambda(\vec{\sigma}\cdot\vec{p})u_A = 4|\vec{p}|^2 u_A\lambda^2\,, \tag{1.233}$$

donde resulta $\lambda = \pm 1/2$ como era de esperar. Vamos agora encontrar os vetores próprios correspondentes a estes valores próprios. Basta encontrar u_A pois usando a equação de Dirac, $(\not{p} - m)u = 0$ obtemos,

$$(\vec{\sigma} \cdot \vec{p})u_A = (E + m)u_B \,, \tag{1.234}$$

e usando agora a Eq. (1.232) obtemos

$$u_B = 2\lambda \frac{|\vec{p}|}{E+m} u_A \ . \tag{1.235}$$

Para encontrar u_A escrevemos

$$\vec{p} \equiv |\vec{p}|\, \vec{n}, \quad \vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,, \tag{1.236}$$

e então encontrar os valores próprios da Eq. (1.232), é equivalente a encontrar os valores próprios de

$$\vec{\sigma} \cdot \vec{n} = \begin{bmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{bmatrix} . \tag{1.237}$$

Este é um problema bem conhecido do spin em mecânica quântica não relativista com o resultado,

$$u_{A\uparrow} = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) e^{i\phi} \end{bmatrix}, \quad u_{A\downarrow} = \begin{bmatrix} -\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) e^{i\phi} \end{bmatrix} , \tag{1.238}$$

onde os vetores estão normalizados e escolhemos as fases globais de tal forma que no limite $\theta \to 0$ recuperamos os resultados da Eq. (1.227). Pondo tudo junto obtemos para os spinores u,

$$u_\uparrow = \sqrt{E+m} \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) e^{i\phi} \\ \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) e^{i\phi} \end{bmatrix}, \; u_\downarrow = \sqrt{E+m} \begin{bmatrix} -\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) e^{i\phi} \\ \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) e^{i\phi} \end{bmatrix} . \tag{1.239}$$

Os estados próprios de v obtêm-se de forma idêntica, não esquecendo que $\vec{S}^{(v)} = -\vec{S}$, e portanto

$$\frac{\vec{\Sigma} \cdot \vec{p}}{2|\vec{p}|} v_\uparrow = -\frac{1}{2} v_\uparrow \,. \tag{1.240}$$

O resultado final é

$$v_\uparrow = \sqrt{E+m} \begin{bmatrix} \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) e^{i\phi} \\ -\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) e^{i\phi} \end{bmatrix}, \; v_\downarrow = \sqrt{E+m} \begin{bmatrix} \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) e^{i\phi} \\ \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) e^{i\phi} \end{bmatrix} . \tag{1.241}$$

Quando estudarmos as colisões em QED, voltaremos a este assunto e mostraremos a sua utilidade nas aplicações.

1.9 Partículas de spin 1/2 sem massa

Na nossa descrição que fizemos da equação de Dirac considerámos sempre o caso de fermiões com massa. Existem contudo na natureza partículas de spin 1/2 com uma massa muito pequena, os neutrinos. De facto, as suas massas são inferiores a 1 eV e em muitas aplicações é uma muito boa aproximação considerá-los sem massa. Além disso, a massa do eletrão é $m_e = 0.511$ MeV o que é muito inferior às energias típicas das colisões nos aceleradores hoje em dia em operação. Assim deverá ser em muito casos, também uma boa aproximação desprezar a massa do eletrão. Por esta razão é importante estudar o caso sem massa.

1.9.1 Descrição em termos de 2-spinores: Equação de Weyl

Para o caso de massa nula, a equação de Dirac escreve-se

$$i\frac{\partial\psi}{\partial t} = -i\vec{\alpha}\cdot\vec{\nabla}\psi \tag{1.242}$$

Vemos assim que a matriz β desaparece do problema. Isto tem uma consequência importante sobre a dimensão mínima do espaço dos spinores. De facto a álgebra

$$\alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij} \tag{1.243}$$

pode ser verificada por matrizes 2×2, por exemplo, as matrizes de Pauli. Existem duas escolhas possíveis

$$\vec{\alpha} = \pm\vec{\sigma} \tag{1.244}$$

Para ver a que correspondem, consideremos soluções da Eq. (1.242) por ondas planas, isto é

$$\psi = \chi(p,s)e^{-ip\cdot x} \tag{1.245}$$

Obtemos então da Eq. (1.242)

$$\pm\vec{\sigma}\cdot\vec{p}\chi(p,s) = E\chi(p,s) \tag{1.246}$$

onde os sinais $\pm$ correspondem aos sinais da Eq. (1.244).

Consideremos primeiro o caso $\alpha = +\vec{\sigma}$. Na representação usual para as matrizes de Pauli e tomando o eixo positivo dos zz segundo $\vec{p}$ a solução da Eq. (1.246) é

$$\chi(p,+) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{1.247}$$

e obtemos ($\mid\vec{p}\mid = E$),

$$\frac{\vec{\sigma}.\vec{p}}{|\vec{p}|}\chi(p,+) = +\chi(p,+) \tag{1.248}$$

Vemos assim que esta solução corresponde a partículas sem massa com *helicidade positiva*[9] (polarização circular direita). Se escolhermos $\vec{\alpha} = -\vec{\sigma}$ temos

$$\chi(p,-) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{1.249}$$

e

$$\frac{\vec{\sigma}.\vec{p}}{|\vec{p}|}\chi(p,-) = -\chi(p,-) \tag{1.250}$$

Esta solução corresponde a *helicidade negativa* (polarização circular esquerda). Os neutrinos observados na Natureza correspondem a esta segunda escolha.

1.9.2 Descrição em termos de 4-spinores

Embora a descrição em termos de spinores a 2 componentes seja suficiente para fermiões sem massa[10], em muitas aplicações é conveniente uma descrição em termos de spinores de 4 componentes. Para se estudar melhor a relação entre os spinores a 2 e 4 componentes é conveniente escolher a *representação quiral* para as matrizes γ:

$$\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix} \quad ; \quad \beta = \gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad ; \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.251}$$

Se escrevermos

$$\psi = \begin{bmatrix} \chi(+) \\ \chi(-) \end{bmatrix} \tag{1.252}$$

obtemos

$$\begin{aligned} i\frac{\partial}{\partial t}\chi(+) &= -i\vec{\sigma}\cdot\vec{\nabla}\chi(+) - m\chi(-) \\ i\frac{\partial}{\partial t}\chi(-) &= i\vec{\sigma}\cdot\vec{\nabla}\chi(-) - m\chi(+) \end{aligned} \tag{1.253}$$

Vemos que as duas equações estão acopladas pelo termo da massa. No limite em que $m \to 0$ as duas equações desacoplam, dando origem à equação de Weyl, Eq. (1.242), para os dois casos $\vec{\alpha} = \pm\vec{\sigma}$. Notemos ainda que

[9] O operador $\frac{\vec{\sigma}.\vec{p}}{|\vec{p}|}$ é definido como a helicidade. Os seus valores próprios são ± 1.

[10] A Eq. (1.242) foi discutida para partículas sem massa por Weyl.

$$\gamma_5\psi(\pm) = \pm\psi(\pm) \tag{1.254}$$

onde

$$\psi(+) = \begin{bmatrix} \chi(+) \\ 0 \end{bmatrix} \quad ; \quad \psi(-) = \begin{bmatrix} 0 \\ \chi(-) \end{bmatrix} \tag{1.255}$$

mostrando que a *quiralidade* iguala a *helicidade* (é oposta para soluções de energia negativa).

1.9.3 Relação entre quiralidade e helicidade com $m = 0$

Vamos ver em mais detalhe a relação entre quiralidade e helicidade, mas usando agora a representação de Dirac. Nesta representação

$$\gamma_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \Longrightarrow \quad P_R = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad ; \quad P_L = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{1.256}$$

Consideremos agora os spinores de helicidade, Eqs. (1.239) e (1.241), no limite $m \to 0$. Obtemos

$$u_\uparrow = \sqrt{E}\begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) e^{i\phi} \\ \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) e^{i\phi} \end{bmatrix}, \; u_\downarrow = \sqrt{E}\begin{bmatrix} -\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) e^{i\phi} \\ \sin\left(\frac{\theta}{2}\right) \\ -\cos\left(\frac{\theta}{2}\right) e^{i\phi} \end{bmatrix}, \tag{1.257}$$

e

$$v_\uparrow = \sqrt{E}\begin{bmatrix} \sin\left(\frac{\theta}{2}\right) \\ -\cos\left(\frac{\theta}{2}\right) e^{i\phi} \\ -\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) e^{i\phi} \end{bmatrix}, \; v_\downarrow = \sqrt{E}\begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) e^{i\phi} \\ \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) e^{i\phi} \end{bmatrix}. \tag{1.258}$$

Usando as Eqs. (1.256), (1.257) e (1.258) podemos mostrar facilmente que

$$P_R u_\uparrow = u_\uparrow \quad ; \quad P_L u_\uparrow = 0 \quad ; \quad P_R u_\downarrow = 0 \quad ; \quad P_L u_\downarrow = u_\downarrow \tag{1.259}$$

$$P_R v_\uparrow = 0 \quad ; \quad P_L v_\uparrow = v_\uparrow \quad ; \quad P_R v_\downarrow = v_\downarrow \quad ; \quad P_L v_\downarrow = 0 \tag{1.260}$$

o que pode ser resumido na forma seguinte

$$\text{Partícula} \quad \Longrightarrow \quad \text{Helicidade} = \text{Quiralidade} \tag{1.261}$$

$$\text{Antipartícula} \quad \Longrightarrow \quad \text{Helicidade} = -\,\text{Quiralidade} \tag{1.262}$$

1.9.4 Relação entre quiralidade e helicidade com $m \neq 0$

Consideremos agora o caso com $m \neq 0$ mas $m \ll E$. Então podemos escrever,

$$u_\uparrow = N \begin{bmatrix} c \\ se^{i\phi} \\ \eta c \\ \eta s e^{i\phi} \end{bmatrix} \tag{1.263}$$

onde para simplificar definimos

$$c = \cos\left(\frac{\theta}{2}\right),\ s = \sin\left(\frac{\theta}{2}\right),\ \eta = \frac{|\vec{p}|}{E+m},\ N = \sqrt{E+m} \tag{1.264}$$

Obtemos então

$$P_R u_\uparrow = \frac{1}{2}(1+\eta)N \begin{bmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{bmatrix} = \frac{1}{2}(1+\eta)\sqrt{\frac{E+m}{E}}\,\sqrt{E} \underbrace{\begin{bmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{bmatrix}}_{u_R} \tag{1.265}$$

e

$$P_L u_\uparrow = \frac{1}{2}(1-\eta)N \begin{bmatrix} c \\ se^{i\phi} \\ -c \\ -se^{i\phi} \end{bmatrix} = \frac{1}{2}(1-\eta)\sqrt{\frac{E+m}{E}}\,\sqrt{E} \underbrace{\begin{bmatrix} c \\ se^{i\phi} \\ -c \\ -se^{i\phi} \end{bmatrix}}_{u_L} \tag{1.266}$$

onde $u_{R,L}$ são estados próprios da quiralidade, satisfazendo

$$\gamma_5 u_R = u_R\ ,\ \gamma_5 u_L = -u_L \tag{1.267}$$

Podemos portanto escrever os spinores de helicidade em termos dos estados próprios de quiralidade,

$$u_\uparrow = (P_R + P_L)u_\uparrow \tag{1.268}$$

$$= \frac{1}{2}(1+\eta)\sqrt{\frac{E+m}{E}}u_R + \frac{1}{2}(1-\eta)\sqrt{\frac{E+m}{E}}u_L \tag{1.269}$$

Se $m \ll E$ temos que

$$\eta \to 1 \quad \Longrightarrow \quad u_\uparrow \to u_R \tag{1.270}$$

De forma semelhante se podia fazer a identificação para os outros casos.

1.10 Acoplamento eletromagnético

A interação com o campo eletromagnético é obtida através da chamada *prescrição mínima*, que consiste na substituição

$$p^\mu \longrightarrow p^\mu - qA^\mu \tag{1.271}$$

para uma partícula de carga q (para o eletrão $q_e = -e < 0$). Fazendo a transcrição quântica dos operadores temos

$$\partial_\mu \longrightarrow \partial_\mu + iqA_\mu \,. \tag{1.272}$$

Esta relação tem origem no formalismo Lagrangiano, tanto para a partícula não relativista como em Teoria de Campo. Nos Complementos 1.6 e 1.7 fazemos uma revisão deste assunto para esses casos.

A equação de Dirac em interação com o campo eletromagnético escreve-se portanto

$$(i\gamma^\mu \partial_\mu - q_e \gamma^\mu A_\mu - m)\,\psi(x) = 0 \tag{1.273}$$

Voltando à forma inicial de Dirac, separando as derivadas em ordem ao tempo e ao espaço obtemos a generalização da Eq. (1.34),

$$\begin{aligned} i\frac{\partial \psi}{\partial t} &= \left[-i\vec{\alpha}\cdot(\vec{\nabla} - iq_e\vec{A}) + \beta m + q_e A^0\right]\psi \\ &= (H_0 + H')\psi \end{aligned} \tag{1.274}$$

onde

$$\begin{cases} H_0 &= -i\vec{\alpha}\cdot\vec{\nabla} + \beta m \\ H' &= -q_e\vec{\alpha}\cdot\vec{A} + q_e A^0 \end{cases} \tag{1.275}$$

Notar a analogia de H' com a expressão clássica para a energia de interação

$$H'_{\text{clássica}} = -q_e \frac{\vec{v}}{c}\cdot\vec{A} + eA^0 \tag{1.276}$$

o que indica a correspondência

$$\vec{v}_{op} = c\vec{\alpha}\ . \tag{1.277}$$

Esta correspondência é também clara da expressão para a corrente (ver também Problema 1.36).

1.11 Limite não relativista da equação de Dirac

1.11.1 Partícula livre

Para vermos qual o limite não relativista da equação de Dirac comecemos pelo caso da partícula livre. Se definirmos

$$\psi = \begin{pmatrix} \hat{\varphi} \\ \hat{\chi} \end{pmatrix} \tag{1.278}$$

onde $\hat{\chi}$ e $\hat{\varphi}$ são spinores de Pauli (2 componentes) e usarmos a representação de Dirac para $\vec{\alpha}$ e β obtemos o seguinte par de equações acopladas para os spinores $\hat{\chi}$ e $\hat{\varphi}$.

$$\begin{cases} i\dfrac{\partial\hat{\varphi}}{\partial t} = -i\vec{\sigma}\cdot\vec{\nabla}\hat{\chi} + m\hat{\varphi} \\ i\dfrac{\partial\hat{\chi}}{\partial t} = -i\vec{\sigma}\cdot\vec{\nabla}\hat{\varphi} - m\hat{\chi} \end{cases} \tag{1.279}$$

No limite não relativista $E - m \ll m$ pelo que fazemos a substituição

$$\begin{pmatrix} \hat{\varphi} \\ \hat{\chi} \end{pmatrix} = e^{-imt}\begin{pmatrix} \varphi \\ \chi \end{pmatrix} . \tag{1.280}$$

Substituindo a Eq. (1.280) na Eq. (1.279) obtemos

$$\begin{cases} i\dfrac{\partial\varphi}{\partial t} = -i\vec{\sigma}\cdot\vec{\nabla}\chi \\ i\dfrac{\partial\chi}{\partial t} = -i\vec{\sigma}\cdot\vec{\nabla}\varphi - 2m\chi \end{cases} \tag{1.281}$$

Como χ varia devagar com o tempo a segunda equação é resolvida, aproximadamente por

$$\chi \simeq -i\frac{\vec{\sigma}\cdot\vec{\nabla}}{2\ m}\varphi = \frac{\vec{\sigma}\cdot\vec{p}}{2\ m}\varphi \ll \varphi \tag{1.282}$$

Substituindo na primeira equação obtemos

$$i\frac{\partial\varphi}{\partial t} = -\frac{\nabla^2}{2\ m}\varphi \tag{1.283}$$

que é a equação de Schrödinger para a partícula livre. Assim, no limite não relativista as grandes componentes φ obedecem à equação não relativista e as pequenas componentes são desprezadas. Notar que desprezar χ corresponde também a desprezar as soluções de energia negativa. Daí o facto de elas nunca terem surgido em mecânica quântica não relativista.

1.11.2 Equação de Pauli

Estamos agora interessados no acoplamento ao campo eletromagnético. Para isso fazemos a substituição da Eq. (1.271) que se escreve mais explicitamente

$$\begin{cases} -i\vec{\nabla} \longrightarrow \vec{\pi} = -i\vec{\nabla} - q_e\vec{A} \\ i\dfrac{\partial}{\partial t} \longrightarrow i\dfrac{\partial}{\partial t} - q_e A^0 \end{cases} \tag{1.284}$$

Então com a separação da Eq. (1.278) obtemos em vez da Eq. (1.281)

$$\begin{cases} i\dfrac{\partial\varphi}{\partial t} = \vec{\sigma}\cdot\vec{\pi}\chi + q_e A^0\varphi \\ i\dfrac{\partial\chi}{\partial t} = \vec{\sigma}\cdot\vec{\pi}\varphi + q_e A^0\chi - 2\ m\chi \end{cases} \tag{1.285}$$

onde se usou a Eq. (1.280). Admitindo que os campos eletrostáticos são fracos (isto é $q_e A^0 \ll 2\ m$ ou seja, 13.6 eV $\ll 1$ MeV) obtemos,

$$\chi = \frac{\vec{\sigma}\cdot\vec{\pi}}{2\ m}\varphi \tag{1.286}$$

e portanto obtemos para as grandes componentes

$$i\frac{\partial\varphi}{\partial t} = \left[\frac{(\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{\pi})}{2\ m} + q_e A^0\right]\varphi\ . \tag{1.287}$$

Para vermos o significado desta equação notemos que (ver Problema 1.45)

$$(\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{\pi}) = \vec{\pi}\cdot\vec{\pi} - q_e\vec{\sigma}\cdot\vec{B} \tag{1.288}$$

Então

$$i\frac{\partial\varphi}{\partial t} = \left[\frac{(\vec{p} - q_e\vec{A})^2}{2\ m} - \frac{q_e}{2\ m}\vec{\sigma}\cdot\vec{B} + q_e A^0\right]\varphi \tag{1.289}$$

que é reconhecida como a equação de Pauli para o eletrão. Pondo os fatores $\hbar$ e c obtemos

$$H_{mag} = -\frac{q_e\hbar}{2mc}\vec{\sigma}\cdot\vec{B} \equiv -\vec{\mu}\cdot\vec{B} \tag{1.290}$$

com

$$\vec{\mu} = \frac{q_e\hbar}{2mc}\vec{\sigma} = -2\left(\frac{e}{2mc}\right)\frac{\hbar\vec{\sigma}}{2} \tag{1.291}$$

o que mostra que o fator giromagnético é $g = 2$. O ser capaz de *prever* o valor correto para g, foi um dos maiores sucessos da teoria de Dirac. Notar que na equação não relativista de Pauli o fator g tinha sido obtido experimentalmente. De facto, como veremos, `QED` permitirá calcular correções a este resultado. Se definirmos

$$a \equiv \frac{g-2}{2} \tag{1.292}$$

a situação atual é tão precisa [13, 14] que se *define*

$$a_e^{th} = a_e^{exp} = (115965218073 \pm 28) \times 10^{-14} \tag{1.293}$$

usando esta definição para determinar a constante de estrutura fina. A situação para o muão [15] apresenta alguma discrepância ao nível de 2σ. No entanto ainda não é claro se é uma flutuação estatística, ou algo de novo. Os resultados são
e

$$a_\mu^{th} = (116591841 \pm 81) \times 10^{-11} \tag{1.294}$$

$$a_\mu^{exp} = (116592080 \pm 58) \times 10^{-11} \tag{1.295}$$

A comparação entre a teoria e experiência pode ser observada na Fig. 1.5.

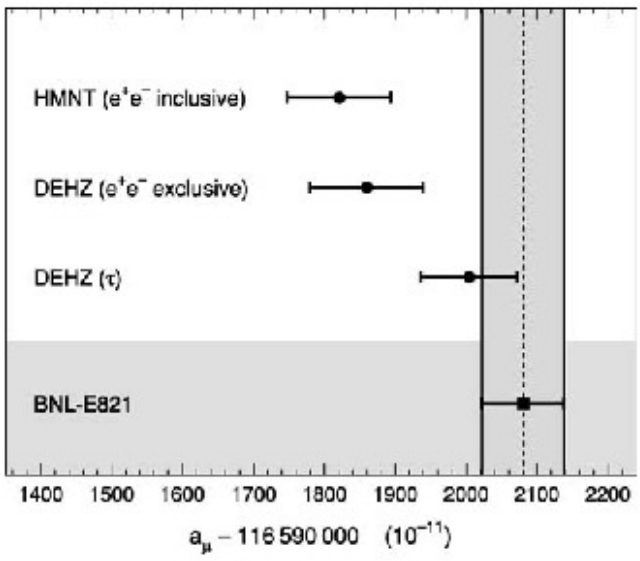


Figure 1.5: Comparação entre a teoria e experiência para o momento magnético anómalo do muão. Fonte: M. Davier and W. Marciano, Annu. Rev. Nucl. Part. Sci. 2004. 54:115

Para o caso importante dum campo magnético uniforme, $\vec{B} = \vec{\nabla} \times \vec{A}$ e $\vec{A} = \frac{1}{2}\vec{B} \times \vec{r}$, a Eq. (1.289) reduz-se a

$$i\frac{\partial \varphi}{\partial t} = \left[\frac{p^2}{2\ m} - \frac{q_e}{2\ m}(\vec{L} + 2\vec{S}) \cdot \vec{B}\right] \varphi \tag{1.296}$$

onde $\vec{L} = \vec{r} \times \vec{p}$ e $\vec{S} = \frac{1}{2}\vec{\sigma}$. Mais uma vez podemos ver o fator $g = 2$ para o spin do eletrão.

Complements

Complement 1.1 Definição de grupo

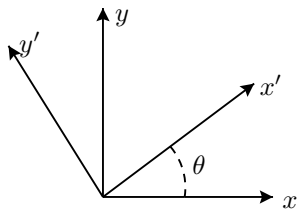
Embora não seja um tópico fundamental nesta disciplina introdutória, façamos uma pequena digressão sobre grupos apresentando a definição e dando um exemplo simples. Consideremos um conjunto

$$\mathcal{C} = \{a, b, c, \ldots\} \tag{1.297}$$

e uma operação que designamos por $\star$. O conjunto $\mathcal{C}$, dotado da operação $\star$, forma um grupo se:

1. Para $\forall a, b \in \mathcal{C}$ temos $a \star b = c \in \mathcal{C}$.
2. $\exists 1 \in \mathcal{C}$ tal que $\forall a \in \mathcal{C}$ então $1 \star a = a \star 1 = a$.
3. $\forall a \in \mathcal{C}, \exists a^{-1}$ tal que $a^{-1} \star a = a \star a^{-1} = 1$.

Em Física têm sobretudo importância os grupos de transformações contínuas (ou grupos de Lie). Vamos dar como exemplo as rotações no plano xy, isto é em torno do eixo dos z.
Consideremos a rotação indicada na Figura. Podemos escrever



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.298}$$

ou, numa forma matricial,

$$\hat{x}' = a(\theta)\hat{x}$$

Podemos facilmente verificar as propriedades

- Identidade:

$$1 = a(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.299}$$

- Inverso:

$$a^{-1}(\theta) = a(-\theta) \quad \rightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.300}$$

- Carácter comutativo:

$$a(\theta_1)a(\theta_2) = a(\theta_1 + \theta_2) = a(\theta_2)a(\theta_1) \tag{1.301}$$

A última propriedade indica que se trata dum grupo abeliano. Costuma designar-se por $O(2)$.

Complement 1.2 Equações de Maxwell na forma covariante

No final desta secção sobre as bases da relatividade restrita é útil revermos as equações fundamentais do eletromagnetismo usando um formalismo quadridimensional. Consideramos só o caso do vazio. A quantidade fundamental é o potencial vetor. A regra é sempre

que os 4-vetores contravariantes, isto é aqueles que se transformam como as coordenadas, têm as dimensões e os nomes da parte espacial. Assim definimos

$$A^\mu = (\frac{\phi}{c}, \vec{A}) \tag{1.302}$$

Podemos facilmente verificar que a condição de gauge de Lorenz [4]

$$\vec{\nabla} \cdot \vec{A} + \epsilon_0 \mu_0 \frac{\partial \phi}{\partial t} = 0 \tag{1.303}$$

se escreve nesta notação (notar que $\epsilon_0 \mu_0 = 1/c^2$),

$$\partial_\mu A^\mu = 0 \ . \tag{1.304}$$

O outro 4-vetor importante é a corrente J^μ definida por

$$J^\mu = (c\rho, \vec{J}) \tag{1.305}$$

satisfazendo a equação da continuidade

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 = \partial_\mu J^\mu \ . \tag{1.306}$$

Os campos eletromagnéticos fazem parte do chamado *tensor de Maxwell* definido por

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.307}$$

Usando as relações usuais [4] entre os potenciais e os campos $\vec{E}$ e $\vec{B}$, obtemos numa conveniente representação matricial

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{1.308}$$

ou ainda

$$F^{0i} = -\frac{1}{c} E^i, \quad F^{ij} = -\epsilon^{ijk} \ B^k \tag{1.309}$$

As equações de Maxwell não homogéneas (isto é com cargas e correntes) obtém-se a partir da equação

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu \tag{1.310}$$

As equações homogéneas são uma consequência do tensor $F_{\mu\nu}$ ser anti-simétrico. De facto, se definirmos o tensor dual (ver Problema 1.13)

$$\mathcal{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \tag{1.311}$$

então o facto do tensor de Maxwell ser anti-simétrico implica que

$$\partial_\mu \mathcal{F}^{\mu\nu} = 0 \tag{1.312}$$

e esta equação é equivalente às equações homogéneas, $\vec{\nabla}\cdot\vec{B}=0$ e $\vec{\nabla}\times\vec{E}+\dfrac{\partial\vec{B}}{\partial t}=0$. Este resultado é conhecido por identidade de Bianchi. Finalmente podemos obter facilmente as transformações dos campos numa mudança de referencial. Devemos ter em geral

$$F'^{\mu\nu}=a^{\mu}{}_{\alpha}a^{\nu}{}_{\beta}\;F^{\alpha\beta} \tag{1.313}$$

onde os coeficientes $a^{\mu}{}_{\nu}$ estão definidos na Eq. (1.9). Consideremos o caso particular duma transformação de Lorentz segundo o eixo do x dada pela Eq. (1.16). Obtemos para o campo elétrico,

$$E'^{i}=-c\;F'^{0i}=-c\;a^{0}{}_{\alpha}a^{i}{}_{\beta}\;F^{\alpha\beta} \tag{1.314}$$

pelo que

$$\begin{aligned}
E'^{1} &= -c\;a^{0}{}_{0}\,a^{1}{}_{1}\;F^{01}-c\;a^{0}{}_{1}\,a^{1}{}_{0}\;F^{10}=-c\;\gamma^{2}(1-\beta^{2})F^{01}=-c\;F^{01}=E^{1}\\
E'^{2} &= -c\;F'^{02}=-c\;a^{0}{}_{0}\,a^{2}{}_{2}\;F^{02}-c\;a^{0}{}_{1}\,a^{2}{}_{2}\;F^{12}=\gamma\left(E^{2}-\beta cB^{3}\right)\\
E'^{3} &= -c\;F'^{03}=-c\;a^{0}{}_{0}\,a^{3}{}_{3}\;F^{03}-c\;a^{0}{}_{1}\,a^{3}{}_{3}\;F^{13}=\gamma\left(E^{3}+\beta cB^{2}\right)\\
B'^{1} &= -F'^{23}=-a^{2}{}_{2}\,a^{3}{}_{3}\;F^{23}=B^{1}\\
B'^{2} &= F'^{13}=a^{1}{}_{0}\,a^{3}{}_{3}\;F^{03}+a^{1}{}_{1}\,a^{3}{}_{3}\;F^{13}=\gamma\left(B^{2}+\frac{\beta}{c}E^{3}\right)\\
B'^{3} &= -F'^{12}=-a^{1}{}_{0}\,a^{2}{}_{2}\;F^{02}-a^{1}{}_{1}\,a^{2}{}_{2}\;F^{12}=\gamma\left(B^{3}-\frac{\beta}{c}E^{2}\right)
\end{aligned} \tag{1.315}$$

Estas relações podem ser reescritas numa forma mais compacta

$$\begin{aligned}
E'_{\parallel} &= E_{\parallel}\\
E'_{\perp} &= \gamma\left(\vec{E}+c\;\vec{\beta}\times\vec{B}\right)_{\perp}
\end{aligned} \tag{1.316}$$

e

$$\begin{aligned}
B'_{\parallel} &= B_{\parallel}\\
B'_{\perp} &= \gamma\left(\vec{B}-\frac{1}{c}\vec{\beta}\times\vec{E}\right)_{\perp}\;.
\end{aligned} \tag{1.317}$$

Finalmente escrevemos a equação covariante para a força de Lorentz. Para isso é conveniente introduzir o 4-vetor velocidade [4], u^{μ}, definido por

$$u^{\mu}=(\gamma\;c,\gamma\vec{\beta}\;c),\quad u^{\mu}u_{\mu}=c^{2},\quad p^{\mu}=m\;u^{\mu} \tag{1.318}$$

Para uma partícula de carga q movendo-se num campo eletromagnético, as equações relativistas são

$$\frac{d\vec{p}}{dt}=q\left(\vec{E}+\vec{v}\times\vec{B}\right),\quad \frac{d\mathcal{E}}{dt}=e\vec{E}\cdot\vec{v} \tag{1.319}$$

e podem ser reunidas numa só equação covariante,

$$\frac{dp^{\mu}}{d\tau}=qF^{\mu\nu}u_{\nu} \tag{1.320}$$

onde $\tau = 1/\gamma t$ é o tempo próprio da partícula.

Complement 1.3 Tensores simétricos e anti-simétricos

Na Eq. (1.36), que conduziu às relações anteriores, simetrizámos o produto $\alpha^i\alpha^j$. Como este tipo de situação vai aparecer várias vezes, expliquemos um pouco mais. Tomemos como exemplo o espaço euclidiano a 3 dimensões com métrica δ_{ij}, mas os resultados são independentes desta hipótese. Seja T_{ij} um tensor de segunda ordem neste espaço (o que quer dizer que se transforma como as coordenadas em cada um dos seus índices), $A_{ij} = -A_{ji}$ um tensor anti-simétrico e $S_{ij} = S_{ji}$ um tensor simétrico. Então

$$\begin{aligned} A_{ij}S_{ij} &= A_{12}S_{12} + A_{21}S_{21} + \cdots \\ &= A_{12}S_{12} - A_{12}S_{12} + \cdots \\ &= 0 \end{aligned} \tag{1.321}$$

pois é sempre possível rearranjar os termos para se cancelarem dois a dois. Dizemos que *a contração dum tensor simétrico com um tensor anti-simétrico é sempre nula.* Por outro lado, um tensor sem simetria definida, pode ser sempre decomposto nas suas partes simétrica e anti-simétrica, isto é,

$$\begin{aligned} T_{ij} &= \frac{1}{2}\left(T_{ij} + T_{ji}\right) + \frac{1}{2}\left(T_{ij} - T_{ji}\right) \\ &= T^S_{ij} + T^A_{ij} \end{aligned} \tag{1.322}$$

Então obtemos facilmente

$$T_{ij}A_{ij} = T^A_{ij}A_{ij} \quad ; \quad T_{ij}S_{ij} = T^S_{ij}S_{ij} \tag{1.323}$$

Complement 1.4 Precessão de Thomas

Podemos usar a forma explícita das transformações de Lorentz para os spinores para discutir a precessão de Thomas que, como se sabe, corrige por um fator de 1/2 o termo do acoplamento spin-órbita. Recordemos o argumento da mecânica quântica não relativista. A interação do spin $\vec{S}$ do eletrão com um campo magnético $\vec{B}_{\rm rp}$, sentido no seu referencial próprio é dada por[11]

$$H = -\vec{\mu}\cdot\vec{B}_{\rm rp} = -\frac{q_e g}{2m}\ \vec{S}\cdot\vec{B}_{\rm rp} = \frac{eg}{2m}\ \vec{S}\cdot\vec{B}_{\rm rp}\,, \tag{1.325}$$

onde g é o famoso fator de Landé. Consideremos agora um átomo de hidrogénio num campo exterior $\vec{B}$. No referencial do laboratório (considerado como aquele em que o protão está em repouso), os campos eletromagnéticos são então (e é a carga do protão)

$$\vec{B},\ \text{(campo exterior)}, \quad \vec{E} = \frac{e}{4\pi\epsilon_0}\frac{\vec{r}}{r^3},\ \text{(campo de Coulomb)}. \tag{1.326}$$

[11] Como esta é a primeira ocasião em que aparece a carga do eletrão, aproveitamos para fixar a nossa notação. Vamos designar sempre por q_e a carga do eletrão e por, $e > 0$, a carga do protão. Temos portanto,

$$q_e = -e = Q_e\, e < 0, \quad \text{onde} \quad e > 0,\, Q_e = -1\,. \tag{1.324}$$

Considerando o referencial em que o eletrão se move com velocidade instantânea $\vec{v}$ como sendo um referencial de inércia e usando as leis de transformação para os campos eletromagnéticos no limite $v \ll c$, obtemos o campo no referencial próprio do eletrão,

$$B_{\rm rp} = \vec{B} - \frac{1}{c^2}\ \vec{v}\times\vec{E} = \vec{B} + \frac{e}{4\pi\epsilon_0 c^2 r^3}\ \vec{r}\times\vec{v} = \vec{B} + \frac{e}{4\pi\epsilon_0 m c^2 r^3}\ \vec{L} \tag{1.327}$$

o que, depois de substituir na Eq. (1.325), dá

$$H = \frac{eg}{2m}\ \vec{S}\cdot\vec{B} + \frac{e^2 g}{8\pi\epsilon_0 m^2 c^2 r^3}\ \vec{S}\cdot\vec{L} \tag{1.328}$$

O problema com esta equação, como é bem conhecido [2,3], é que para explicar o efeito de Zeeman devemos ter $g = 2$, enquanto que para explicar corretamente o desdobramento fino o acoplamento spin-órbita da Eq. (1.328) é o dobro do observado experimentalmente para $g = 2$. Assim parecia que a explicação correta do efeito de Zeeman estava em contradição com o desdobramento fino.

Em 1926 e 1927 L. H. Thomas [16, 17] identificou e resolveu o problema. A origem do problema tem a ver com o facto do referencial próprio do eletrão não ser um referencial de inércia, pois o eletrão tem aceleração. Isto faz com que o referencial próprio esteja a ter um movimento de precessão, a chamada *precessão de Thomas*. Como veremos na secção XXsec:FW, a equação de Dirac prevê o acoplamento spin-órbita correto, mas podemos usar as transformações de Lorentz para spinores para compreender o argumento de Thomas. O efeito está relacionado com o facto de que duas transformações de Lorentz em *direções diferentes* serem equivalentes a uma rotação e a uma transformação de Lorentz. É esta rotação no referencial próprio do eletrão que causa a precessão de Thomas.

Seja $|0\rangle$ o spinor do eletrão no seu referencial próprio (isto é onde $\vec{v} = 0$). Se quisermos representar o eletrão num referencial em que ele se move com velocidade $\vec{\beta} \equiv \vec{v}/c$, devemos ter,

$$\left|\vec{\beta}\right\rangle = S_L(-\vec{\beta})\,|0\rangle \tag{1.329}$$

onde $S_L(\vec{\beta})$ é dada pela Eq. (1.88), isto é,

$$S_L(\vec{\beta}) = \cosh\frac{\omega}{2} - \hat{\beta}\cdot\vec{\alpha}\sinh\frac{\omega}{2} \tag{1.330}$$

com

$$\tanh\omega = \beta, \qquad \hat{\beta} = \frac{\vec{\beta}}{\beta} \tag{1.331}$$

No instante $t + \delta t$ o eletrão tem velocidade $\vec{\beta} + \delta\vec{\beta}$ e portanto

$$\left|\vec{\beta} + \delta\vec{\beta}\right\rangle = S_L(-\vec{\beta} - \delta\vec{\beta})\,|0\rangle \tag{1.332}$$

Se quisermos relacionar o estado em $\vec{\beta}+\delta\vec{\beta}$ com o estado em $\vec{\beta}$ podemos escrever, invertendo a Eq. (1.329),

$$\left|\vec{\beta} + \delta\vec{\beta}\right\rangle = S_L(-\vec{\beta} - \delta\vec{\beta})S_L(\vec{\beta})\left|\vec{\beta}\right\rangle \tag{1.333}$$

Calculemos agora o produto das duas transformações de Lorentz da Eq. (1.333). Definindo

$$\tanh\omega' = |\vec{\beta} + \delta\vec{\beta}| \simeq \beta + \hat{\beta}\cdot\delta\vec{\beta} \tag{1.334}$$

onde no último passo se desprezaram termos de ordem superior em $\delta\vec{\beta}$, obtemos,

$$S_L(-\vec{\beta}-\delta\vec{\beta})S_L(\vec{\beta}) = \left(\cosh\frac{\omega'}{2} + \frac{\vec{\beta}+\delta\vec{\beta}}{|\vec{\beta}+\delta\vec{\beta}|}\cdot\vec{\alpha}\sinh\frac{\omega'}{2}\right)\left(\cosh\frac{\omega}{2} - \hat{\beta}\cdot\vec{\alpha}\sinh\frac{\omega}{2}\right) \quad (1.335)$$

Usando agora as aproximações (desprezando termos de ordem superior em $\delta\vec{\beta}$),

$$\begin{aligned}
&\frac{\vec{\beta}+\delta\vec{\beta}}{|\vec{\beta}+\delta\vec{\beta}|} \simeq \hat{\beta} + \frac{\delta\vec{\beta}}{\beta} - \frac{\hat{\beta}\cdot\delta\vec{\beta}}{\beta}\hat{\beta} \\
&\cosh\frac{\omega'}{2} \simeq \cosh\frac{\omega}{2} + \frac{\gamma^2}{2}\hat{\beta}\cdot\delta\vec{\beta}\ \sinh\frac{\omega}{2} \\
&\sinh\frac{\omega'}{2} \simeq \sinh\frac{\omega}{2} + \frac{\gamma^2}{2}\hat{\beta}\cdot\delta\vec{\beta}\ \cosh\frac{\omega}{2}
\end{aligned} \quad (1.336)$$

obtemos

$$\begin{aligned}
S_L(-\vec{\beta}-\delta\vec{\beta})S_L(\vec{\beta}) \;=\; &\left[\cosh\frac{\omega}{2} + \frac{\gamma^2}{2}\hat{\beta}\cdot\delta\vec{\beta}\ \sinh\frac{\omega}{2}\right. \\
&\left. + \left(\hat{\beta} + \frac{\delta\vec{\beta}}{\beta} - \frac{\hat{\beta}\cdot\delta\vec{\beta}}{\beta}\hat{\beta}\right)\cdot\vec{\alpha}\left(\sinh\frac{\omega}{2} + \frac{\gamma^2}{2}\hat{\beta}\cdot\delta\vec{\beta}\ \cosh\frac{\omega}{2}\right)\right] \\
&\left(\cosh\frac{\omega}{2} - \hat{\beta}\cdot\vec{\alpha}\sinh\frac{\omega}{2}\right)
\end{aligned} \quad (1.337)$$

O caso geral será estudado no Problema 1.43. Aqui basta considerar o caso particular em que $\vec{v}\perp\delta\vec{v}$ e em que $v \ll c$. Usando

$$\alpha_i\alpha_j = \delta_{ij} + i\epsilon_{ijk}\Sigma_k \quad (1.338)$$

obtemos,

$$S_L(-\vec{\beta}-\delta\vec{\beta})S_L(\vec{\beta}) \simeq 1 + \vec{\Delta\beta}\cdot\frac{\vec{\alpha}}{2} - i\vec{\Delta\theta}\cdot\frac{\vec{\Sigma}}{2} \simeq S_L(-\vec{\Delta\beta})\ S_R(-\vec{\Delta\theta}) \quad (1.339)$$

onde, para $v \ll c$

$$\vec{\Delta\theta} = \frac{\delta\vec{\beta}\times\vec{\beta}}{2}, \quad \vec{\Delta\beta} = \delta\vec{\beta} \quad (1.340)$$

A interpretação é agora fácil. Consideremos o estado $\left|\vec{\beta}+\vec{\Delta\beta}\right\rangle$ obtido no instante $t+\delta t$ por uma transformação de Lorentz *sem rotação* a partir de $\left|\vec{\beta}\right\rangle$, isto é

$$\left|\vec{\beta}+\vec{\Delta\beta}\right\rangle = S_L(-\vec{\Delta\beta})\left|\vec{\beta}\right\rangle = S_R(\vec{\Delta\theta})\left|\vec{\beta}+\delta\vec{\beta}\right\rangle \quad (1.341)$$

onde usámos a Eq. (1.339), isto é, o referencial obtido sem rotação no instante $t+\delta t$, onde se devem aplicar as leis da evolução do spin (ver a Eq. (1.343 em baixo) está rodado dum ângulo $\vec{\Delta}\theta$ em relação ao referencial que segue o eletrão no mesmo instante $t+\delta t$. Isto

que dizer que o referencial próprio do eletrão parece estar a precessar com uma velocidade angular

$$\vec{\Omega}_{\rm T} = \frac{\vec{\dot{v}} \times \vec{v}}{2c^2} \tag{1.342}$$

Para vermos como isto afeta o acoplamento spin-órbita recordemos que no referencial próprio temos

$$\left(\frac{d\vec{S}}{dt}\right)_{\rm rp} = \vec{\mu} \times \vec{B}_{\rm rp}, \quad H = -\vec{\mu} \cdot \vec{B}_{\rm rp} \tag{1.343}$$

com $B_{\rm rp}$ dado pela Eq. (1.327). Pelo facto do referencial do eletrão precessar devemos ter num referencial sem rotação,

$$\left(\frac{d\vec{S}}{dt}\right)_{\rm sr} = \left(\frac{d\vec{S}}{dt}\right)_{\rm rp} + \vec{\Omega}_T \times \vec{S} \tag{1.344}$$

ou seja

$$\left(\frac{d\vec{S}}{dt}\right)_{\rm sr} = \vec{S} \times \left(\frac{eg}{2m}\ \vec{B}_{\rm rp} - \vec{\Omega}_T\right) \tag{1.345}$$

o que, comparando com a equação para o Hamiltoniano, nos diz que o Hamiltoniano correto deve ser

$$H = \frac{eg}{2m}\ \vec{S} \cdot \vec{B} + \frac{e^2 g}{8\pi\epsilon_0 m^2 c^2 r^3}\ \vec{S} \cdot \vec{L} + \vec{S} \cdot \vec{\Omega}_T \tag{1.346}$$

Para o eletrão no campo de Coulomb temos, na aproximação $v \ll c$,

$$m\dot{v} = -\frac{e^2}{4\pi\epsilon_0 r^3}\vec{r} \tag{1.347}$$

pelo que

$$\Omega_T = -\frac{e^2}{8\pi\epsilon_0 m^2 r^3 c^2}\vec{L} \tag{1.348}$$

Substituindo na Eq. (1.346) obtemos finalmente o Hamiltoniano correto

$$H = \frac{eg}{2m}\ \vec{S} \cdot \vec{B} + \frac{e^2(g-1)}{8\pi\epsilon_0 m^2 c^2 r^3}\ \vec{S} \cdot \vec{L} \tag{1.349}$$

que substitui a Eq. (1.328) e que explica corretamente o efeito de Zeeman e o acoplamento spin órbita.

Muitas vezes diz-se que o efeito da relatividade dá um fator de 1/2 na taxa de precessão do eletrão e isto parecia estranho na altura. Por exemplo Uhlenbeck escreveu:

> ...when I first heard about [the Thomas precession], it seemed unbelievable that a relativistic effect could give a fator of 2 instead of something of order v/c... Even the cognoscenti of relativity theory (Einstein included!) were quite surprised.

No entanto esta descrição é um pouco enganadora já que há dois efeitos de ordem v/c que se adicionam, como os cálculos acima mostram. Voltaremos a esta questão no Complemento 1.8.

Complement 1.5 Propriedades do operador de Pauli-Lubanski

Mostremos em maior detalhe algumas propriedades deste operador W_μ. Começamos por definir os dois vetores

$$\vec{J} \equiv \left(J^{23}, J^{31}, J^{12}\right), \quad \vec{K} \equiv \left(J^{01}, J^{02}, J^{03}\right) \tag{1.350}$$

Então das relações de comutação do grupo de Lorentz (ver Problema 1.26) podemos obter as relações de comutação para estes dois operadores vetoriais. Obtemos

$$\left[J^i, J^j\right] = i\epsilon^{ijk}\, J^k \tag{1.351}$$

$$\left[K^i, K^j\right] = -i\epsilon^{ijk}\, J^k \tag{1.352}$$

$$\left[K^i, J^j\right] = i\epsilon^{ijk}\, K^k \tag{1.353}$$

Estas relações que o operador $\vec{J}$ está associado com as rotações no espaço a 3 dimensões, obedece à álgebra do momento angular e portanto o seu quadrado J^2 deve ter os valores próprios $j(j+1)$ ($\hbar = 1$). O outro operador deve estar associado às transformações de Lorentz propriamente ditas (*boosts*). A Eq. (1.353) mostra que se transforma como um vetor para rotações e a Eq. (1.352) mostra que a diferença entre duas transformações de Lorentz é uma rotação.

Voltando ao vetor de Pauli-Lubanski podemos escrever,

$$\begin{aligned} W^0 &= -\frac{1}{2}\epsilon^{0ijk}\; J_{ij}P_k = \frac{1}{2}\epsilon^{0ijk}\; J^{ij}P^k \\ &= J^{23}P^1 + J^{31}P^2 + J^{12}P^3 \\ &= \vec{J}\cdot\vec{P} \end{aligned} \tag{1.354}$$

e

$$\begin{aligned} W^i &= -\frac{1}{2}\epsilon^{i\nu\rho\sigma}\; J_{\nu\rho}P_\sigma = -\epsilon^{i0jk}\; J_{0j}P_k - \frac{1}{2}\epsilon^{ijk0}\; J_{jk}P_0 \\ &= \epsilon^{0ijk}\; J^{0j}P^k + \frac{1}{2}\epsilon^{0ijk}\; J^{jk}P^0 \\ &= \left(\vec{K}\times\vec{P}\right)^i + P^0 J^i \end{aligned} \tag{1.355}$$

ou seja

$$W^0 = \vec{J}\cdot\vec{P}, \quad \vec{W} = P^0\vec{J} + \vec{K}\times\vec{P} \tag{1.356}$$

Calculemos agora os valores próprios de W^2. Para isso vamos para o referencial próprio da partícula de massa m, onde os valores próprios do operador P^μ devem ser

$$P^\mu = (m, \vec{0}) \tag{1.357}$$

Obtemos para W^μ,

$$W^0 = 0, \quad \vec{W} = m\vec{J} \tag{1.358}$$

e portanto

$$\begin{aligned} W^2 &= \left(W^0\right)^2 - \vec{W}\cdot\vec{W} \\ &= -m^2J^2 = -m^2 s(s+1) \end{aligned} \tag{1.359}$$

onde a última passagem resulta do facto de que no referencial próprio não há momento angular orbital e portanto $\vec{J}$ deve ser identificado com o spin da partícula. Como W^2 é um invariante de Lorentz o resultado da Eq. (1.359) deve ser verdade em todos os referenciais de inércia.
Para demonstrar que W^2 é de facto um invariante de Casimir (ver Problema 1.26) é conveniente obter as relações seguintes

$$\begin{aligned} [W_\mu, P_\alpha] &= -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\left[J^{\nu\rho}, P_\alpha\right]P^\sigma \\ &= -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\left(iP^\nu g^\rho_\alpha - iP^\rho g^\nu_\alpha\right)P^\sigma \\ &= 0 \end{aligned} \tag{1.360}$$

e

$$[W_\mu, J_{\alpha\beta}] = i\left(g_{\mu\alpha}W_\beta - g_{\mu\beta}W_\alpha\right) \ . \tag{1.361}$$

A Eq. (1.361) mostra que W_μ se transforma como um 4-vetor.

Complement 1.6 Prescrição mínima em física não relativista

Consideremos uma partícula, não relativista, de massa m e carga q, em interação com o campo eletromagnético. O Lagrangiano para essa partícula é

$$L = \frac{1}{2}mv^2 - q\phi + q\vec{A}\cdot\vec{v} \tag{1.362}$$

Vejamos que isto é verdade. Obtemos facilmente (estamos no espaço euclidiano a 3 dimensões, pelo que não fazemos distinção entre a posição dos índices. Assim escrevemos todos em baixo).

$$\frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + qA_i \tag{1.363}$$

e

$$\frac{\partial L}{\partial x_i} = -q\partial_i\phi + q\partial_i A_j \dot{x}_j \tag{1.364}$$

Obtemos então

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = m\ddot{x}_i + q\frac{\partial A_i}{\partial t} + q\partial_j A_i \dot{x}_j \tag{1.365}$$

Portanto a equação de Euler-Lagrange

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0 \tag{1.366}$$

dá

$$m\ddot{x}_i + q\partial_i\phi + q\frac{\partial A_i}{\partial t} + q\left(\partial_j A_i \dot{x}_j - \partial_i A_j \dot{x}_j\right) = 0 \tag{1.367}$$

ou seja

$$m\ddot{x}_i = q\left(-\partial_i\phi - \frac{\partial A_i}{\partial t}\right) + q\left(\partial_i A_j - \partial_j A_i\right)\dot{x}_j \tag{1.368}$$

Usando agora

$$E_i = -\partial_i\phi - \frac{\partial A_i}{\partial t} \tag{1.369}$$

e

$$\begin{aligned} \epsilon_{ijk}B_k &= \epsilon_{ijk}\,\epsilon_{kmn}\,\partial_m A_n \\ &= \left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right)\partial_m A_n \\ &= \partial_i A_j - \partial_j A_i \end{aligned} \tag{1.370}$$

obtemos finalmente

$$m\ddot{x}_i = qE_i + q\epsilon_{ijk}\dot{x}_j B_k \tag{1.371}$$

ou numa notação mais familiar

$$m\frac{d\vec{v}}{dt} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) \tag{1.372}$$

o que confirma que a Eq. (1.362) é de facto o Lagrangiano para a partícula não relativista num campo eletromagnético exterior. Como subproduto deste cálculo vemos que o momento canónico, conjugado da variável x_i é

$$p_i \equiv \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + qA_i = p_i^{\mathbf{mec}} + qA_i \tag{1.373}$$

o que nos leva à regra da prescrição mínima, pois,

$$\begin{aligned} \vec{p}^{\,\mathbf{mec}} &\rightarrow \vec{p} - q\vec{A} \\ -i\hbar\vec{\nabla} &\rightarrow -i\hbar\vec{\nabla} - q\vec{A} \\ i\hbar\partial^\mu &\rightarrow i\hbar\partial^\mu - qA^\mu \\ \partial_\mu &\rightarrow \partial_\mu + i\frac{q}{\hbar}A_\mu \end{aligned} \tag{1.374}$$

onde q é a carga da partícula. Isto justifica a Eq. (1.271).

Complement 1.7 Lagrangianos em teoria do campo

Na passagem dum sistema com um número finito de graus de liberdade para a situação em teoria do campo onde temos infinitos graus de liberdade é conveniente estabelecer o seguinte dicionário:

Sistemas Finitos graus liberdade	Teoria do Campo
t	x^μ
q	$\phi(x)$
$\dot{q}$	$\partial_\mu\phi(x)$
$S=\int dtL(q,\dot{q})$	$S=\int d^4x\mathcal{L}(\phi,\partial_\mu\phi)$
$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}}-\frac{\partial L}{\partial q}=0$	$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}-\frac{\partial\mathcal{L}}{\partial\phi}=0$

É fácil de verificar que para o campo real de Klein-Gordon a seguinte densidade Lagrangiana

$$\mathcal{L}=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}m^2\phi^2 \tag{1.375}$$

reproduz a Eq. (1.29). Para o campo de Dirac temos que tratar o spinor e o seu adjunto como graus de liberdade independentes (tal como acontece no campo escalar complexo, ver Problema 1.19). Assim é fácil de ver que a seguinte densidade Lagrangiana

$$\mathcal{L}=i\overline{\psi}\gamma^\mu\partial_\mu\psi-m\overline{\psi}\psi \tag{1.376}$$

conduz à equação de Dirac. De facto As equações de Euler-Lagrange são, para o caso do campo de Dirac,

$$\partial^\mu\frac{\partial\mathcal{L}}{\partial\left(\partial^\mu\psi\right)}-\frac{\partial\mathcal{L}}{\partial\psi}=0,\quad \partial^\mu\frac{\partial\mathcal{L}}{\partial\left(\partial^\mu\overline{\psi}\right)}-\frac{\partial\mathcal{L}}{\partial\overline{\psi}}=0 \tag{1.377}$$

Do Lagrangiano, Eq. (1.376), obtemos

$$\frac{\partial\mathcal{L}}{\partial\left(\partial^\mu\overline{\psi}\right)}=0,\quad \frac{\partial\mathcal{L}}{\partial\overline{\psi}}=i\,\gamma^\mu\partial_\mu\psi-m\psi \tag{1.378}$$

e portanto

$$\left(i\,\gamma^\mu\partial_\mu-m\right)\psi=0, \tag{1.379}$$

como queríamos mostrar. A densidade Lagrangiana de Dirac tem uma propriedade notável. É invariante para as transformações

$$\psi'(x)=e^{iq\alpha}\;\psi(x)\qquad;\qquad\overline{\psi}'(x)=\overline{\psi}\;e^{-iq\alpha} \tag{1.380}$$

com α constante. Isto corresponde a redefinir a fase da função de onda, que certamente é arbitrária. Contudo o facto de α ser constante apresenta um problema potencial. Como é que eu sei que fase escolher para comparar experiências em dois laboratórios diferentes? Seria melhor que a fase pudesse ser escolhida independentemente em qualquer sítio. Matematicamente isto quer dizer que devíamos ter $\alpha=\alpha(x)$. Contudo neste caso a Eq. (1.376) deixa de ser invariante devido ao termo da derivada. Contudo se modificarmos a densidade Lagrangiana para

$$\mathcal{L}=i\overline{\psi}\gamma^\mu D_\mu\psi-m\overline{\psi}\psi\qquad;\qquad D_\mu=\partial_\mu+iqA_\mu \tag{1.381}$$

ela é invariante para as transformações da Eq. (1.380) se

$$A'_\mu = A_\mu - \partial_\mu \alpha \tag{1.382}$$

Ora esta transformação é já nossa conhecida. Ela é a transformação de gauge do eletromagnetismo [4]. Esta transformação deixa invariante o tensor de Maxwell, Eq. (1.307), pois

$$F'_{\mu\nu} = F_{\mu\nu} - \partial_\mu\partial_\nu\alpha + \partial_\nu\partial_\mu\alpha = F_{\mu\nu} \tag{1.383}$$

Somos portanto conduzidos ao resultado que se generalizarmos a derivada para incluir o campo eletromagnético mantemos a densidade Lagrangiana invariante para transformações de gauge locais. Temos que completar adicionando o termo de Maxwell ao Lagrangiano. Somos assim conduzidos à densidade Lagrangiana de `QED` que é

$$\mathcal{L} = i\overline{\psi}\gamma^\mu D_\mu\psi - m\overline{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1.384}$$

Vemos assim que a prescrição mínima da Eq. (1.272) corresponde à definição da derivada covariante da Eq. (1.381). Por outro lado a densidade Lagrangiana de interação, isto é, com mais de dois campos, é

$$\mathcal{L}_{int} = -q_e\overline{\psi}\gamma^\mu\psi A_\mu = e\overline{\psi}\gamma^\mu\psi A_\mu = -J^\mu_{\rm em}\, A_\mu \tag{1.385}$$

onde se definiu

$$J^\mu_{\rm em} = q_e\overline{\psi}\gamma^\mu\psi = -e\overline{\psi}\gamma^\mu\psi \tag{1.386}$$

Ora da física clássica sabe-se que a interação é precisamente dada por $-J^\mu A_\mu$. A interpretação de $J^\mu_{\rm em}$ como a densidade de corrente elétrica também é intuitiva pois é o produto de carga pela corrente de probabilidade.

Complement 1.8 A equação BMT para o spin

Vamos voltar à equação para a evolução no tempo do spin e obter uma equação covariante. Essa equação permitirá compreender como é que é possível medir com tanta precisão o momento magnético anómalo do eletrão e do muão. No referencial próprio a equação de movimento para o spin é (ver Eq. (1.343)

$$\frac{d\vec{S}'}{dt'} = \vec{\mu}\times\vec{B}' = \frac{gq_e}{2m}\vec{S}'\times\vec{B}' \tag{1.387}$$

onde $\vec{S}'$, t' e $\vec{B}'$ são medidos no referencial em que a partícula está em repouso.

i) A equação de movimento covariante (equação BMT)

Pretendemos escrever uma equação covariante que se reduza à Eq. (1.387) no referencial próprio. Como vimos na secção 1.5.2 a generalização covariante de $\vec{S}$ é o 4-vetor S^μ que satisfaz $S^\mu p_\mu = 0$ ou seja

$$S^\mu u_\mu = 0 \tag{1.388}$$

Usando as leis de transformação dos 4-vetores devemos ter num outro referencial

$$\begin{cases} S^0 = \gamma\vec{\beta}\cdot\vec{S}' \\ \vec{S} = \vec{S}' + \dfrac{\gamma^2}{\gamma+1}\left(\vec{\beta}\cdot\vec{S}'\right)\vec{\beta} \end{cases} \tag{1.389}$$

A generalização óbvia do lado esquerdo da Eq. (1.387) é $dS^\mu/d\tau$ (ver Eq. (1.320)), onde τ é o tempo próprio. Para o lado direito devemos exigir que seja um 4-vetor linear no spin e no campo exterior. Da Eq. (1.320) resulta que

$$\frac{du^\mu}{d\tau} = \frac{q_e}{m}\, F^{\mu\nu}u_\nu \tag{1.390}$$

pelo que podemos escrever a forma mais geral[12]

$$\frac{dS^\mu}{d\tau} = k_1 F^{\mu\nu}S_\nu + k_2 S_\alpha F^{\alpha\beta}u_\beta\; u^\mu + k_3 S_\alpha \frac{du^\alpha}{d\tau}\; u^\mu \tag{1.391}$$

para determinar as constantes k_i notemos que da Eq. (1.388) resulta

$$\frac{dS^\mu}{d\tau}u_\mu + S^\mu\frac{du_\mu}{d\tau} = 0 \tag{1.392}$$

o que dá, usando a Eq. (1.391),

$$\left(k_1 - k_2c^2\right) S_\alpha F^{\alpha\beta}u_\beta + \left(k_3c^2+1\right) S^\mu\frac{du_\mu}{d\tau} = 0 \tag{1.393}$$

Para que esta equação seja válida mesmo quando a origem da força não é eletromagnética, devemos exigir que os dois termos se anulem separadamente. Isto dá

$$k_1 = k_2c^2, \quad k_3c^2 = -1 \tag{1.394}$$

Substituindo na Eq. (1.391) obtemos

$$\frac{dS^\mu}{d\tau} = k_1\left(F^{\mu\nu}S_\nu + \frac{1}{c^2}S_\alpha F^{\alpha\beta}u_\beta\; u^\mu\right) - \frac{1}{c^2}\; S^\nu\frac{du_\nu}{d\tau}u^\mu \tag{1.395}$$

Para obter k_1 tomamos o limite do referencial próprio, $\vec{\beta} = 0$. Então $u^\mu = (c, 0)$ e $S^\mu = (0, \vec{S}')$. Obtemos a partir da Eq. (1.395)

$$\frac{d\vec{S}'}{dt'} = k_1\vec{S}'\times\vec{B}' \tag{1.396}$$

pelo que, comparando com a Eq. (1.387),

$$k_1 = \frac{gq_e}{2m} \tag{1.397}$$

[12]Pode-se mostrar [18] que outros termos ou são nulos, ou se podem escrever em termos dos escolhidos.

Particularizando para o caso em que a força é de origem eletromagnética, Eq. (1.390), obtemos finalmente a equação BMT de Bargmann, Michel e Telegdi [19],

$$\frac{dS^\mu}{d\tau} = \frac{q_e}{m}\left[\frac{g}{2}F^{\mu\nu}S_\nu + \frac{1}{c^2}\left(\frac{g}{2}-1\right)u^\mu\ S_\alpha F^{\alpha\beta}u_\beta\right] \tag{1.398}$$

ii) **Relação com a precessão de Thomas**

Usando o resultado

$$S_\mu\frac{du^\mu}{d\tau} = -\gamma c\vec{S}\cdot\frac{d\vec{\beta}}{d\tau} \tag{1.399}$$

podemos escrever a partir da Eq. (1.395)

$$\begin{aligned}\frac{dS^0}{d\tau} &= F^0 - \gamma^2\left(\vec{S}\cdot\frac{d\vec{\beta}}{d\tau}\right)\\ \frac{d\vec{S}}{d\tau} &= \vec{F} - \gamma^2\left(\vec{S}\cdot\frac{d\vec{\beta}}{d\tau}\right)\vec{\beta}\end{aligned} \tag{1.400}$$

onde o 4-vetor F^μ é dado por

$$F^\mu = \frac{gq_e}{2m}\left(F^{\mu\nu}S_\nu + \frac{1}{c^2}S_\alpha F^{\alpha\beta}u_\beta\ u^\mu\right) \tag{1.401}$$

Usando as Eq. (1.400) Eq. (1.389) podemos finalmente escrever

$$\frac{d\vec{S}'}{d\tau} = \vec{F} - \frac{\gamma\vec{\beta}}{\gamma+1}F^0 + \frac{\gamma^2}{\gamma+1}\left[\left(\frac{d\vec{\beta}}{d\tau}\times\vec{\beta}\right)\times\vec{S}'\right] \tag{1.402}$$

No referencial próprio obtemos

$$\frac{d\vec{S}'}{dt'} = \frac{1}{\gamma}\vec{F}' + \Omega_T\times\vec{S}' \tag{1.403}$$

onde o primeiro termo corresponde ao binário, Eq. (1.387), e o segundo termo é a precessão de Thomas (ver Problema 1.43). Para movimento em campos eletromagnéticos onde

$$\frac{d\vec{\beta}}{dt} = \frac{q_e}{\gamma mc}\left[\vec{E} + c\vec{\beta}\times\vec{B} - \vec{\beta}\left(\vec{\beta}\cdot\vec{E}\right)\right] \tag{1.404}$$

e

$$\frac{1}{\gamma}\vec{F}' = \frac{gq_e}{2m}\vec{S}'\times\left[\vec{B} - \frac{\gamma}{\gamma+1}\left(\vec{\beta}\cdot\vec{B}\right)\vec{\beta} - \frac{1}{c}\vec{\beta}\times\vec{E}\right] \tag{1.405}$$

obtemos finalmente a equação de Thomas (ver Problema 1.44)

$$\frac{d\vec{S}'}{dt} = \frac{q_e}{m}\vec{S}'\times\left[\left(\frac{g}{2}-1+\frac{1}{\gamma}\right)\vec{B} - \left(\frac{g}{2}-1\right)\frac{\gamma}{\gamma+1}\left(\vec{\beta}\cdot\vec{B}\right)\vec{\beta} - \left(\frac{g}{2}-\frac{\gamma}{\gamma+1}\right)\frac{\vec{\beta}\times\vec{E}}{c}\right] \tag{1.406}$$

iii) **A constância da polarização longitudinal**

Usemos agora a equação de Thomas, Eq. (1.406), para determinar a taxa de variação da componente longitudinal da polarização ou helicidade. Temos

$$\frac{d}{dt}\left(\hat{\beta}\cdot\vec{S}'\right)=\hat{\beta}\cdot\frac{d\vec{S}'}{dt}+\frac{1}{\beta}\left[\vec{S}'-\left(\hat{\beta}\cdot\vec{S}'\right)\hat{\beta}\right]\cdot\frac{d\vec{\beta}}{dt} \tag{1.407}$$

Usando a Eq. (1.404) e Eq. (1.406) podemos escrever

$$\frac{d}{dt}\left(\hat{\beta}\cdot\vec{S}'\right)=-\frac{q_e}{m}\vec{S}'_{\perp}\cdot\left[\left(\frac{g}{2}-1\right)\hat{\beta}\times\vec{B}+\left(\frac{g\beta}{2}-\frac{1}{\beta}\right)\frac{\vec{E}}{c}\right] \tag{1.408}$$

Esta equação mostra uma propriedade notável duma partícula com $g=2$. Num campo magnético a partícula movimenta-se de tal forma que a sua polarização longitudinal permanece constante. Isto permite medir com grande precisão qualquer pequeno desvio de $g=2$. Notar que no limite ultra-relativístico ($\beta\to 1$), o mesmo acontece também em campos elétricos.

Problemas

Natural units and relativistic kinematics

1.1 Considere o sistema de unidades natural utilizado em Física de Altas Energias, isto é $\hbar = 1$, $c = 1$. Neste sistema todas as grandezas físicas podem ser expressas em unidades de energia ou suas potências.

a) Exprima 1 s, 1 Kg e 1 m em MeV.

b) Exprima o seu peso (72 Kg), altura (1.70m) e idade (71 anos) em MeV.

1.2 O tempo de vida média τ duma partícula instável (que decai noutra) é definido como o tempo ao fim do qual o número de partículas é reduzido a $1/e$ do seu valor inicial, ou seja

$$N(t) = N_0 \; e^{-\frac{t}{\tau}}$$

onde N_0 é o número de partículas no instante inicial e τ é referido ao referencial no qual a partícula se encontra em repouso. Sabendo que os piões carregados têm $\tau_\pi = 2.6 \times 10^{-8}$ s e $m_\pi = 140$ MeV calcule:

a) O fator γ para um feixe de piões de 200 GeV.

b) O tempo de vida média no referencial do Laboratório.

c) Calcule a percentagem de piões que decaiu ao fim de percorrerem 300 m no Laboratório. Se não houvesse dilatação no tempo qual seria a percentagem ao fim da mesma distância?

1.3 Uma partícula de massa M e 4-momento P^α decai em duas partículas de massas m_1 e m_2.

a) Use conservação de energia e momento ($P^\alpha = p_1^\alpha + p_2^\alpha$) e a invariância dos produtos escalares para mostrar que no referencial em que a partícula que decai está em repouso temos para as energias das partículas 1 e 2

$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} \quad ; \quad E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M}$$

b) Mostre que a energia cinética da partícula i no mesmo referencial é

$$T_i = \Delta M \left(1 - \frac{m_i}{M} - \frac{\Delta M}{2M}\right)$$

onde $\Delta M = M - m_1 - m_2$ é o excesso de massa.

c) O pião carregado ($M_\pi = 139.6$ MeV) decai num muão ($m_1 = 105.7$ MeV) e num antineutrino ($m_2 = 0$). Calcule as energias cinéticas do muão e do neutrino no referencial do pião. (Nota: Foi usando o resultado deste exercício que foi descoberto o pião em 1947).

1.4
Considere o declínio $\pi^- \to \mu^- + \overline{\nu}$ descrito no problema 1.3. Calcule:

a) O momento linear do μ^- e do $\overline{\nu}$ no referencial do centro de massa, isto é onde o π^- está em repouso.

b) O momento linear do μ^- e do $\overline{\nu}$ no referencial do Laboratório ($p_\pi = 10\ GeV$) supondo que o $\overline{\nu}$ tinha sido emitido no CM na direção e sentido do π^-.

c) Repita b) supondo que era o μ^- que tinha sido emitido no CM na direção e sentido do π^-.

1.5 Um fotão pode ser descrito como uma partícula de massa zero e 4-momento $k^\alpha = (\omega, \vec{k})$ onde $\omega = 2\pi\nu = 2\pi/\lambda$ e $|\vec{k}| = \omega$ no sistema de unidades onde $\hbar = c = 1$. Se o fotão colidir com um eletrão de massa m_e em repouso, será difundido com um ângulo θ e com uma nova energia ω'.

a) Mostre que

$$\lambda' - \lambda = 2\lambda_c \ \sin^2 \frac{\theta}{2} \qquad \text{onde} \qquad \lambda_c = \frac{2\pi}{m}$$

b) Mostre que a energia cinética de recuo do eletrão é

$$T = \omega \frac{2\frac{\lambda_c}{\lambda} \sin^2 \frac{\theta}{2}}{1 + 2\frac{\lambda_c}{\lambda} \sin^2 \frac{\theta}{2}}$$

1.6 Considere os dois processos

$$\gamma \ \to \ e^+ + e^-$$

$$\gamma + e^- \ \to \ e^- + e^- + e^+$$

a) Mostre que a reação (1) nunca pode ter lugar.

b) Calcule a energia mínima dum fotão incidente num eletrão em repouso para que a reação (2) possa ter lugar.

1.7
Suponha que a energia do ν_μ no referencial do laboratório (onde o eletrão está em repouso) é o dobro da energia mínima necessária para que o processo $\nu_\mu + e^- \to \mu^- + \nu_e$ seja possível. Sabendo que o tempo de vida média do muão é $2.2 \times 10^{-6}\, s$ qual é a distância média percorrida no laboratório pelo muão antes de decair?

1.8 Considere o processo

$$A + B \to C_1 + C_2 + \cdots + C_n$$

O feixe de partículas A tem energia E_A no referencial do **Lab**, onde a partícula B está em repouso. Escreva a expressão para a energia mínima, $E_A^{\min}$, necessária para que a reação possa ter lugar, em função de m_A, m_B e $M \equiv m_{C_1} + m_{C_2} + \cdots m_{C_n}$.

1.9 Um feixe de mesões K^+ incide num alvo de hidrogénio líquido para se estudar o processo

$$K^+ + p \to K^+ + p + \pi^+ + \pi^-$$

No referencial do **Lab**, onde o protão está em repouso, determine a energia mínima do feixe de mesões K^+ para que o processo possa ter lugar. Dados: $m_{K^+} = 493$ MeV, $m_p = 938$ MeV, $m_{\pi^\pm} = 140$ MeV.

1.10 Um feixe de eletrões com energia $E_e = 50$ GeV, colide frontalmente com um feixe dum laser com energia $E_\gamma = 1$ eV. Qual é a energia dos fotões que são difundidos para trás, isto é, na direção do feixe de eletrões.

1.11 O acelerador HERA, que funcionou no DESY em Hamburg a partir de 1991, fazia colidir um feixe de protões com energia $E_p = 920$ GeV, com um feixe de eletrões de $E_e = 27.5$ GeV. Calcule a energia do feixe de eletrões que seria necessária para se ter a mesma energia no centro de massa, se os protões estivessem em repouso (experiência de alvo fixo).

1.12 Considere uma colisão elástica na qual uma partícula de massa M com momento p_{Lab} incide numa partícula de massa m em repouso no laboratório. Mostre que a perda de energia da partícula incidente na colisão se pode escrever

$$\Delta E = \frac{m p_{\text{Lab}}^2}{s} \left(1 - \cos\theta_{\text{CM}}\right)$$

onde s é o quadrado da energia no `CM` e θ_{CM} é o ângulo de difusão no referencial do `CM`.

Classical Field Theory

1.13 Considere o tensor do campo eletromagnético $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. A partir deste tensor define-se o chamado *tensor dual*

$$\mathcal{F}^{\mu\nu} = \frac{1}{2}\,\epsilon^{\mu\nu\rho\sigma}\,F_{\rho\sigma}\ .$$

a) Mostre que as equações de Maxwell são

$$\partial_\mu F^{\mu\nu} = J^\nu$$

e que estas reproduzem as leis de Gauss e Ampère (incluindo a corrente de deslocamento introduzida por Maxwell).

b) Mostre que se tem

$$\partial_\mu \mathcal{F}^{\mu\nu} = 0$$

Verifique que esta equação contém as chamadas equações de Maxwell homogéneas, isto é, $\vec{\nabla}\cdot\vec{B} = 0$, e $\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$. Verifique que aquela relação é equivalente à forma mais usual (identidade de Bianchi)

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$$

c) Exprima os invariantes $F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu}\mathcal{F}^{\mu\nu}$ e $\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$ em termos dos campos $\vec{E}$ e $\vec{B}$.

d) Mostre que se $\vec{E}$ e $\vec{B}$ são perpendiculares num dado referencial, então são perpendiculares em todos os referenciais de inércia.

e) Considere um referencial S onde se tem $\vec{E} \neq 0$ e $\vec{B} = 0$. será possível encontrar um referencial S' onde $\vec{E} = 0$ e $\vec{B} \neq 0$? Justifique.

f) Defina

$$f^2 \equiv F_{\mu\nu}F^{\mu\nu}, \quad f^4 \equiv F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} \tag{1.409}$$

Mostre que

$$\left(F_{\mu\nu}\mathcal{F}^{\mu\nu}\right)^2 = 4f^4 - 2\left(f^2\right)^2 \tag{1.410}$$

1.14 Considere o Lagrangiano seguinte

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi$$

a) Verifique que as equações de movimento são as equações de Klein-Gordon.

b) Verifique que o Lagrangiano é invariante para as transformações

$$\phi' = e^{iq\alpha}\ \phi \qquad ; \qquad \alpha = \text{constante}$$

c) Mostre que se a ação

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)$$

é invariante para uma transformação

$$\phi_i' = \phi_i - i\varepsilon\lambda_{ij}\ \phi_j$$

onde ε é infinitesimal e λ_{ij} são constantes, então existe uma corrente conservada, isto é

$$\partial_\mu J^\mu = 0$$

onde

$$J^\mu = -i\lambda_{ij}\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\ \phi_j$$

Este resultado é conhecido pelo nome de *teorema de Noether* .

d) Aplique este resultado ao Lagrangiano dado e obtenha a corrente J^μ. Notar que para o caso do Lagrangeano dadao em a) com as transformações em b) temos $\lambda = -q$ com $\epsilon = \alpha$ infinitesimal.

e) Mostre que se $\alpha = \alpha(x)$ o Lagrangiano

$$\mathcal{L} = (\partial_\mu - iqA_\mu)\,\phi^*\,(\partial_\mu + iqA_\mu)\,\phi - m^2\phi^*\phi$$

é invariante para as transformações

$$\phi' = e^{iq\alpha(x)}\ \phi$$

se A_μ se transformar de forma apropriada. Qual? Comente.

	1	γ_5	γ^μ	$\gamma_5\gamma^\mu$	$\sigma^{\mu\nu}$
1					
γ_5					
γ^α					
$\gamma_5\gamma^\alpha$					
$\sigma^{\alpha\beta}$					

(1.411)

Table 1.1: Tabela de multiplicação de matrizes γ

Spinors and Dirac γ matrices

1.15 Preencha as entradas da *tabela de multiplicação* das matrizes γ indicada na Tabela 1.1. Esta tabela torna-se muito útil em cálculos práticos. Para estabelecer a tabela tenha em atenção que qualquer produto de matrizes γ se pode escrever em termos das 16 matrizes independentes e que com as nossas convenções

$$\varepsilon^{0123} = +1$$

$$\varepsilon_{\mu\nu\rho\sigma} = -\varepsilon^{\mu\nu\rho\sigma}$$

$$\varepsilon_{\alpha\beta_1\gamma_1\delta_1}\varepsilon^{\alpha\beta_2\gamma_2\delta_2} = -\sum_P (-1)^P g_{\beta_1}^{P[\beta_2} g_{\gamma_1}^{\gamma_2} g_{\delta_1}^{\delta_2]}$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma_0\gamma_1\gamma_2\gamma_3$$

$$\varepsilon_{\alpha\beta\gamma_1\delta_1}\varepsilon^{\alpha\beta\gamma_2\delta_2} = -2\left(g_{\gamma_1}^{\gamma_2} g_{\delta_1}^{\delta_2} - g_{\gamma_1}^{\delta_2} g_{\delta_1}^{\gamma_2}\right)$$

$$\varepsilon_{\alpha\beta\gamma\delta_1}\varepsilon^{\alpha\beta\gamma\delta_2} = -6 g_{\delta_1}^{\delta_2}$$

onde $(-1)^P$ é $+1$ (-1) para uma permutação par (ímpar) de $(\beta_2\gamma_2\delta_2)$.

1.16 Mostre as relações seguintes:

$$(\Gamma^a)^2 = \pm 1$$

$$\mathrm{Tr}\,(\Gamma^a) = 0\ , \forall a \neq s$$

$$\gamma^\mu\gamma_\mu = 4\ ;\ \gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu\ ;\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho = g^{\mu\nu}\gamma^\rho - g^{\mu\rho}\gamma^\nu + g^{\nu\rho}\gamma^\mu + i\varepsilon^{\mu\nu\rho\alpha}\ \gamma_\alpha\gamma_5$$

1.17 Mostre que os spinores $w^r(\vec{p})$ satisfazem as relações

$$(\not{p} - \varepsilon_r m)w^r(\vec{p}) = 0\ ;\ \overline{w}^r(\vec{p})\ (\not{p} - \varepsilon_r m) = 0$$

$$\overline{w}^r(\vec{p})w^{r'}(\vec{p}) = 2m\ \delta_{rr'}\varepsilon_r$$

$$\sum_{r=1}^{4} \varepsilon_r w^r_\alpha(\vec{p})\overline{w}^r_\beta(\vec{p}) = 2m\ \delta_{\alpha\beta}$$

$$w^{r\,\dagger}(\varepsilon_r\vec{p})w^{r'}(\varepsilon_{r'}\vec{p}) = 2E\ \delta_{rr'}$$

1.18 Admita que γ_μ se transforma como um 4- vetor, isto é, $\gamma'^\mu = a^\mu{}_\nu\gamma^\nu$. Mostre que

$$\gamma'^\mu = U^{-1}\gamma^\mu_{\mathrm{Dirac}}\ U$$

Sugestão: Considere o caso particular duma transformação segundo o eixo x^1, isto é

$$\begin{aligned} x'^0 &= \cosh\omega\ x^0 - \sinh\omega\ x^1 \\ x'^1 &= \cosh\omega\ x^1 - \sinh\omega\ x^0 \\ x'^2 &= x^2 \\ x'^3 &= x^3 \end{aligned}$$

onde $\tanh\omega = V/c$.

1.19 Representação de Majorana e representação quiral.

a) Mostre que é possível encontrar uma representação, dita representação de Majorana, onde $Re\left(\gamma^\mu_{\mathrm{Maj}}\right) = 0$ e portanto a equação de Dirac é real. Verifique a matriz U

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \sigma_2 \\ \sigma_2 & -1 \end{pmatrix} \tag{1.412}$$

leva através duma transformação de semelhança

$$\gamma^{\mu}_{\text{Maj}} = U^{-1}\gamma^{\mu}_{\text{Dirac}}\, U$$

a uma solução do problema.

b) Mostre que é possível encontrar uma representação, dita representação quiral, onde

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Verifique a matriz U

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{1.413}$$

leva através duma transformação de semelhança

$$\gamma^{\mu}_{\text{Quiral}} = U^{-1}\gamma^{\mu}_{\text{Dirac}}\, U$$

a uma solução do problema.

1.20 Prove a decomposição de Gordon:

$$\overline{u}(p_1, s_1)\gamma^{\mu}u(p_2, s_2) = \frac{1}{2m}\,\overline{u}(p_1, s_1)\left[(p_1 + p_2)^{\mu} + i\sigma^{\mu\nu}(p_1 - p_2)_{\nu}\right] u(p_2, s_2)$$

Sugestão: Use a identidade

$$\not{a}\not{b} = a^{\mu}b_{\mu} - ia^{\mu}b^{\nu}\sigma_{\mu\nu}$$

1.21 Verifique as seguintes relações

$$\overline{u}(p, s)u(p, s') = 2m\ \delta_{ss'}$$

$$\overline{v}(p, s)v(p, s') = -2m\ \delta_{ss'}$$

$$u^{\dagger}(p, s)u(p, s') = 2E_p\ \delta_{ss'}$$

$$v^{\dagger}(p, s)v(p, s') = 2E_p\ \delta_{ss'}$$

$$\overline{v}(p, s)u(p, s') = 0$$

$$v^{\dagger}(p, s)u(-p, s') = 0$$

$$\sum_s [u_{\alpha}(p, s)\overline{u}_{\beta}(p, s)] = (\not{p} + m)_{\alpha\beta}$$

$$\sum_s [v_\alpha(p,s)\overline{v}_\beta(p,s)] = -(-\not{p} + m)_{\alpha\beta}$$

$$\sum_s [u_\alpha(p,s)\overline{u}_\beta(p,s) - v_\alpha(p,s)\overline{v}_\beta(p,s)] = 2m\ \delta_{\alpha\beta}$$

Dirac equation

1.22

a) Construa o Hamiltoniano H da equação de Dirac para partículas livres no espaço dos momentos.

b) Calcule o comutador $\left[H, \vec{L}\right]$, onde $\vec{L} = \vec{r} \times \vec{p}$ é o momento angular orbital.

c) Calcule o comutador $\left[H, \vec{S}\right]$, onde $\vec{S} = \frac{1}{2}\vec{\Sigma}$ é o momento angular intrínseco ou spin.

d) Use os resultados anteriores para calcular $\left[H, \vec{J}\right]$, onde $\vec{J} = \vec{L} + \vec{S}$. Comente.

1.23 Considere um eletrão descrito pela equação de Dirac.

a) Mostre que no caso do eletrão livre se tem,

$$\frac{d(\vec{\Sigma} \cdot \vec{p})}{dt} = 0$$

onde

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

Qual o significado desta lei de conservação?

b) Considere agora que o eletrão está num campo eletromagnético exterior A^μ, independente do tempo. Calcule agora

$$\frac{d(\vec{\Sigma} \cdot \vec{\pi})}{dt}$$

onde $\vec{\pi} = \vec{p} - e\vec{A}$ é o momento canónico.

c) Em que condições

$$\frac{d(\vec{\Sigma} \cdot \vec{\pi})}{dt} = 0\,?$$

Qual o interesse prático deste resultado?

Sugestão: Para um operador $\mathcal{O}$ que não dependa do tempo tem-se

$$\frac{d\mathcal{O}}{dt} = i\Big[H, \mathcal{O}\Big]$$

onde H é o Hamiltoniano do sistema. Não esquecer que H é diferente nas alíneas a) e b).

1.24 Considere a equação de Dirac para um eletrão livre de massa m. As soluções de onda plana são estados próprios do Hamiltoniano $H = \vec{\alpha} \cdot \vec{p} + \beta m$ e do momento $\vec{p}$. Mostre que

$$\frac{d\vec{\Sigma}}{dt} = -2\vec{\alpha} \times \vec{p}, \quad \text{com} \quad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

Utilize este resultado para mostrar que o operador helicidade $(\vec{\Sigma} \cdot \vec{p})/|\vec{p}|$ é uma constante do movimento com valores próprios ± 1.

1.25 Utilize as expressões explícitas

$$\begin{aligned} S_R &= \cos\frac{\theta}{2} + i\hat{\theta} \cdot \vec{\Sigma} \sin\frac{\theta}{2} \\ S_L &= \cosh\frac{\omega}{2} - \hat{\omega} \cdot \vec{\alpha} \sinh\frac{\omega}{2} \end{aligned}$$

para verificar que para transformações finitas também temos

$$S^{-1}\gamma^\mu S = a^\mu{}_\nu \gamma^\nu$$

1.26 O grupo de Poincaré é constituído pelo grupo de Lorentz mais as translações. Se $J_{\mu\nu}$ designarem os geradores do grupo de Lorentz e P_μ os geradores das translações, as relações de comutação são

$$[J_{\mu\nu}, J_{\rho\sigma}] = i\left(g_{\nu\rho}J_{\mu\sigma} - g_{\nu\sigma}J_{\mu\rho} - g_{\mu\rho}J_{\nu\sigma} + g_{\mu\sigma}J_{\nu\rho}\right) \tag{1.414}$$

$$[P_\alpha, J_{\mu\nu}] = i\left(g_{\mu\alpha}P_\nu - g_{\nu\alpha}P_\mu\right) \tag{1.415}$$

$$[P_\mu, P_\nu] = 0$$

Mostre que

$$\big[P^2, J_{\mu\nu}\big] = \big[P^2, P_\mu\big] = 0 \tag{1.416}$$

$$\left[W^2, J_{\mu\nu}\right] = \left[W^2, P_\mu\right] = \left[W^2, P^2\right] = 0$$

onde

$$W_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\ J^{\nu\rho}\ P^\sigma$$

é o vetor (operador) de Pauli-Lubanski.

1.27 Mostre que o operador de spin

$$-\frac{W\cdot s}{m} = \frac{1}{2m}\gamma_5 \not{s}\not{p}\varepsilon_r \tag{1.417}$$

comuta com o operador $(\not{p}+\varepsilon_r m)$ e portanto as soluções para a partícula livre $u(p,s)$ e $v(p,s)$ são funções próprias simultâneas do Hamiltoniano e do spin.

1.28 Mostre que para a equação de Dirac o valor próprio de W^2 é

$$W^2 = -\frac{3}{4}\ m^2$$

1.29 Mostre que as Eq. (1.136) e Eq. (1.135) são compatíveis. Para isso mostre que se definirmos

$$(\epsilon^1, \epsilon^2, \epsilon^3) \equiv (\omega^2{}_3, \omega^3{}_1, \omega^2{}_2), \quad \vec{L} \equiv (\hat{J}_{23}, \hat{J}_{31}, \hat{J}_{12}) \tag{1.418}$$

onde

$$\hat{J}_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) \tag{1.419}$$

é a parte *orbital* do gerador do grupo de Lorentz, então para rotações obtemos

$$-\frac{1}{2}J_{ij}\,\omega^{ij} = \vec{L}\cdot\vec{\epsilon} \tag{1.420}$$

1.30 Mostre que a equação de Klein-Gordon descreve spin zero.

1.31 Construa os spinores normalizados u^+ e u^- representando eletrões com energia positiva e momento $\vec{p}$ e helicidade ± 1, isto é, que são vetores próprios do operador $(\vec{p}\cdot\vec{\Sigma})/|\vec{p}|$, com valor próprio ± 1. Para isso siga os passos seguintes:

a) Considere o caso de helicidade $+1$. Seja $u^+ = \begin{pmatrix} u_A^+ \\ u_B^+ \end{pmatrix}$ com $u_A^+ = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ e $u_B^+ = \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}$. Mostre que $\dfrac{u_1}{u_2} = \dfrac{p_z + |\vec{p}|}{p_x + ip_y}$.

b) Use a equação de Dirac para escrever u_B^+ em função de u_A^+.

c) Normalize o spinor u^+ de acordo com a condição $(u^+)^\dagger\, u^+ = 2E$ ou se preferir, $\overline{u^+}u^+ = 2m$. Obtenha assim a expressão final para u^+.

d) Quais as alterações para u^-?

1.32

a) Prove a identidade (definimos $\hat{p} = \vec{p}/|\vec{p}|$):

$$\left(1 - \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right) = \left(1 - \frac{|\vec{p}|}{E+m}\right)\frac{1+\vec{\sigma}\cdot\hat{p}}{2} + \left(1 + \frac{|\vec{p}|}{E+m}\right)\frac{1-\vec{\sigma}\cdot\hat{p}}{2}$$

b) Considere um campo fermiónico com massa, com quiralidade esquerda, definido por

$$\psi_L = \frac{1-\gamma_5}{2}\psi$$

onde ψ é um spinor de energia positiva. Mostre que se pode escrever,

$$\psi_L = N \begin{pmatrix} 1 \\ -1 \end{pmatrix} \left[\alpha_P \ \frac{1+\vec{\sigma}\cdot\hat{p}}{2} + \alpha_N \ \frac{1-\vec{\sigma}\cdot\hat{p}}{2}\right] \chi \ e^{-ip\cdot x}$$

onde N é uma normalização e χ um spinor de duas componentes. Determine α_P e α_N (a menos duma normalização). Qual o seu significado?

c) Define-se a polarização do fermião quiral ψ_L como sendo

$$P = \frac{|\alpha_P|^2 - |\alpha_N|^2}{|\alpha_P|^2 + |\alpha_N|^2}$$

Mostre que $P = -|\vec{p}|/E = -\beta$. Discuta o valor limite quando $|\vec{p}| \gg m$.

1.33 Considere que para $t = 0$ temos uma solução de energia positiva com *spin up* localizada, dada pela expressão

$$\psi(\vec{r}, 0, s) = (\pi d^2)^{-3/4} \ e^{-\frac{r^2}{2d^2}} \ w^1(0) \tag{1.421}$$

Num tempo $t > 0$ será

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \ \frac{1}{2E} \ \sum_s \left[b(p,s)u(p,s) \ e^{-ip\cdot x} + d^*(p,s)v(p,s) \ e^{ip\cdot x}\right]$$

a) Determine $b(p,s)$ e $d^*(p,s)$.

b) Mostre quando é que as amplitudes correspondentes a energia negativa são importantes.

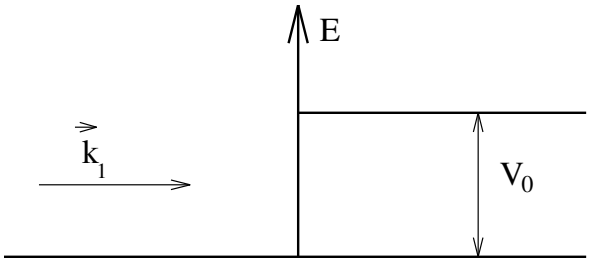


Figure 1.6:

1.34 Considere um eletrão incidente da região I com energia E conforme indicado na Figura 1.6. Admita que a partícula incidente tem a função de onda

$$\psi_{\text{inc}} = a\ e^{ik_1 z} \begin{pmatrix} 1 \\ 0 \\ \frac{k_1}{E+m} \\ 0 \end{pmatrix}$$

a) Calcule a onda refletida e a onda transmitida.

b) Mostre que a corrente refletida e transmitida obedecem a

$$\frac{J_{\text{trans}}}{J_{\text{inc}}} = \frac{4r}{(1+r)^2} \qquad ; \qquad \frac{J_{\text{refl}}}{J_{\text{inc}}} = \frac{(1-r)^2}{(1+r)^2}$$

isto é, aparentemente tudo bem pois

$$J_{\text{inc}} = J_{\text{trans}} + J_{\text{refl}}$$

contudo

$$r = \frac{k_2}{k_1}\ \frac{E+m}{E-V_0+m} \qquad \text{e se} \qquad V_0 > E+m \qquad \text{então} \qquad r < 0$$

Portanto

$$J_{\text{ref}} > J_{\text{inc}}$$

Comente este resultado.

1.35 Mostre que a densidade de probabilidade e a densidade de corrente se podem escrever na forma (decomposição de Gordon)

$$\begin{cases} \rho &= \rho_{\text{convectiva}} + \rho_{\text{interna}} \\ \vec{J} &= \vec{J}_{\text{convectiva}} + \vec{J}_{\text{interna}} \end{cases}$$

onde

$$\begin{cases} \rho_{\text{convectiva}} &= \frac{i}{2m}\left[\overline{\psi}\,\frac{\partial\psi}{\partial t} - \frac{\partial\overline{\psi}}{\partial t}\,\psi\right] \\ \vec{J}_{\text{convectiva}} &= \frac{i}{2m}\left[\overline{\psi}\,\vec{\nabla}\psi - (\vec{\nabla}\overline{\psi})\,\psi\right] \end{cases}$$

e

$$\begin{cases} \rho_{\text{interna}} &= -\vec{\nabla}\cdot\vec{P} \\ \vec{J}_{\text{interna}} &= \vec{\nabla}\times\vec{M} + \dfrac{\partial\vec{P}}{\partial t} \end{cases}$$

Determine $\vec{P}$ e $\vec{M}$. Compare com a equação de Klein-Gordon. Comente. Mostre que as correntes conectivas e internas são conservadas separadamente.

1.36 Demonstre as relações de Ehrenfest

$$\frac{d}{dt}\vec{r}_{\text{op}} = i\,[H,\vec{r}_{\text{op}}] = c\vec{\alpha}$$
$$\frac{d}{dt}\vec{\pi}_{\text{op}} = i\,[H,\vec{\pi}_{\text{op}}] + \frac{\partial}{\partial t}\vec{\pi}_{\text{op}} = q_e\left(\vec{E} + \vec{v}_{\text{op}}\times\vec{B}\right)$$

onde

$$\begin{cases} \vec{\pi}_{\text{op}} = -i\vec{\nabla} - q_e\vec{A} \\ H = -i\vec{\alpha}\cdot\vec{\nabla} + \beta m - q_e\vec{\alpha}\cdot\vec{A} + q_e A^0 \end{cases}$$

1.37 Suponha que para a partícula em repouso o vetor polarização é dado por

$$s^\mu = (0,\vec{\eta}) \qquad \text{com} \qquad \vec{\eta}\cdot\vec{\eta} = 1$$

a) Mostre que no referencial onde a partícula se move com velocidade $\vec{\beta}$ o vetor polarização é dado por

$$s^\mu = \left(\gamma\vec{\eta}\cdot\vec{\beta},\ \vec{\eta} + \frac{\gamma^2\vec{\beta}\,(\vec{\eta}\cdot\vec{\beta})}{\gamma+1}\right)$$

b) Mostre que satisfaz $s^2 = -1$ e $s \cdot p = 0$ com $p = m(\gamma, \gamma\vec{\beta})$.

c) Mostre que o vetor polarização longitudinal, isto é, $\vec{s}_L \parallel \vec{\beta}$, é dado por

$$s^\mu = \left(\gamma\beta, \gamma\vec{\beta}/\beta\right)$$

1.38 Resolva a equação de Dirac num campo magnético uniforme. **Sugestão:** Faça $\psi = e^{-iEt}(\varphi, \chi)^T$ e elimine χ. Obtém assim a equação para um oscilador harmónico.

Hidrogen Atom

1.39 Introduza na equação de Klein-Gordon o acoplamento mínimo

$$i\hbar\partial_\mu \to i\hbar\partial_\mu - q_e A_\mu$$

e considere as soluções estacionárias do átomo de hidrogénio, isto é ($\hbar = c = 1$)

$$\psi(\vec{r}, t) = \phi(\vec{r})\ e^{-iEt} \quad ; \quad A_0 = -\frac{q_e}{4\pi r}$$

a) Mostre que a equação de Klein-Gordon se escreve

$$\left[-\nabla^2 + m^2 - \left(E + \frac{\alpha}{r}\right)^2\right]\phi(\vec{r}) = 0$$

b) Mostre que esta equação se pode resolver exatamente pelos métodos usuais dando as energias

$$E_{n\ell} = \frac{m}{\sqrt{1 + \frac{\alpha^2}{(n-\varepsilon_\ell)^2}}} \quad ; \quad \begin{cases} n & = 1, 2, \cdots \\ \ell & = 0, 1, \cdots, n-1 \end{cases}$$

onde

$$\varepsilon_\ell = \ell + \frac{1}{2} - \left[\left(\ell + \frac{1}{2}\right)^2 - \alpha^2\right]^{1/2}$$

c) Expandindo em potências de α compare com os resultados da teoria de Schrödinger incluindo correções relativistas.

Non-relativistic limit of Dirac equation

1.40 Faça os cálculos da transformação de Foldy- Wouthuysen para obter

$$\begin{aligned} H''' &= \beta\left(m+\frac{(\vec{p}-q_e\vec{A})^2}{2m}-\frac{p^4}{8m^3}\right)+q_eA^0-\frac{q_e}{2m}\ \beta\ \vec{\sigma}\cdot\vec{v} \\ &+\left(-\frac{iq_e}{8m^2}\ \vec{\sigma}\cdot\vec{\nabla}\times\vec{E}-\frac{q_e}{4m^2}\ \vec{\sigma}\cdot\vec{E}\times\vec{p}\right)-\frac{q_e}{8m^2}\ \vec{\nabla}\cdot\vec{E} \end{aligned}$$

1.41 O momento magnético do eletrão é dado por $\vec{\mu}=g\ \frac{q_e}{2m}\ \vec{S}$ onde $\vec{S}=\frac{1}{2}\vec{\sigma}$ e o fator $g=2$. Então poderia pensar-se que a interação spin-órbita era dada por

$$\tilde{H}_{\rm SO}=-\vec{\mu}\cdot\vec{B}_{\rm rest}\qquad\text{onde}\qquad\vec{B}_{\rm rest}=-\vec{v}\times\vec{E}=-\frac{1}{m}\ \frac{1}{r}\ \frac{\partial V}{\partial r}\vec{L}$$

Nesta última expressão admitiu-se um potencial central e $v\ll c$. Assim obtemos

$$\tilde{H}_{\rm SO}=g\frac{q_e}{2m}\ \frac{1}{r}\ \frac{\partial V}{\partial r}\vec{S}\cdot\vec{L} \tag{1.422}$$

que é o dobro do valor dado pela experiência e pela equação de Dirac. A explicação está na *precessão de Thomas* e deve-se ao facto do referencial do eletrão não ser de inércia. Mostre que devido à não comutatividade de duas transformações de Lorentz o referencial próprio do eletrão precessa com uma velocidade angular

$$\vec{\omega}_T=\frac{1}{2}\dot{\vec{v}}\cdot\vec{v}$$

para $v\ll c$. Então

$$H_{\rm SO}=\tilde{H}_{\rm SO}+\vec{L}\cdot\vec{\omega}_T=\frac{q_e}{2m}\ (g-1)\frac{1}{r}\ \frac{\partial V}{\partial r}\vec{S}\cdot\vec{L}$$

que é o resultado dado pela teoria de Dirac.

1.42 Calcule o desdobramento hiperfino do estado $^1S_{1/2}$ do átomo de hidrogénio.

1.43 Mostre que, quando não se faz a aproximação não relativista, as expressões corretas para o movimento de precessão do eletrão (ver Complemento 1.4), são

$$\Delta\vec{\beta}=\gamma^2\delta\vec{\beta}_{\|}+\gamma\delta\vec{\beta}_{\perp},\quad\Delta\vec{\theta}=\frac{\gamma-1}{\beta}\delta\vec{\beta}\times\hat{\beta} \tag{1.423}$$

O que conduz à expressão relativista para a frequência de precessão de Thomas,

$$\vec{\Omega}_{\rm T}=\frac{\gamma-1}{\beta^2}\dot{\vec{\beta}}\times\vec{\beta} \tag{1.424}$$

1.44 Deduza a equação de Thomas, Eq. (1.406),

$$\frac{d\vec{S}'}{dt} = \frac{q_e}{m}\vec{S}' \times \left[\left(\frac{g}{2} - 1 + \frac{1}{\gamma} \right) \vec{B} - \left(\frac{g}{2} - 1 \right) \frac{\gamma}{\gamma + 1} \left(\vec{\beta} \cdot \vec{B} \right) \vec{\beta} - \left(\frac{g}{2} - \frac{\gamma}{\gamma + 1} \right) \frac{\vec{\beta} \times \vec{E}}{c} \right]$$

para isso reproduza todos os cálculos do Complemento 1.8.

1.45 Mostre que

$$(\vec{\sigma} \cdot \vec{\pi})(\vec{\sigma} \cdot \vec{\pi}) = \vec{\pi} \cdot \vec{\pi} - q_e \vec{\sigma} \cdot \vec{B}$$

onde

$$\begin{cases} \vec{\pi} = -i\vec{\nabla} - q_e \vec{A} \\ \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{cases}$$

1.46 Considere o Lagrangiano não mínimo para a interação do eletrão com o fotão

$$\mathcal{L} = i\overline{\psi}\gamma^\mu(\partial_\mu + iq_e A_\mu)\psi - m\overline{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\Delta g}{2}\ \frac{q_e}{4m}\ \overline{\psi}\sigma_{\mu\nu}\psi\ F^{\mu\nu}$$

a) Qual a equação de movimento para o eletrão?

b) Mostre que o fator g do eletrão será então $g = 2 + \Delta g$. **Sugestão:** Encontre o limite não relativista de a).

Chapter 2

Quantization of Free Fields

In the last chapter we obtained the Feynman rules for QED using the initial method of Feynman based on Green's functions. As we mention before, there are two other methods of tackling the same problem. One is through the so-called second quantized field theory, the other is using the Feynman path integral. Although this last method is the only one possible for the non-abelian gauge theories that are the basis of the standard model, it is beyond the scope of this course to study this subject here. However, QED is an example of a theory where you can apply any of the three methods. So, in this chapter we will see how to apply the second quantized formalism to derive the Feynman rules for QED. In the process we will learn Wick's theorem and understand those signs that were difficult to understand by other methods. We will follow the text of Mandl and Shaw [20], although our notation can differ slightly at some places.

2.1 The Schrödinger, Heisenberg and Interaction Pictures

These three different ways of dealing with quantum systems differ in the way they describe the time evolution of a system. The Schrödinger Picture (SP) is the usual representation used in non-relativistic QM where the states are time dependent and their evolution is governed by the Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi, t\rangle_{\rm S} = H |\psi, t\rangle_{\rm S} \ , \tag{2.1}$$

for a given state $|\psi, t\rangle$. This equation can be formally solved to find the time evolution of the state

$$|\psi, t\rangle_{\rm S} = U |\psi, t_0\rangle_{\rm S} \ , \tag{2.2}$$

where U is the unitary operator

$$U = U(t, t_0) = e^{-\frac{i}{\hbar} H(t-t_0)} \, . \tag{2.3}$$

We can use this unitary operator to go to the Heisenberg picture where the time dependence is transferred to the operators and the states are time independent. To do this we define the state in the Heisenberg Picture (HP) through

$$|\psi,t\rangle_{\rm H} \equiv U^\dagger\,|\psi,t\rangle_{\rm S} = |\psi,t_0\rangle_{\rm S}\ . \tag{2.4}$$

At $t=t_0$ the states are the same in the two pictures, showing that the HP state vectors are constant in time. The time dependence is carried by the operators. To find the way they change in time we require that the matrix elements are the same in the two representations, that is,

$${}_{\rm S}\langle\psi_2,t|\,O^{\rm S}\,|\psi_1,t\rangle_{\rm S} = {}_{\rm H}\langle\psi_2,t|\,O^{\rm H}(t)\,|\psi_1,t\rangle_{\rm H} = {}_{\rm S}\langle\psi_2,t|\,UO^{\rm H}(t)U^\dagger\,|\psi_1,t\rangle_{\rm S}\ , \tag{2.5}$$

and therefore

$$O^{\rm H}(t) = U^\dagger O^{\rm S} U\,. \tag{2.6}$$

The time evolution of the operators is obtained from Eq. (2.6) by taking the time derivative. Assuming that the operator is time-independent in the SP we get in the HP,

$$i\hbar\frac{d}{dt}O^{\rm H}(t) = \left[O^{\rm H}(t),H\right]\ . \tag{2.7}$$

For the study of the second quantized free fields the HP is the one we should use, as the fields are now operators, evolving in time. However, for the study of the interactions and to setup a perturbation theory it is important to introduce a new picture, the Interaction Picture (IP). To get the IP we split the Hamiltonian in its free and interaction parts, that is

$$H = H_0 + H_{\rm int}\ . \tag{2.8}$$

The states in the IP are defined as evolving in time only with the free Hamiltonian in relation to the sates in the SP, that is, we define

$$|\psi,t\rangle_{\rm I} = U_0^\dagger\,|\psi,t\rangle_{\rm S}\ , \tag{2.9}$$

where

$$U_0 = e^{-\frac{i}{\hbar}H_0(t-t_0)}\,. \tag{2.10}$$

By requiring that

$${}_{\rm S}\langle\psi_2,t|\,O^{\rm S}\,|\psi_1,t\rangle_{\rm S} = {}_{\rm I}\langle\psi_2,t|\,O^{\rm I}(t)\,|\psi_1,t\rangle_{\rm I} = {}_{\rm S}\langle\psi_2,t|\,U_0O^{\rm I}(t)U_0^\dagger\,|\psi_1,t\rangle_{\rm S}\ , \tag{2.11}$$

we obtain

$$O^{\rm I}(t) = U_0^\dagger O^{\rm S} U_0\,. \tag{2.12}$$

Thus the relation between IP and SP is similar to the relation between HP and SP, with the only difference that the time evolution in the IP is governed by the unperturbed Hamiltonian H_0. We also note from Eq. (2.12) that we have

$$H_0^I = H_0^S = H_0\,. \tag{2.13}$$

By differentiating Eq. (2.12) we get the time evolution of the operator in the IP,

$$i\hbar\frac{d}{dt}O^I(t) = \left[O^I(t), H_0\right] \ . \tag{2.14}$$

As for the states, from Eq. (2.1) and Eq. (2.9) we get

$$\begin{aligned} i\hbar\frac{d}{dt}\left|\psi,t\right\rangle_{\rm I} =& i\hbar\frac{d}{dt}\left(U_0^\dagger\left|\psi,t\right\rangle_{\rm S}\right) \\ =& i\hbar\frac{d}{dt}(U_0^\dagger)\left|\psi,t\right\rangle_{\rm S} + i\hbar U_0^\dagger\frac{d}{dt}\left|\psi,t\right\rangle_{\rm S} \\ =& -H_0U_0^\dagger\left|\psi,t\right\rangle_{\rm S} + U_0^\dagger H\left|\psi,t\right\rangle_{\rm S} \\ =& \left(-H_0+H^I\right)\left|\psi,t\right\rangle_{\rm I} \ , \end{aligned} \tag{2.15}$$

and therefore

$$i\hbar\frac{d}{dt}\left|\psi,t\right\rangle_{\rm I} = H^I_{\rm int}\left|\psi,t\right\rangle_{\rm I} \ . \tag{2.16}$$

We will see in section 3.1 how to use this representation to set up a perturbation theory for the S-matrix.

2.2 Brief review of second quantized free fields

To fix the notation we will review here the basic results from the second quantization of free fields. We will not derive any result, just collect them. Some of these derivations are left as exercises. For those that are not familiar with the subject we refer to any standard text in Quantum Field Theory. I will follow the conventions of my text *Advanced Quantum Field Theory* [12] that differ, mainly in normalizations, with respect to the text of Mandl and Shaw [20] that I follow here more closely.

2.2.1 Real scalar field

The real scalar field is described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi\varphi \, . \tag{2.17}$$

The conjugate momentum is

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \dot{\varphi} \, , \tag{2.18}$$

and the equal time commutation relations are

$$\begin{aligned} &[\varphi(\vec{x},t),\varphi(\vec{x}',t)] = [\pi(\vec{x},t),\pi(\vec{x}',t)] = 0 \\ &[\pi(\vec{x},t),\varphi(\vec{x}',t)] = -i\delta^3(\vec{x}-\vec{x}') \, . \end{aligned} \tag{2.19}$$

The Hamiltonian is given by,

$$H = P^0 = \int d^3x \mathcal{H} = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}|\vec{\nabla}\varphi|^2 + \frac{1}{2}m^2\varphi^2\right] , \tag{2.20}$$

and the linear momentum is

$$\vec{P} = -\int d^3x \pi \vec{\nabla}\varphi \, . \tag{2.21}$$

In order to define the states of the theory it is convenient to have eigenstates of energy and momentum. To build these states we start by making a spectral Fourier decomposition of $\varphi(\vec{x}, t)$ in plane waves[1]:

$$\varphi(\vec{x}, t) = \int \widetilde{dk} \ \left[a(k)e^{-ik\cdot x} + a^\dagger(k)e^{ik\cdot x}\right] , \tag{2.22}$$

where

$$\widetilde{dk} \equiv \frac{d^3k}{(2\pi)^3 2\omega_k} \ ; \ \omega_k = +\sqrt{|\vec{k}|^2 + m^2} \, , \tag{2.23}$$

is the Lorentz invariant integration measure[2]. As in the quantum theory φ is an operator, also $a(k)$ e $a^\dagger(k)$ should be operators. As φ is real, then $a^\dagger(k)$ should be the hermitian conjugate to $a(k)$. Their commutation relations are

$$[a(k), a^\dagger(k')] = (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}') \tag{2.24}$$

$$[a(k), a(k')] = [a^\dagger(k), a^\dagger(k')] = 0 \, , \tag{2.25}$$

We then see that, except for a small difference in the normalization, $a(k)$ e $a^\dagger(k)$ should be interpreted as annihilation and creation operators of states with momentum k^μ. To show this, we observe that

$$H = \frac{1}{2}\int \widetilde{dk} \ \omega_k \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] , \tag{2.26}$$

$$\vec{P} = \frac{1}{2}\int \widetilde{dk} \ \vec{k} \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] . \tag{2.27}$$

Using these explicit forms we can then obtain

$$[P^\mu, a^\dagger(k)] = k^\mu a^\dagger(k), \quad [P^\mu, a(k)] = -k^\mu a(k) \, , \tag{2.28}$$

showing that $a^\dagger(k)$ adds momentum k^μ and that $a(k)$ destroys momentum k^μ. That the quantization procedure has produced an infinite number of oscillators should

[1]The operators a and $a^\dagger$ are really functions of $\vec{k}$, that is $a(\vec{k})$ and $a^\dagger(\vec{k})$. However to simplify notation we use $a(k)$ and $a^\dagger(k)$ being understood that this is simply a notation.

[2]Beware that are many different conventions in the literature.

come as no surprise. In fact $a(k), a^\dagger(k)$ correspond to the quantization of the normal modes of the classical Klein-Gordon field.

By analogy with the harmonic oscillator, we are now in position of finding the eigenstates of H. We start by defining the base state, that in quantum field theory is called the vacuum. We have

$$a(k)\,|0\rangle_k = 0 \quad ; \quad \forall_k\,. \tag{2.29}$$

Then the vacuum, that we will denote by $|0\rangle$, will be formally given by

$$|0\rangle = \Pi_k\,|0\rangle_k\ , \tag{2.30}$$

and we will assume that it is normalized, that is $\langle 0|0\rangle = 1$. If now we calculate the vacuum energy, we find immediately the first problem with infinities in Quantum Field Theory (QFT). In fact

$$\begin{aligned}
\langle 0|H|0\rangle &= \frac{1}{2}\int \widetilde{dk}\ \omega_k \left\langle 0\right|\left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right]\left|0\right\rangle \\
&= \frac{1}{2}\int \widetilde{dk}\ \omega_k \left\langle 0\right|\left[a(k), a^\dagger(k)\right]\left|0\right\rangle \\
&= \frac{1}{2}\int d^3k\ \omega_k \delta^3(0) = \infty\,.
\end{aligned} \tag{2.31}$$

This infinity can be understood as the (infinite) sum of the zero point energy of all quantum oscillators. In the discrete case we would have, $\sum_k \frac{1}{2}\omega_k = \infty$. This infinity can be easily removed. We start by noticing that we only measure energies as differences with respect to the vacuum energy, and those will be finite. We will then define the energy of the vacuum as being zero. Technically this is done as follows. We define a new operator $P^\mu_{N.O.}$ as

$$\begin{aligned}
P^\mu_{N.O.} &\equiv \frac{1}{2}\int \widetilde{dk}\ k^\mu \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] \\
&\quad -\frac{1}{2}\int \widetilde{dk}\ k^\mu \left[a(k), a^\dagger(k)\right] \\
&= \int \widetilde{dk}\ k^\mu a^\dagger(k)a(k)\,.
\end{aligned} \tag{2.32}$$

Now $\langle 0|P^\mu_{N.O.}|0\rangle = 0$. The ordering of operators where the annihilation operators appear on the right of the creation operators is called *normal ordering* and the usual notation is

$$:\frac{1}{2}(a^\dagger(k)a(k) + a(k)a^\dagger(k)) :\equiv a^\dagger(k)a(k)\,. \tag{2.33}$$

Therefore removing the infinity of the energy and momentum corresponds to choosing the normal ordering to our operators. We will adopt this convention in the

following dropping the subscript "$N.O.$" to simplify the notation. This should not appear as an *ad hoc* procedure. In fact, in going from the classical theory where we have products of fields into the quantum theory where the fields are operators, we should have a prescription for the correct ordering of such products. We have just seen that this should be the normal ordering.

Once we have the vacuum we can build the states by applying the the creation operators $a^\dagger(k)$. As in the case of the harmonic oscillator, we can define the *number* operator,

$$N = \int \widetilde{dk}\; a^\dagger(k) a(k)\,. \tag{2.34}$$

It is easy to see that N commutes with H and therefore the eigenstates of H are also eigenstates of N. The state with one particle of momentum k^μ is obtained as $a^\dagger(k)\,|0\rangle$. More precisely

$$|k\rangle = N_k a^\dagger(k)\,|0\rangle \tag{2.35}$$

where N_k is some normalization (that we will discuss in Section 3.4.1 below). We then have

$$\begin{aligned} P^\mu\,|k\rangle =& P^\mu N_k a^\dagger(k)\,|0\rangle = k^\mu N_k a^\dagger(k)\,|0\rangle = k^\mu\,|k\rangle\ , \\ N\,|k\rangle =& N N_k a^\dagger(k)\,|0\rangle = N_k a^\dagger(k)\,|0\rangle = |k\rangle\ . \end{aligned} \tag{2.36}$$

In a similar way, the state $a^\dagger(k_1)...a^\dagger(k_n)\,|0\rangle$ would be a state with n particles. Notice that because of the commutation relations such a state is symmetric under the interchange of any two particles and the particles described are bosons.

2.2.2 Charged scalar field

The description in terms of real fields does not allow the distinction between particles and antiparticles. It applies only to those cases were the particle and antiparticle are identical, like the π^0. For the more usual case where particles and antiparticles are distinct, it is necessary to have some charge (electric or other) that allows us to distinguish them. For this we need complex fields with the Lagrangian given by

$$\mathcal{L} =: \partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi :\,, \tag{2.37}$$

where we are already assuming the normal ordering. The classical theory given in Eq. (2.37) has, at the classical level, a conserved current, $\partial_\mu J^\mu = 0$, with[3]

$$J^\mu = i\varphi^\dagger \overset{\leftrightarrow}{\partial}{}^\mu \varphi\,. \tag{2.39}$$

[3]We define

$$\phi_1 \overset{\leftrightarrow}{\partial}{}^\mu \phi_2 \equiv \phi_1 \partial^\mu \phi_2 - (\partial^\mu \phi_1)\phi_2\ . \tag{2.38}$$

Therefore we expect, at the quantum level, the charge Q

$$Q = \int d^3x : i(\varphi^\dagger \dot{\varphi} - \dot{\varphi}^\dagger \varphi) : , \tag{2.40}$$

to be conserved, that is, $[H, Q] = 0$. To show this we need to know the commutation relations for the field φ. These are

$$\begin{aligned} &[\varphi(x), \varphi(y)] = [\varphi^\dagger(x), \varphi^\dagger(y)] = 0 \\ &[\pi(\vec{x}, t), \varphi(\vec{y}, t)] = [\pi^\dagger(\vec{x}, t), \varphi^\dagger(\vec{y}, t)] = -i\delta^3(\vec{x} - \vec{y}) , \end{aligned} \tag{2.41}$$

where

$$\pi = \dot{\varphi}^\dagger \quad ; \quad \pi^\dagger = \dot{\varphi} . \tag{2.42}$$

The plane waves expansion is then

$$\begin{aligned} \varphi(x) &= \int \widetilde{dk} \ \left[a_+(k) e^{-ik\cdot x} + a_-^\dagger(k) e^{ik\cdot x} \right] , \\ \varphi^\dagger(x) &= \int \widetilde{dk} \ \left[a_-(k) e^{-ik\cdot x} + a_+^\dagger(k) e^{ik\cdot x} \right] , \end{aligned} \tag{2.43}$$

where the operators $a_\pm(k)$ and is $a_\pm^\dagger(k)$ have the following non-vanishing commutation relations,

$$[a_+(k), a_+^\dagger(k')] = [a_-(k), a_-^\dagger(k')] = (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}') , \tag{2.44}$$

therefore allowing us to interpret a_+ and $a_+^\dagger$ as annihilation and creation operators of quanta of type $+$, and similarly for the quanta of type $-$. We can construct the number operators for those quanta:

$$N_\pm = \int \widetilde{dk} \ a_\pm^\dagger(k) a_\pm(k) . \tag{2.45}$$

The energy-momentum operator can be written in terms of the $+$ and $-$ operators,

$$P^\mu = \int \widetilde{dk} \ k^\mu \left[a_+^\dagger(k) a_+(k) + a_-^\dagger(k) a_-(k) \right] , \tag{2.46}$$

where we have already considered the normal ordering. Using the decomposition in Eq. (2.43), we obtain for the charge Q:

$$\begin{aligned} Q &= \int d^3x : i(\varphi^\dagger \dot{\varphi} - \dot{\varphi}^\dagger \dot{\varphi}) : \\ &= \int \widetilde{dk} \ \left[a_+^\dagger(k) a_+(k) - a_-^\dagger(k) a_-(k) \right] \\ &= N_+ - N_- . \end{aligned} \tag{2.47}$$

Using the commutation relation in Eq. (2.44) one can easily verify that

$$[H, Q] = 0\,, \tag{2.48}$$

proving that the charge Q is conserved. The Eq. (2.47) allows us to interpret the $\pm$ quanta as having charge ± 1. However, before introducing interactions, the theory is symmetric, and we can not distinguish between the two types of quanta. From the commutation relations (2.44) we obtain,

$$[P^\mu, a_\pm^\dagger(k)] = k^\mu a_\pm^\dagger(k)\,, \quad [Q, a_\pm^\dagger(k)] = \pm a_\pm^\dagger(k)\,, \tag{2.49}$$

showing that $a_+^\dagger(k)$ creates a quanta with 4-momentum k^μ and charge $+1$. In a similar way we can show that $a_-^\dagger$ creates a quanta with charge -1 and that $a_\pm(k)$ annihilate quanta of charge ± 1, respectively.

2.2.3 Time ordered product and the Feynman propagator

The operator $\varphi^\dagger$ creates a particle with charge $+1$ or annihilates a particle with charge -1. In both cases it adds a total charge $+1$. In a similar way φ annihilates one unit of charge. Let us construct a state of one particle (not normalized) with charge $+1$ by application of $\varphi^\dagger$ in the vacuum:

$$|\Psi_+(\vec{x}, t)\rangle \equiv \varphi^\dagger(\vec{x}, t)\,|0\rangle\ . \tag{2.50}$$

The amplitude to propagate the state $|\Psi_+\rangle$ into the future to the point $(\vec{x}', t')$ with $t' > t$ is given by

$$\theta(t'-t)\,\langle\Psi_+(\vec{x}', t')|\Psi_+(\vec{x}, t)\rangle = \theta(t'-t)\,\big\langle 0|\varphi(\vec{x}', t')\varphi^\dagger(\vec{x}, t)|0\big\rangle\ . \tag{2.51}$$

In $\varphi^\dagger(\vec{x}, t)\,|0\rangle$ only the operator $a_+^\dagger(k)$ is active, while in $\langle 0|\,\varphi(\vec{x}', t')$ the same happens to $a_+(k)$. Therefore Eq. (2.51) is the matrix element that creates a quanta of charge $+1$ in $(\vec{x}, t)$ and annihilates it in $(\vec{x}', t')$ with $t' > t$.

There exists another way of increasing the charge by $+1$ unit in $(\vec{x}, t)$ and decreasing it by -1 in $(\vec{x}', t')$. This is achieved if we create a quanta of charge -1 in $\vec{x}'$ at time t' and let it propagate to $\vec{x}$ where it is absorbed at time $t > t'$. The amplitude is then,

$$\theta(t-t')\,\langle\Psi_-(\vec{x}, t)|\Psi_-(\vec{x}', t')\rangle = \big\langle 0|\varphi^\dagger(\vec{x}, t)\varphi(\vec{x}', t')|0\big\rangle\,\theta(t-t')\,. \tag{2.52}$$

Since we can not distinguish the two paths we must sum of the two amplitudes in Eqs. (2.51) e (2.52). This is the so-called *Feynman propagator*. It can be written in a more compact way if we introduce the time ordered product. Given two operators $a(x)$ and $b(x')$ we define the time ordered product T by,

$$Ta(x)b(x') = \theta(t-t')a(x)b(x') + \theta(t'-t)b(x')a(x)\,. \tag{2.53}$$

In this prescription the older times are always to the right of the more recent times. It can be applied to an arbitrary number of operators. With this definition, the Feynman propagator reads,

$$\Delta_F(x'-x) = \left\langle 0|T\varphi(x')\varphi^\dagger(x)|0\right\rangle \ . \tag{2.54}$$

Using the φ and $\varphi^\dagger$ decomposition we can calculate Δ_F (for free fields, of course)

$$\begin{aligned}
\Delta_F(x'-x) &= \int \widetilde{dk}\ \left[\theta(t'-t)e^{-ik\cdot(x'-x)} + \theta(t-t')e^{ik\cdot(x'-x)}\right] &(2.55)\\
&= \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\varepsilon}e^{-ik\cdot(x'-x)} &(2.56)\\
&\equiv \int \frac{d^4k}{(2\pi)^4}\Delta_F(k)e^{-ik\cdot(x'-x)}\,,
\end{aligned}$$

where

$$\Delta_F(k) \equiv \frac{i}{k^2-m^2+i\varepsilon}\,. \tag{2.57}$$

$\Delta_F(k)$ is the propagator in momenta space (Fourier transform). The equivalence between Eq. (2.56) and Eq. (2.55) is done using integration in the complex plane of the time component k^0, with the help of the residue theorem. The $i\,\epsilon$ prescription assures that positive energy propagates to the future and negative energy to the past. The position of the poles to perform the integration in is shown in Fig. 2.1

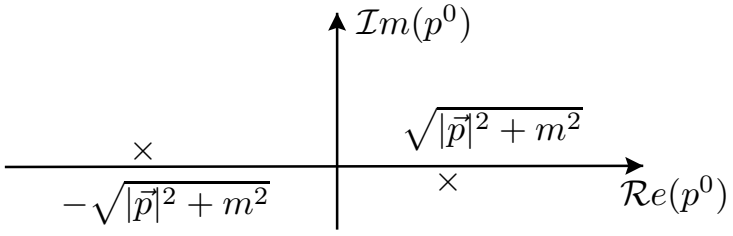


Figure 2.1: Position of the poles in the complex variable k^0 plane.

Then we get

$$\begin{aligned}
\Delta_F(x'-x) &= \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\varepsilon}e^{-ik\cdot(x'-x)}\\
&= \int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}'-\vec{x})}\int\frac{dk^0}{2\pi}\frac{i}{k^2-m^2+i\varepsilon}e^{-ik^0(x'^0-x^0)}\\
&= \int \frac{d^3k}{(2\pi)^3}\theta(t'-t)\ e^{i\vec{k}\cdot(\vec{x}'-\vec{x})}\frac{e^{-ik^0(x'^0-x^0)}}{2k^0}\\
&\quad + \int \frac{d^3k}{(2\pi)^3}\theta(t-t')\ e^{i\vec{k}\cdot(\vec{x}'-\vec{x})}\frac{e^{ik^0(x'^0-x^0)}}{2k^0}
\end{aligned}$$

$$= \int \widetilde{dk} \ \left[\theta(t'-t)e^{-ik\cdot(x'-x)} + \theta(t-t')e^{ik\cdot(x'-x)}\right] \tag{2.58}$$

where in the second term we have done the change of variables $\vec{k} \to -\vec{k}$. This completes the proof of the equivalence between between Eq. (2.56) and Eq. (2.55).

Also applying the operator $(\Box'_x + m^2)$ to $\Delta_F(x'-x)$ in any of the forms of Eq. (2.55) one can show that

$$(\Box'_x + m^2)\Delta_F(x'-x) = -i\delta^4(x'-x)\,, \tag{2.59}$$

that is, $\Delta_F(x'-x)$ is the Green's function for the Klein-Gordon equation with Feynman boundary conditions.

In the presence of interactions, the Feynman propagator loses the simple form of Eq. (2.57). However, as we will see, the free propagator plays a key role in perturbation theory.

2.2.4 Second quantization of the Dirac field

The Lagrangian density that leads to the Dirac equation is

$$\mathcal{L} =: i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi : \ . \tag{2.60}$$

The conjugate momentum to ψ_α is

$$\pi_\alpha = \frac{\partial\mathcal{L}}{\partial\dot{\psi}_\alpha} = i\psi^\dagger_\alpha\,, \tag{2.61}$$

while the conjugate momentum to $\psi^\dagger_\alpha$ vanishes. The Hamiltonian density is then

$$\mathcal{H} =: \pi\dot{\psi} - \mathcal{L} :=: \psi^\dagger(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi :\,, \tag{2.62}$$

which gives the Hamiltonian

$$H =: \int d^3x\psi^\dagger(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi : \ . \tag{2.63}$$

We also obtain for the linear momentum [12]

$$\vec{P} =: \int d^3x\psi^\dagger(-i\vec{\nabla})\psi : \ . \tag{2.64}$$

We can also identify a conserved current, $\partial_\mu j^\mu = 0$, with $j^\mu = \overline{\psi}\gamma^\mu\psi$, which will give the conserved charge

$$Q =: \int d^3x\psi^\dagger\psi : \ . \tag{2.65}$$

The plane wave expansions are

$$\psi(x) = \int \widetilde{dp} \sum_s \left[b(p,s)u(p,s)e^{-ip\cdot x} + d^\dagger(p,s)v(p,s)e^{ip\cdot x} \right] , \tag{2.66}$$

$$\psi^\dagger(x) = \int \widetilde{dp} \sum_s \left[b^\dagger(p,s)u^\dagger(p,s)e^{+ip\cdot x} + d(p,s)v^\dagger(p,s)e^{-ip\cdot x} \right] , \tag{2.67}$$

where $u(p,s)$ and $v(p,s)$ are the spinors for positive and negative energy, respectively, introduced in the study of the Dirac equation and $b, b^\dagger, d$ and $d^\dagger$ are operators obeying the anticommutation relations (fermions)[4]

$$\begin{aligned} \{b^\dagger(p,s), b(k,s)\} &= (2\pi)^3 2k^0 \delta^3(\vec{p}-\vec{k})\delta_{ss'} , \\ \{d^\dagger(p,s'), d(k,s)\} &= (2\pi)^3 2k^0 \delta^3(\vec{p}-\vec{k})\delta_{ss'} , \end{aligned} \tag{2.68}$$

and all the other anticommutators vanish. With the anticommutator relations both contributions to P^μ have the same sign. As in boson case we have to subtract the zero point energy. This is done, as usual, by taking all quantities normal ordered. Therefore we have for P^μ,

$$\begin{aligned} P^\mu &= \int \widetilde{dk}\ k^\mu \sum_s : \left(b^\dagger(k,s)b(k,s) - d(k,s)d^\dagger(k,s) \right) : \\ &= \int \widetilde{dk}\ k^\mu \sum_s \left(b^\dagger(k,s)b(k,s) + d^\dagger(k,s)d(k,s) \right) , \end{aligned} \tag{2.69}$$

and for the charge

$$\begin{aligned} Q &= \int d^3x : \psi^\dagger(x)\psi(x) : \\ &= \int \widetilde{dk} \sum_s \left[b^\dagger(k,s)b(k,s) - d^\dagger(k,s)d(k,s) \right] , \end{aligned} \tag{2.70}$$

which means that the quanta of b type have charge $+1$ while those of d type have charge -1. It is interesting to note that it was the second quantization of the Dirac field that introduced the $-$ sign in Eq. (2.70), making the charge operator without a definite sign, while in Dirac theory was the probability density that was positive defined. The reverse is true for bosons. We can easily show that

$$\begin{aligned} [Q, b^\dagger(k,s)] &= b^\dagger(k,s) \qquad [Q, d(k,s)] = d(k,s) , \\ [Q, b(k,s)] &= -b(k,s) \qquad [Q, d^\dagger(k,s)] = -d^\dagger(k,s) , \end{aligned} \tag{2.71}$$

[4]We do not go here into the arguments that forced us to use anticommutation relations for fermions. The important point is that the signs in Eq. (2.69) and Eq. (2.70) are correct. For a more detailed discussion on the relation between spin and statistics see Complement 2.1.

and then

$$[Q,\psi] = -\psi \ ; \ [Q,\overline{\psi}] = \overline{\psi}\,. \tag{2.72}$$

In QED the charge is given by $q_e Q = -eQ$ ($e = |e| > 0$). Therefore we see that ψ creates positrons and annihilates electrons and the opposite happens with $\overline{\psi}$.

We can introduce the number operators

$$N^+(p,s) = b^\dagger(p,s)b(p,s) \quad ; \quad N^-(p,s) = d^\dagger(p,s)d(p,s)\,, \tag{2.73}$$

and we can rewrite

$$P^\mu = \int \widetilde{dk}\ k^\mu \sum_s (N^+(k,s) + N^-(k,s))\,, \tag{2.74}$$

$$Q = \int \widetilde{dk}\ \sum_s (N^+(k,s) - N^-(k,s))\,. \tag{2.75}$$

2.2.5 Feynman propagator for the Dirac Field

For the Dirac field, as in the case of the charged scalar field, there are two ways of increasing the charge by one unit in x' and decrease it by one unit in x (note that the electron has negative charge). These ways are

$$\theta(t'-t)\left\langle 0|\psi_\alpha(x')\overline{\psi}_\beta(x)|0\right\rangle\,, \tag{2.76}$$

$$\theta(t-t')\left\langle 0|\overline{\psi}_\beta(x)\psi_\alpha(x')|0\right\rangle\,. \tag{2.77}$$

In Eq. (2.76) an electron of positive energy is created at $\vec{x}$ in the instant t, propagates until $\vec{x}'$ where is annihilated at time $t' > t$. In Eq. (2.77) a positron of positive energy is created in x' and annihilated at x with $t > t'$. The Feynman propagator is obtained summing the two amplitudes. Due the exchange of ψ_β and $\overline{\psi}_\alpha$ there must be a minus sign between these two amplitudes. We get for the Feynman propagator,

$$\begin{aligned} S_F(x'-x)_{\alpha\beta} =& \theta(t'-t)\left\langle 0|\psi_\alpha(x')\overline{\psi}_\beta(x)|0\right\rangle - \theta(t-t')\left\langle 0|\overline{\psi}_\beta(x)\psi_\alpha(x')|0\right\rangle \\ \equiv& \left\langle 0|T\psi_\alpha(x')\overline{\psi}_\beta(x)|0\right\rangle\,, \end{aligned} \tag{2.78}$$

where we have defined the time ordered product for fermion fields,

$$T\eta(x)\chi(y) \equiv \theta(x^0-y^0)\eta(x)\chi(y) - \theta(y^0-x^0)\chi(y)\eta(x)\,. \tag{2.79}$$

Inserting in Eq. (2.78) the expansions for ψ and $\overline{\psi}$ we get,

$$\begin{aligned} S_F(x'-x)_{\alpha\beta} =& \int \widetilde{dk}\ \left[(\not{k}+m)_{\alpha\beta}\theta(t'-t)e^{-ik\cdot(x'-x)} + (-\not{k}+m)_{\alpha\beta}\theta(t-t')e^{ik\cdot(x'-x)}\right] \\ =& \int \frac{d^4k}{(2\pi)^4}\frac{i(\not{k}+m)_{\alpha\beta}}{k^2-m^2+i\varepsilon}e^{-ik\cdot(x'-x)} \end{aligned}$$

$$\equiv \int \frac{d^4k}{(2\pi)^4} S_F(k)_{\alpha\beta} e^{-ik\cdot(x'-x)}\,, \tag{2.80}$$

where $S_F(k)$ is the Feynman propagator in momentum space. We can also verify that Feynman's propagator is the Green function for the Dirac equation, that is,

$$(i\partial\!\!\!/' - m)_{\lambda\alpha}\, S_F(x'-x)_{\alpha\beta} = i\delta_{\lambda\beta}\delta^4(x'-x)\,. \tag{2.81}$$

2.2.6 Electromagnetic field quantization

The free electromagnetic field is described by the classical Lagrangian,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu}\,, \tag{2.82}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{2.83}$$

When we try to apply the canonical quantization to the potentials A^μ we immediately run into difficulties. For instance, if we define the conjugate momentum as,

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\dot{A}_\mu)}\,, \tag{2.84}$$

we get

$$\begin{aligned} \pi^k &= \frac{\partial \mathcal{L}}{\partial(\dot{A}_k)} = -\dot{A}^k - \frac{\partial A^0}{\partial x^k} = E^k\,, \\ \pi^0 &= \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0\,. \end{aligned} \tag{2.85}$$

Therefore the conjugate momentum to the coordinate A^0 vanishes and does not allow us to use directly the canonical formalism. The problem has its origin in the fact that the photon, that we want to describe, has only two degrees of freedom (positive or negative helicity) but we are using a field A^μ with four degrees of freedom. In fact, we have to impose constraints on A^μ in such a way that it describes the photon. This problem can be addressed in three different ways:

i) *Radiation Gauge*

Historically, this was the first method to be used. It is based in the fact that it is always possible to choose a gauge, called the *radiation* or *Coulomb gauge*, where

$$A^0 = 0 \quad ; \quad \vec{\nabla}\cdot\vec{A} = 0\,, \tag{2.86}$$

that is, the potential $\vec{A}$ is transverse. The conditions in Eq. (2.86) reduce the number of degrees of freedom to two, the transverse components of $\vec{A}$. It is then

possible to apply the canonical formalism to these transverse components and quantize the electromagnetic field in this way. The problem with this method is that we lose explicit Lorentz covariance. It is then necessary to show that this is recovered in the final result. This method is followed in many text books, for instance in Bjorken and Drell [11].

ii) *Quantization of systems with constraints*

It can be shown that the electromagnetism is an example of an Hamilton generalized system, that is a system where there are constraints among the variables. The way to quantize these systems was developed by Dirac for systems of particles with n degrees of freedom. The generalization to quantum field theories is done using the formalism of path integrals. This is the only method that can be applied to non-abelian gauge theories, like the standard model, but it is beyond the level of this introductory course. There are many excellent textbooks (see for instance my text [12]).

iii) *Undefined metric formalism*

There is another method that works for the electromagnetism, called the formalism of the *undefined metric*, developed by Gupta and Bleuler [21, 22]. In this formalism, that we will study below, Lorentz covariance is kept, that is we will always work with the 4-vector A_μ, but the price to pay is the appearance of states with negative norm. We have then to define the Hilbert space of the physical states as a sub-space where the norm is positive. We see that in all cases, in order to maintain the explicit Lorentz covariance, we have to complicate the formalism. We will follow the books of Silvan Schweber [23] and Mandl and Shaw [20].

2.2.7 Undefined metric formalism

To solve the difficulty of the vanishing of π^0, we will start by modifying the Maxwell Lagrangian introducing a new term,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial \cdot A)^2 \,, \tag{2.87}$$

where ξ is a dimensionless parameter that acts as a Lagrange multiplier for the Lorenz condition $\partial_\mu\, A^\mu = 0$. Now we obtain for the conjugate momenta

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0} - \frac{1}{\xi} g^{\mu 0}(\partial \cdot A) \,, \tag{2.88}$$

that is

$$\begin{cases} \pi^0 &= -\frac{1}{\xi}(\partial \cdot A) \\ \pi^k &= E^k \end{cases} .$$

Now we can proceed with the quantization that leads to (we take $\xi = 1$, the so-called Feynman gauge),

$$[A_\mu(\vec{x},t), A_\nu(\vec{y},t)] = [\dot{A}_\mu(\vec{x},t), \dot{A}_\mu(\vec{y},t)] = 0\,,$$
$$[\dot{A}_\mu(\vec{x},t), A_\nu(\vec{y},t)] = i g_{\mu\nu}\delta^3(\vec{x}-\vec{y})\,. \tag{2.89}$$

If we compare these relations with the corresponding ones for the real scalar field, where the only one non-vanishing is,

$$[\dot{\varphi}(\vec{x},t), \varphi(\vec{y},t)] = -i\delta^3(\vec{x}-\vec{y})\,, \tag{2.90}$$

we see [$g_{\mu\nu} = \mathrm{diag}(+,-,-,-)$] that the relations for space components are equal but they differ for the time component (hence the name of indefinite metric). This sign will be the source of the difficulties previously mentioned in quantizing the electromagnetic field. It turns out that we can keep going with this covariant formalism and that for physical states only the two transverse photon polarizations will contribute. The contributions from the time component will cancel those from the longitudinal one just leaving the transverse degrees of freedom.

So we do not worry about this sign, and expand $A_\mu(x)$ in plane waves,

$$A^\mu(x) = \int \widetilde{dk} \sum_{\lambda=0}^{3} \left[a(k,\lambda)\varepsilon^\mu(k,\lambda)e^{-ik\cdot x} + a^\dagger(k,\lambda)\varepsilon^{\mu *}(k,\lambda)e^{ik\cdot x}\right]\,, \tag{2.91}$$

where $\varepsilon^\mu(k,\lambda)$ are a set of four independent 4-vectors that we can assume to be real, without loss of generality[5]. We will now make a choice for these 4-vectors. For this we need to introduce, besides the photon four momentum k^μ, another four-vector n^μ linearly independent of the previous four-vectors. We choose $\varepsilon^\mu(1)$ and $\varepsilon^\mu(2)$ orthogonal to k^μ and n^μ, such that

$$\varepsilon^\mu(k,\lambda)\varepsilon_\mu(k,\lambda') = -\delta_{\lambda\lambda'} \text{ for } \lambda,\lambda' = 1,2\,. \tag{2.92}$$

After, we choose $\varepsilon^\mu(k,3)$ in the plane (k^μ, n^μ) orthogonal to n^μ and normalized, that is

$$\varepsilon^\mu(k,3)n_\mu = 0 \quad ; \quad \varepsilon^\mu(k,3)\varepsilon_\mu(k,3) = -1\,. \tag{2.93}$$

Finally we choose $\varepsilon^\mu(k,0) = n^\mu$. The vectors $\varepsilon^\mu(k,1)$ and $\varepsilon^\mu(k,2)$ are called transverse polarizations, while $\varepsilon^\mu(k,3)$ and $\varepsilon^\mu(k,0)$ longitudinal and scalar polarizations, respectively. We can give an example. In the frame where $n^\mu = (1,0,0,0)$ and $\vec{k}$ is along the z axis we have

$$\varepsilon^\mu(k,0) = (1,0,0,0); \varepsilon^\mu(k,1) = (0,1,0,0); \varepsilon^\mu(k,2) = (0,0,1,0); \varepsilon^\mu(k,3) = (0,0,0,1). \tag{2.94}$$

[5]As they can also be taken as complex, in general we take the complex conjugation.

In general we can show that

$$\varepsilon(k,\lambda)\cdot\varepsilon^*(k,\lambda') = g^{\lambda\lambda'}, \quad \sum_\lambda g^{\lambda\lambda}\varepsilon^\mu(k,\lambda)\varepsilon^{*\nu}(k,\lambda) = g^{\mu\nu}\,. \tag{2.95}$$

Inserting the expansion (2.91) in (2.89) we get

$$[a(k,\lambda), a^\dagger(k',\lambda')] = -g^{\lambda\lambda'}2k^0(2\pi)^3\delta^3(\vec{k}-\vec{k}')\,, \tag{2.96}$$

showing, once more, that the quanta associated with $\lambda = 0$ has a commutation relation with the wrong sign. See, for instance, Refs. [12, 20] for more detailed discussion. For our purposes we will use this covariant formalism without showing all the details.

2.2.8 Feynman propagator for the Maxwell field

The Feynman propagator is defined as the vacuum expectation value of the time ordered product of the fields, that is

$$\begin{aligned} D_{\mu\nu}(x,y) &\equiv \langle 0|TA_\mu(x)A_\nu(y)|0\rangle \\ &= \theta(x^0-y^0)\,\langle 0|A_\mu(x)A_\nu(y)|0\rangle + \theta(y^0-x^0)\,\langle 0|A_\nu(y)A_\mu(x)|0\rangle\,. \end{aligned} \tag{2.97}$$

Inserting the expansions for $A_\mu(x)$ and $A_\nu(y)$ we get

$$\begin{aligned} D_{\mu\nu}(x-y) &= -g_{\mu\nu}\int \widetilde{dk}\;\left[e^{-ik\cdot(x-y)}\theta(x^0-y^0) + e^{ik\cdot(x-y)}\theta(y^0-x^0)\right] \\ &= -g_{\mu\nu}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2+i\varepsilon}e^{-ik\cdot(x-y)} \\ &\equiv \int \frac{d^4k}{(2\pi)^4}D_{\mu\nu}(k)e^{-ik\cdot(x-y)}\,, \end{aligned} \tag{2.98}$$

where $D_{\mu\nu}(k)$ is the Feynman propagator on the momentum space

$$D_{\mu\nu}(k) \equiv \frac{-ig_{\mu\nu}}{k^2+i\varepsilon}\,. \tag{2.99}$$

It is easy to verify that $D_{\mu\nu}(x-y)$ is the Green's function of the equation of motion, that for $\xi = 1$ is the wave equation, that is

$$\Box_x D_{\mu\nu}(x-y) = ig_{\mu\nu}\delta^4(x-y)\,. \tag{2.100}$$

These expressions for $D_{\mu\nu}(x-y)$ and $D_{\mu\nu}(k)$ correspond to the particular case of $\xi = 1$, the so-called *Feynman gauge*. For the general case where $\xi \neq 0$ the equation of motion reads

$$\left[\Box_x g^\mu_\rho - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial_\rho\right]A^\rho(x) = 0\,. \tag{2.101}$$

For this case the equal times commutation relations are more complicated (see Problem 2.5). Using those relations one can show (see Problem 2.6) that the Feynman propagator is still the Green's function of the equation of motion, that is

$$\left[\Box_x g^\mu_\rho - \left(1 - \frac{1}{\xi}\right)\partial^\mu \partial_\rho\right] \langle 0|TA^\rho(x)A^\nu(y)|0\rangle = ig^{\mu\nu}\delta^4(x-y)\,. \tag{2.102}$$

Using this equation we can then obtain in an arbitrary ξ gauge (of the Lorenz type),

$$D_{\mu\nu}(k) = -i\frac{g_{\mu\nu}}{k^2 + i\varepsilon} + i(1-\xi)\frac{k_\mu k_\nu}{(k^2+i\varepsilon)^2}\ . \tag{2.103}$$

Complements

Complement 2.1 Spin-Statistics Theorem

In simple terms the spin-statistics theorem states that integer spin particles should be quantized with commutation relations and those with half-integer spin should be quantized with anticommutation relations. The integer spin particles are called bosons and obey the Bose-Einstein statistics and those with half-integer spin are called fermions and obey Fermi-Dirac statistics.

The complete proof of the theorem is quite complicated [24–27]. Normally it consists in showing that we get a pathological flawed theory if we do not follow the spin-statistics assignment. There are three main ways of going about this:

1. Using the **Lorentz invariance of the S-matrix**
2. Requiring the energy to bounded from below (**stability**)
3. Requiring **causality** to hold.

We will not say anything about Lorentz invariance, but we can easily indicate the problems with stability and causality. We will follow the arguments of Mathew Schwartz [28].

The Stability Argument

This is the simplest argument, and in fact we have already discussed it although we did not emphasize it. Let us start with the real scalar field. We have seen that the Hamiltonian density when written in terms of the annihilation and creation operators was given by Eq. (2.26),

$$H = \frac{1}{2}\int \widetilde{dk}\ \omega_k \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] . \tag{2.104}$$

The next step was to remove the infinite energy of the vacuum. This was done normal ordering the Hamiltonian, that is bringing the annihilation operators to the right of the creation operators. Let us fro a moment assume that we do not know if we have to use commutation or anticommutation rules for the operators. Then one would get, after discarding the c-number that would result from the commutation (or anticommutation)

$$H = \frac{1}{2}\int \widetilde{dk}\ \omega_k \left[a^\dagger(k)a(k) \pm a^\dagger(k)a(k)\right] \tag{2.105}$$

where the $+$ is for commutation and the $-$ for anticommutation. It is clear that we only get a sensible result if we use commutation for the scalar field.
Now consider the case of the complex scalar field. There we would get, after bringing the operators to normal order

$$P^\mu = \int \widetilde{dk}\ k^\mu \left[a_+^\dagger(k)a_+(k)\frac{1}{2}(1\pm 1) + a_-^\dagger(k)a_-(k)\frac{1}{2}(1\pm 1)\right] \tag{2.106}$$

It is interesting to note that for the charge

$$Q = \int \widetilde{dk}\ :\left[a_+^\dagger(k)a_+(k) - a_-(k)a_-^\dagger(k)\right]:$$

$$= \int \widetilde{dk} \ \left[a_+^\dagger(k) a_+(k) \mp a_-(k) a_-^\dagger(k) \right] \tag{2.107}$$

where here the $-$ is for commutation and the $+$ for anticommutation. So we see that for scalar fields we have to have commutation relations for the theory to make sense.
Now consider the case of fermions. Before using any commutation or anticommutation we get,

$$\begin{aligned} P^\mu &= \int \widetilde{dk} \ k^\mu : \left[b^\dagger(k,s) b(k,s) - d(k,s) d^\dagger(k,s) \right] : \\ Q &= \int \widetilde{dk} \ : \left[b^\dagger(k,s) b(k,s) + d(k,s) d^\dagger(k,s) \right] : \end{aligned} \tag{2.108}$$

Then immediately we see that if we use commutation relations, the energy is not bound from below, the creation of antiparticles will lower it. Also the charge looses its meaning. In summary we have to quantize bosons with commutation rules and fermions using anticommutators, for the theories to be well behaved.

The Causality Argument

Causality here means that the commutator of two observables must vanish outside the light cone, that is for spacelike separation. For spin zero fields this means.

$$[\phi(x), \phi(y)] = 0, \quad (x-y)^2 < 0 \tag{2.109}$$

There is no equivalent for spinors as they are not observables. We can construct observables out of bilinears like

$$\left\{ \overline{\psi}(x)\psi(x), \overline{\psi}(y)\psi(y) \right\} = 0, \quad (x-y)^2 < 0 \tag{2.110}$$

However this does not imply that spinors have to anticommute, as Eq. (2.110) can be satisfied if the spinors commute or anticommute at spacelike separations.

Now these commutators and anticommutators can be calculated and look to see if the above conditions are verified. We get

$$[\phi(x), \phi(y)] \equiv i\Delta(x-y) \tag{2.111}$$

where

$$i\Delta(x-y) = \int \widetilde{dk} \left[e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)} \right] \tag{2.112}$$

This function can be evaluated although it is not a simple problem. We change variables to $t = x^0 - y^0, \vec{r} = \vec{x} - \vec{y}$ and $r = |\vec{r}|$. The integration is then done in spherical coordinates in momentum space,

$$\begin{aligned} i\Delta(x-y) &= \int \widetilde{dk} \left[e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)} \right] \\ &= \frac{1}{(2\pi)^2} \int \frac{dk k^2}{2\omega_k} \int_{-1}^{1} d\cos\theta \left[e^{-i\omega_k t} e^{ikr\cos\theta} - e^{i\omega_k t} e^{-ikr\cos\theta} \right] \end{aligned}$$

$$= -i\frac{1}{2\pi^2}\int_0^\infty dk k^2 \frac{\sin(\sqrt{k^2+m^2}t)}{\sqrt{k^2+m^2}}\frac{\sin(kr)}{kr} \tag{2.113}$$

The result can be expressed in terms of the Bessel Function J_0. We have [28]

$$\Delta(t,r) = \frac{1}{4\pi r}\frac{\partial}{\partial r}\begin{cases} J_0(m\sqrt{t^2-r^2}) & t > r \\ 0 & t < |r| \\ -J_0(m\sqrt{t^2-r^2}) & t < -r \end{cases} \tag{2.114}$$

As the function $\Delta(x-y)$ is Lorentz invariant we have

$$\Delta(\vec{x}-\vec{y},0) = 0 \tag{2.115}$$

which ensures that the commutator of two spin zero fields vanishes for space-like separations. Note that $\Delta(x-y)$ also satisfies the relations

$$\begin{aligned} &(\Box_x + m^2)\Delta(x-y) = 0 \\ &\Delta(x-y) = -\Delta(y-x) \end{aligned} \tag{2.116}$$

We also note that

$$\partial^0\Delta(x-y)|_{x^0=y^0} = -\delta^3(\vec{x}-\vec{y}) \tag{2.117}$$

ensuring the equal time commutation relation. If we had quantized with anticommutators one would get

$$\begin{aligned} \{\phi(x),\phi(y)\} =& \int \widetilde{dk}\left[e^{-ik\cdot(x-y)} + e^{ik\cdot(x-y)}\right] \\ =& \frac{1}{(2\pi)^2}\int \frac{dkk^2}{2\omega_k}\int_{-1}^{1} d\cos\theta \left[e^{-i\omega_k t}e^{ikr\cos\theta} + e^{i\omega_k t}e^{-ikr\cos\theta}\right] \\ =& \frac{1}{2\pi^2}\int_0^\infty dk k^2 \frac{\cos(\sqrt{k^2+m^2}t)}{\sqrt{k^2+m^2}}\frac{\sin(kr)}{kr} \\ \equiv& i\Delta_1(t,r) \end{aligned} \tag{2.118}$$

The function $\Delta_1(t,r)$ is given by [28]

$$\Delta_1(t,r) = -\frac{1}{4\pi r}\frac{\partial}{\partial r}\begin{cases} iY_0(m\sqrt{t^2-r^2}) & t > r \\ H_0(i\sqrt{r^2-t^2}) & t < |r| \\ iY_0(m\sqrt{t^2-r^2}) & t < -r \end{cases} \tag{2.119}$$

where Y_0 is the Bessel function of second kind and $H_0 = J_0 + iY_0$ is the Hankel function. We see that it does not vanish for spacelike separations, and the causality would be violated.

Now for fermions. We start with commutators. We get

$$[\psi(x),\psi(y)] = \int \widetilde{dk}\left[(\not{k}+m)e^{-ik\cdot(x-y)} - (\not{k}-m)e^{ik\cdot(x-y)}\right]$$

$$
\begin{aligned}
&=(i\gamma^\mu\partial_\mu + m)\int \widetilde{dk}\left[e^{-ik\cdot(x-y)} + e^{ik\cdot(x-y)}\right] \\
&=(i\gamma^\mu\partial_\mu + m)i\Delta_1(t,r) \qquad (2.120)
\end{aligned}
$$

and therefore the commutator does not vanish for space-like separations. However for the anticommutator

$$
\begin{aligned}
\{\psi(x),\psi(y)\} &= \int \widetilde{dk}\left[(\not{k} + m)e^{-ik\cdot(x-y)} + (\not{k} - m)e^{ik\cdot(x-y)}\right] \\
&=(i\gamma^\mu\partial_\mu + m)\int \widetilde{dk}\left[e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}\right] \\
&=(i\gamma^\mu\partial_\mu + m)i\Delta(t,r) \qquad (2.121)
\end{aligned}
$$

which vanishes outside the light cone. This is a sufficient but not a necessary condition for causality to hold.

Problems

2.1 Show that Eq. (2.22) can be inverted to obtain

$$a(k) = i \int d^3x e^{ik\cdot x} \overleftrightarrow{\partial}_0 \varphi(x)$$
$$a^\dagger(k) = -i \int d^3x e^{-ik\cdot x} \overleftrightarrow{\partial}_0 \varphi(x) \tag{2.122}$$

Use this result to derive the commutation rules in Eq. (2.24).

2.2 Verify the results of Eq. (2.28).

2.3 For the charged scalar field derive the results in Eq. (2.47) and Eq. (2.49).

2.4 For the Dirac field derive the results in Eq. (2.70) and Eq. (2.71).

2.5 Show that for the general case of $\xi \neq 1$ we have

$$\begin{aligned}
&[A_\mu(\vec{x},t), A_\nu(\vec{y},t)] = 0 \\
&[\dot{A}_\mu(\vec{x},t), A_\nu(\vec{y},t)] = i g_{\mu\nu} \left[1 - (1-\xi) g_{\mu 0}\right] \delta^3(\vec{x}-\vec{y}) \\
&[\dot{A}_i(\vec{x},t), \dot{A}_j(\vec{y},t)] = [\dot{A}_0(\vec{x},t), \dot{A}_0(\vec{y},t)] = 0 \\
&[\dot{A}_0(\vec{x},t), \dot{A}_i(\vec{y},t)] = i(1-\xi)\partial_i \delta^3(\vec{x}-\vec{y})
\end{aligned} \tag{2.123}$$

2.6 Use the results of Problem 2.5 to show that, in the general gauge with $\xi \neq 1$ we have

$$\left[\Box_x g^\mu{}_\rho - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial_\rho\right] \langle 0|T A^\rho(x) A^\nu(y)|0\rangle = i g^{\mu\nu} \delta^4(x-y) \tag{2.124}$$

where

$$\left(\Box g^\mu \rho - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial_\rho\right) A^\rho = 0 \tag{2.125}$$

Chapter 3

S-Matrix, Wick's Theorem and Feynman Rules for QED

3.1 The S-matrix

In the previous section we described the free field quantization using the Heisenberg Picture (HP). Now we want to study the interacting fields. For this we begin by separating the Lagrangian in its free and interacting part,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\rm int}\,. \tag{3.1}$$

For instance for QED we have

$$\mathcal{L}_0 =: i\overline{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - m\overline{\psi}(x)\psi(x) - \frac{1}{4}F^{\mu\nu}(x)F_{\mu\nu}(x) :\,, \tag{3.2}$$

and

$$\mathcal{L}_{\rm int} =: e\overline{\psi}(x)\gamma_\mu\psi A^\mu(x) :\ . \tag{3.3}$$

The separation in Eq. (3.1) leads to a corresponding separation in the Hamiltonian,

$$H = H_0 + H_{\rm int}\,. \tag{3.4}$$

As we saw in section 2.1, this is precisely the separation that is at the basis of the Interaction Picture (IP). In the IP the operators satisfy equations of motion similar with those of the HP, but evolving with the free Hamiltonian H_0, Eq. (2.14). The other important point is that if the interaction Lagrangian does not have derivatives, then the momenta canonically conjugate to the free fields and interacting fields are equal[1]. As the IP and HP are related by an unitary transformation, the interacting fields in the IP satisfy the same commutation relation as the free fields in the HP. These means that we can use the algebra of the free fields for the interacting

[1]We will comment on section 3.6 on this assumption, but for the moment we will only consider this case, as it happens in QED.

fields in the IP. They share the same plane wave expansion and the same Feynman propagators.

The equation of motion for states in the IP is Eq. (2.16) that we write now as (we go back to $\hbar = 1$)

$$i\frac{d}{dt}\left|\Phi(t)\right\rangle = H_{\rm int}\left|\Phi(t)\right\rangle \,, \tag{3.5}$$

where the interaction Hamiltonian in the IP is (see Eq. (2.12)),

$$H_{\rm int} = e^{iH_0(t-t_0)}H^{\rm S}_{\rm int}e^{-iH_0(t-t_0)}\,, \tag{3.6}$$

with $H^{\rm S}_{\rm int}$ being the interaction Hamiltonian in the SP. In this equation and from now on, we will drop the mention to the IP as we will only be considering this case and this simplifies the notation. Eq. (3.5) is the Schrödinger equation for a time dependent Hamiltonian. As the Hamiltonian is hermitian the probability of the sates is preserved,

$$\langle\Phi(t)|\Phi(t)\rangle = \langle i|i\rangle \,, \tag{3.7}$$

where at some initial time, t_i, we have $|\Phi(t_i)\rangle = |i\rangle$.

This formalism is particularly adapted to the scattering processes we are interested in. At $t = -\infty$ we prepare some initial state $|i\rangle$ that we then let evolve with time and interact with other fields. At time $t = \infty$ all these interactions took place. The S-matrix is defined as the relation between $|\Phi(\infty)\rangle$ and $|\Phi(-\infty)\rangle = |i\rangle$. More precisely,

$$|\Phi(\infty)\rangle = S\,|\Phi(-\infty)\rangle = S\,|i\rangle \,. \tag{3.8}$$

The state $|\Phi(\infty)\rangle$ contains all the final states, but normally we are interested in the probability of going into a particular final state $|f\rangle$. This amplitude is therefore,

$$\langle f|\Phi(\infty)\rangle = \langle f|S|i\rangle = S_{fi}\,. \tag{3.9}$$

Using the expansion of $|\Phi(\infty)\rangle$ in a complete set of final states one can show the unitarity of the S-matrix, namely

$$\sum_f |S_{fi}|^2 = 1\,, \tag{3.10}$$

ensuring the conservation of probability (not of particles) (see problem 3.1).

To find the S-matrix we have to solve the differential equation, Eq. (3.5) with the initial condition $|\Phi(-\infty)\rangle = |i\rangle$. We can write the integral equation

$$|\Phi(t)\rangle = |i\rangle + (-i)\int_{-\infty}^{t} dt_1 H_{\rm int}(t_1)\,|\Phi(t_1)\rangle \,, \tag{3.11}$$

which is equivalent to Eq. (3.5) with the proper initial condition as $t \to -\infty$. It should be clear that we have not solved the problem, because the state $|\Phi(t)\rangle$ appears on both sides of the equation. However, this form is particularly useful to set up

perturbation theory. If the interaction Hamiltonian has some small dimensionless parameter (like the fine structure constant $\alpha = 1/137$ in QED), then we can solve Eq. (3.11) by iteration. In a first step we get

$$|\Phi(t)\rangle = |i\rangle + (-i)\int_{-\infty}^{t} dt_1 H_{\rm int}(t_1)\,|i\rangle + (-i)^2 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 H_{\rm int}(t_1)H_{\rm int}(t_2)\,|\Phi(t_2)\rangle\ . \tag{3.12}$$

Iterating further and noticing that $t_n \to -\infty$ in this process, we get

$$|\Phi(\infty)\rangle = \sum_{n=0}^{\infty}(-i)^n \int_{-\infty}^{t\to\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n H_{\rm int}(t_1)H_{\rm int}(t_2)\cdots H_{\rm int}(t_n)\,|\Phi(t_n)\rangle\ , \tag{3.13}$$

and using the definition of the S-matrix we get

$$S = \sum_{n=0}^{\infty}(-i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n H_{\rm int}(t_1)H_{\rm int}(t_2)\cdots H_{\rm int}(t_n)\,. \tag{3.14}$$

Noticing that in Eq. (3.14) the times are time ordered, because $t > t_1 > t_2 > \cdots > t_n$, we can use this to write the S-matrix in the form (see problem 3.2)

$$S = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \cdots \int_{-\infty}^{\infty} dt_n T\left(H_{\rm int}(t_1)H_{\rm int}(t_2)\cdots H_{\rm int}(t_n)\right)\ . \tag{3.15}$$

The final step is to write the S-matrix in terms of the Hamiltonian density $\mathcal{H}_{\rm int}$,

$$S = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!} \int d^4x_1 d^4x_2 \cdots d^4x_n T\left(\mathcal{H}_{\rm int}(x_1)\mathcal{H}_{\rm int}(x_2)\cdots \mathcal{H}_{\rm int}(x_n)\right)\ , \tag{3.16}$$

where now the integrations are over all spacetime. This equation was derived by Dyson [29, 30] and is known as the Dyson expansion of the S-matrix. This will the starting point to establish the perturbative series.

There are some fine points regarding the specification of the initial and final states $|i\rangle$ and $|f\rangle$. These are eigenstates of the unperturbed Hamiltonian H_0. The question that arises is how these states get perturbed before the transition we are studying. We will comment on this further on section 3.6 but for now it is enough to say that if we keep to the lowest order in the non-vanishing contribution to a given process there is no problem.

3.2 Wick's Theorem

In doing actual calculations we will be dealing with the problem of calculating S-matrix elements between given initial and final states,

$$S_{fi} = \langle f|S|i\rangle\ , \tag{3.17}$$

where the states $|i\rangle$ and $|f\rangle$ are obtained from the vacuum by use of appropriate creation operators and for the S-matrix we use the Dyson expansion in Eq. (3.16). For instance, the state with one electron with momentum p and spin s will be given by

$$|i\rangle = |p\rangle = N_p b^\dagger(p,s)\,|0\rangle\ , \tag{3.18}$$

where N_p is some normalization. In section 3.4 we will discuss the normalization and we will discover the value of N_p in order to make contact with the usual definitions. For the purpose of this section we just want to look at the structure of the matrix elements and relative signs, and therefore we can simplify the expressions by setting[2] $N_p = 1$.

Now consider, for definiteness, that we want to calculate the scattering the Møller scattering, $e^-(p_1) + e^-(p_2) \to e^-(p_3) + e^-(p_4)$, in second order in QED. We then want to look at the following term in the expansion of the S-matrix

$$S^{(2)}_{fi} = \frac{(-i)^2}{2!}\int d^4x_1 d^4x_2\,\left\langle 0|b(p_3)b(p_4)T\left(\mathcal{H}_{\rm int}(x_1)\mathcal{H}_{\rm int}(x_2)\right)b^\dagger(p_2)b^\dagger(p_1)|0\right\rangle\ , \tag{3.19}$$

where, for simplicity we have suppressed the spin indices and the interaction Hamiltonian density is given by[3]

$$\mathcal{H}_{\rm int}(x) = -e :\overline{\psi}(x)\gamma^\mu\psi(x)A_\mu(x):\ . \tag{3.20}$$

As the fields in Eq. (3.19) have the free field plane wave expansion, the method to evaluate that expression is to move all the annihilation operators to the right and all creation to the left until they hit the vacuum. This is straightforward but a bit lengthy to do in every case. That is where the theorem of Wick comes to rescue us. This theorem relates the time ordered product that appears in Eq. (3.19) with a sum of normal ordered terms and some c-numbers. As the normal ordered terms have already the annihilation operators to the right and the creation operators to the left, the process is very much simplified.

Before given the general form, let us consider first the case of just two fields. This will enable us to introduce the appropriate notation to state the theorem. We start with two scalar fields. Then

$$T\left(\varphi(x_1)\varphi(x_2)\right) =: \varphi(x_1)\varphi(x_2) : + \text{c-number}\,, \tag{3.21}$$

because each of the fields have a creation and annihilation part, and the process of doing the normal ordering will give the commutators that are c-numbers (that is, they are not operators). In fact it is very easy to calculate this c-number. For

[2] In fact, in section 3.4 we will see that it is possible to choose a consistent normalization where $N_p = 1$.

[3] To simplify notation, in this chapter we are already assuming QED and that we are dealing with electrons, with $Q_e = -1$ already included explicitly and $e > 0$ as everywhere.

that we take the matrix element between two vacuum states and use the results $\langle 0| : .... : |0\rangle = 0$ and $\langle 0|0\rangle = 1$, to write

$$\text{c-number} = \langle 0|T\left(\varphi(x_1)\varphi(x_2)\right)|0\rangle = \Delta_F(x_1 - x_2)\,, \tag{3.22}$$

that is, the c-number is the free field Feynman propagator (Δ_F is the propagator for the real scalar). It is conventional to use a name, *contraction*, and a notation for this operation, that is,

$$T\left(\varphi(x_1)\varphi(x_2)\right) =: \varphi(x_1)\varphi(x_2) : + \underbrace{\varphi(x_1)\varphi}(x_2)\,, \tag{3.23}$$

where, clearly, the contraction is the Feynman propagator,

$$\underbrace{\varphi(x_1)\varphi}(x_2) = \langle 0|T\left(\varphi(x_1)\varphi(x_2)\right)|0\rangle = \Delta_F(x_1 - x_2)\,. \tag{3.24}$$

For the other types of fields we have an expression similar to Eq. (3.23) with the following correspondence,

$$\underbrace{\varphi(x_1)\varphi}(x_2) = \Delta_F(x_1 - x_2)\,, \tag{3.25}$$

$$\underbrace{\varphi(x_1)\varphi^\dagger}(x_2) = \underbrace{\varphi^\dagger(x_2)\varphi}(x_1) = \Delta_F(x_1 - x_2)\,, \tag{3.26}$$

$$\underbrace{\psi_\alpha(x_1)\overline{\psi}_\beta}(x_2) = -\underbrace{\overline{\psi}_\beta(x_2)\psi_\alpha}(x_1) = S_{F\,\alpha\beta}(x_1 - x_2)\,, \tag{3.27}$$

$$\underbrace{A^\mu(x_1)A^\nu}(x_2) = D_F^{\mu\nu}(x_1 - x_2)\,, \tag{3.28}$$

where the minus sign in Eq. (3.27) comes from the anticommutation rules for fermionic fields.

We are now in position to state the theorem of Wick. To simplify we omit all indices and spacetime coordinates. We have

$$\begin{aligned} T(ABCD\ldots WXYZ) =:& ABCD\ldots WXYZ : \\ &+ : \underbrace{AB}CD\ldots WXYZ : + : \underbrace{ABC}D\ldots WXYZ : + \cdots + : ABCD\ldots WX\underbrace{YZ} : \\ &+ : \underbrace{AB}\underbrace{CD}\ldots WXYZ : + \cdots + : ABCD\ldots \underbrace{WX}\underbrace{YZ} : \\ &+ \cdots\,, \end{aligned} \tag{3.29}$$

where in the first, second, third line, we have respectively no contractions, one contraction, two contractions and so on in all possible ways. As the contractions are c-numbers they can be taken out of the normal products. When we actually substitute the contractions for the Feynman propagators, the correct signs for fermions have to be taken in account, as in Eq. (3.27). Wick also proved an extension of the theorem for the cases where some of the operators inside the T-product were already

normal ordered, as it happens with $\mathcal{H}_{\rm int}$. In this case we should not do contractions among the fields inside the normal ordered product, at the same spacetime point.

Wick's theorem is proved by induction. We will leave the proof for appendix B. Here we just give a non-trivial case to show how it works. Let us consider a case that will be useful in the next section, the term that comes from second order in the QED interaction, where we have,

$$\begin{aligned} T\left(\mathcal{H}_{\rm int}(x_1)\mathcal{H}_{\rm int}(x_2)\right) &= T\left(:-e\overline{\psi}(x_1)\gamma_\mu\psi(x_1)A^\mu(x_1)::-e\overline{\psi}(x_2)\gamma_\nu\psi(x_2)A^\nu(x_2):\right) \\ &= (-e)^2\left[T\left(:\overline{\psi}_1\gamma_\mu\psi_1A_1^\mu::\overline{\psi}_2\gamma_\nu\psi_2A_2^\nu:\right)\right], \end{aligned} \tag{3.30}$$

where we are using a simplified notation. We then get

$$\begin{aligned} T\big((-1/e)\mathcal{H}_{\rm int}(x_1)\ (-1/e)\mathcal{H}_{\rm int}(x_2)\big) &= T\left(:\overline{\psi}_1\gamma_\mu\psi_1A_1^\mu::\overline{\psi}_2\gamma_\nu\psi_2A_2^\nu:\right) \\ &= :\overline{\psi}_1\gamma_\mu\psi_1A_1^\mu\overline{\psi}_2\gamma_\nu\psi_2A_2^\nu: \\ &\quad + :\overline{\psi}_1\gamma_\mu\psi_1A_1^\mu\overline{\psi}_2\gamma_\nu\psi_2A_2^\nu: + :\overline{\psi}_1\gamma_\mu\psi_1A_1^\mu\overline{\psi}_2\gamma_\nu\psi_2A_2^\nu: \\ &\quad\quad + :\overline{\psi}_1\gamma_\mu\psi_1A_1^\mu\overline{\psi}_2\gamma_\nu\psi_2A_2^\nu: \\ &\quad + :\overline{\psi}_1\gamma_\mu\psi_1A_1^\mu\overline{\psi}_2\gamma_\nu\psi_2A_2^\nu: + :\overline{\psi}_1\gamma_\mu\psi_1A_1^\mu\overline{\psi}_2\gamma_\nu\psi_2A_2^\nu: \\ &\quad\quad + :\overline{\psi}_1\gamma_\mu\psi_1A_1^\mu\overline{\psi}_2\gamma_\nu\psi_2A_2^\nu: \\ &\quad + :\overline{\psi}_1\gamma_\mu\psi_1A_1^\mu\overline{\psi}_2\gamma_\nu\psi_2A_2^\nu: \\ &= :\overline{\psi}_1\gamma_\mu\psi_1A_1^\mu\overline{\psi}_2\gamma_\nu\psi_2A_2^\nu: \\ &\quad - S_{F\,\beta\alpha}(x_2-x_1):\left(\gamma_\mu\psi_1\right)_\alpha A_1^\mu\left(\overline{\psi}_2\gamma_\nu\right)_\beta A_2^\nu: \\ &\quad\quad + S_{F\,\alpha\beta}(x_1-x_2):\left(\overline{\psi}_1\gamma_\mu\right)_\alpha A_1^\mu\left(\gamma_\nu\psi_2\right)_\beta A_2^\nu: \\ &\quad\quad + D_F^{\mu\nu}(x_1-x_2):\overline{\psi}_1\gamma_\mu\psi_1\overline{\psi}_2\gamma_\nu\psi_2: \\ &\quad\quad - S_{F\,\beta\alpha}(x_2-x_1)\left(\gamma_\mu\right)_{\alpha\alpha'}S_{F\,\alpha'\beta'}(x_1-x_2)\left(\gamma_\nu\right)_{\beta'\beta}:A_1^\mu A_2^\nu: \\ &\quad\quad - S_{F\,\beta\alpha}(x_2-x_1)D_F^{\mu\nu}(x_1-x_2):\left(\gamma_\mu\psi_1\right)_\alpha\left(\overline{\psi}_2\gamma_\nu\right)_\beta: \\ &\quad\quad + S_{F\,\alpha\beta}(x_1-x_2)D_F^{\mu\nu}(x_1-x_2):\left(\overline{\psi}_1\gamma_\mu\right)_\alpha\left(\gamma_\nu\psi_2\right)_\beta: \\ &\quad\quad - S_{F\,\beta\alpha}(x_2-x_1)\left(\gamma_\mu\right)_{\alpha\alpha'}S_{F\,\alpha'\beta'}(x_1-x_2)\left(\gamma_\nu\right)_{\beta'\beta}D_F^{\mu\nu}(x_1-x_2), \end{aligned} \tag{3.31}$$

where no contractions were made between fields in the same interaction Hamiltonian as they are already normal ordered.

We will use this expansion in the next section, but let us remark, that if we take the amplitude of Eq. (3.31) between two vacuum states only the last term gives a non-zero result, that is

$$\begin{aligned}&\langle 0|T\big(\mathcal{H}_{\rm int}(x_1)\ \mathcal{H}_{\rm int}(x_2)\big)|0\rangle \\ &\qquad = (-1)(-e)^2\,{\rm Tr}\left[S_F(x_2-x_1)\gamma_\mu S_F(x_1-x_2)\gamma_\nu\right] D_F^{\mu\nu}(x_1-x_2)\,, \end{aligned} \tag{3.32}$$

that corresponds to the diagram of Fig. 3.1. We will discuss this type of diagrams

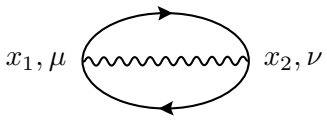


Figure 3.1: Bubble corresponding to Eq. (3.32).

in section 3.6. Notice also the minus sign connected to a closed loop of fermions. We will come back to this in the next section.

3.3 Feynman Diagrams in configuration space

Now we will calculate the S-matrix elements using the Dyson expansion[4]. We want to calculate the amplitudes

$$S_{fi} = \langle f|S|i\rangle\ , \tag{3.33}$$

for a given final and initial state using the expansion in Eq. (3.16), for the QED interaction

$$\mathcal{H}_{\rm int}(x) = -e:\overline{\psi}(x)\gamma_\mu\psi(x)A^\mu(x):\ . \tag{3.34}$$

The idea is to perform a few examples and then the complete set of rules will emerge clearly. To this end we realize that in order to obtain a non-zero result, a certain combination of creation and annihilation operators have to appear in S to annihilate or create the particles in $|i\rangle$ or $|f\rangle$. Obviously the $n = 0$ term in the S-matrix corresponds to no interaction. In QED, the $n = 1$ term will also not contribute to physical processes. This is because it could only contribute to processes like

$$e^- \to e^- + \gamma, \quad \gamma \to e^- + e^+,\ \cdots\,, \tag{3.35}$$

that are forbidden by conservation of four-momentum[5]. So the lowest order term contribution is the second order term. In the previous section we have already illustrated the use of Wick's theorem using precisely this term. Let us write separately

[4]For a derivation based on the Feynman approach see, for instance chapter 3 of Ref. [1].

[5]This only true if we consider the photon as a quantum field. If we consider scattering of an external classical field these processes are allowed. See section 3.5 for a discussion of the corresponding Feynman rules.

the various terms grouping them in a way that they will contribute to different physical processes. We have

$$S^{(2)} = \sum_{i=A}^{F} S_i^{(2)} \,, \tag{3.36}$$

where

$$S_A^{(2)} = -\frac{e^2}{2!}\int d^4x_1 d^4x_2 : \overline{\psi}_1\gamma_\mu\psi_1 A_1^\mu \overline{\psi}_2\gamma_\nu\psi_2 A_2^\nu : , \tag{3.37}$$

$$S_B^{(2)} = -\frac{e^2}{2!}\int d^4x_1 d^4x_2 \left[: \overline{\psi}_1\gamma_\mu\psi_1 A_1^\mu \overline{\psi}_2\gamma_\nu\psi_2 A_2^\nu : + : \overline{\psi}_1\gamma_\mu\psi_1 A_1^\mu \overline{\psi}_2\gamma_\nu\psi_2 A_2^\nu : \right] , \tag{3.38}$$

$$S_C^{(2)} = -\frac{e^2}{2!}\int d^4x_1 d^4x_2 : \overline{\psi}_1\gamma_\mu\psi_1 A_1^\mu \overline{\psi}_2\gamma_\nu\psi_2 A_2^\nu : , \tag{3.39}$$

$$S_D^{(2)} = -\frac{e^2}{2!}\int d^4x_1 d^4x_2 \left[: \overline{\psi}_1\gamma_\mu\psi_1 A_1^\mu \overline{\psi}_2\gamma_\nu\psi_2 A_2^\nu : + : \overline{\psi}_1\gamma_\mu\psi_1 A_1^\mu \overline{\psi}_2\gamma_\nu\psi_2 A_2^\nu : \right] , \tag{3.40}$$

$$S_E^{(2)} = -\frac{e^2}{2!}\int d^4x_1 d^4x_2 : \overline{\psi}_1\gamma_\mu\psi_1 A_1^\mu \overline{\psi}_2\gamma_\nu\psi_2 A_2^\nu : , \tag{3.41}$$

$$S_F^{(2)} = -\frac{e^2}{2!}\int d^4x_1 d^4x_2 : \overline{\psi}_1\gamma_\mu\psi_1 A_1^\mu \overline{\psi}_2\gamma_\nu\psi_2 A_2^\nu : , \tag{3.42}$$

Now let us discuss the various types of processes that correspond to these different terms

Processes in $S_A^{(2)}$

This term does not contribute to any physical process. The two vertices with coordinates x_1 and x_2 are not connected to each other and can only be connected to external particles like in $S^{(1)}$, being forbidden by energy-momentum conservation.

Processes in $S_B^{(2)}$

Let us look now at processes in $S_B^{(2)}$. There are two terms in Eq. (3.38) but they are not independent. In fact we can relabel $(x_1,\mu) \leftrightarrow (x_2,\nu)$ in the first term (just a change of integration variables) and then we get

$$: \overline{\psi}_2\gamma_\nu\psi_2 A_2^\nu \overline{\psi}_1\gamma_\mu\psi_1 A_1^\mu : = : \overline{\psi}_1\gamma_\mu\psi_1 A_1^\mu \overline{\psi}_2\gamma_\nu\psi_2 A_2^\nu : , \tag{3.43}$$

and therefore it is equal to the second term, cancelling the factor 1/2!. This is in general true, in QED the factors $1/n!$ will always cancel. In getting Eq. (3.43) there

was an even permutation of fermion fields and therefore no minus sign. So after this we get

$$S_B^{(2)} = -e^2 \int d^4x_1 d^4x_2 : \overline{\psi}_1 \gamma_\mu \underbrace{\psi_1 A_1^\mu \overline{\psi}_2}\gamma_\nu \psi_2 A_2^\nu : \,. \tag{3.44}$$

In Eq. (3.44) we have a fermion propagator connecting the internal vertices with coordinates x_1, x_2 and two uncontracted fermion and two uncontracted photon fields. The creation and annihilation operators in these fields have to annihilate the corresponding annihilation and creation operators in the initial and final state. Therefore this term contributes to processes with two fermions (electrons or positrons) and two photons. The possible processes (conserving electric charge) are

$$e^- + \gamma \to e^- + \gamma, \quad e^+ + \gamma \to e^+ + \gamma, \quad e^- + e^+ \to \gamma + \gamma, \quad \gamma + \gamma \to e^- + e^+ \,. \tag{3.45}$$

To see how this works let us look at the first process, $e^- + \gamma \to e^- + \gamma$, the Compton scattering. This process corresponds to select the positive frequency part $\psi^+(x_2)$ of $\psi(x_2)$ to annihilate the initial electron and the negative energy part $\overline{\psi}^-(x_1)$ of $\overline{\psi}(x_1)$ to create the final electron. But for the photons we have two possibilities, either $A_\mu^+(x_1)$ or $A_\mu^+(x_2)$ can absorb the initial photon, and $A_\mu^-(x_1)$ or $A_\mu^-(x_2)$ can emit the final photon. So we have two possibilities,

$$S^{(2)}(e^- + \gamma \to e^- + \gamma) = S_a^{(2)} + S_b^{(2)} \,, \tag{3.46}$$

where

$$S_a^{(2)} = -e^2 \int d^4x_1 d^4x_2 \; \overline{\psi}^-(x_1) \gamma^\mu S_F(x_1 - x_2) \gamma^\nu A_\mu^-(x_1) A_\nu^+(x_2) \psi^+(x_2) \,, \tag{3.47}$$

where we have written the contraction in terms of the Feynman propagator and have put the operators in normal order, so that we can take out the : symbol. For the other possibility we have

$$S_b^{(2)} = -e^2 \int d^4x_1 d^4x_2 \; \overline{\psi}^-(x_1) \gamma^\mu S_F(x_1 - x_2) \gamma^\nu A_\nu^-(x_2) A_\mu^+(x_1) \psi^+(x_2) \,. \tag{3.48}$$

These two possibilities correspond to the Feynman diagrams[6] of Fig. 3.2.

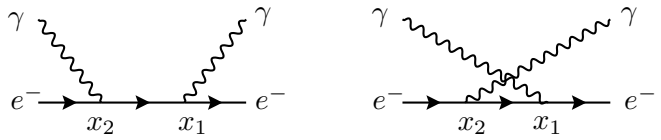


Figure 3.2: Diagrams for Compton scattering corresponding to Eq. (3.47) and Eq. (3.48).

The other processes in Eq. (3.45) could be obtained in the same way. In all of them there is a plus sign between the two diagrams as they only differ by the exchange of the photon field that is a boson. We leave to the next section how to proceed in finding the Feynman rules in momentum space.

[6]From now on, we will use the usual convention of drawing the diagrams with time flowing from left to right.

Processes in $S_C^{(2)}$

Now we look at $S_C^{(2)}$. As we can see from Eq. (3.39) there is one contracted photon line, that will give an internal photon propagator and four uncontracted fermion lines. So we can have all processes with two electrons or positrons in the initial and final state. The only requirement is to conserve charge. So the possible processes are

$$e^- + e^- \to e^- + e^-, \quad e^+ + e^+ \to e^+ + e^+, \quad e^- + e^+ \to e^- + e^+ \,. \tag{3.49}$$

To see how this works let us look at the first process in Eq. (3.49), $e^- + e^- \to e^- + e^-$, known as Møller scattering. The initial state is defined as

$$|i\rangle = b^\dagger(p_2) b^\dagger(p_1) \,|0\rangle \ . \tag{3.50}$$

Now the electron 1 with momentum p_1 can be absorbed either by $\psi^+(x_1)$ or $\psi^+(x_2)$. In the end both possibilities will give the same result after an interchange $x_1 \leftrightarrow x_2$ and will cancel the 1/2! factor. So we assume, for definiteness, that electron 1 is absorbed by $\psi^+(x_1)$ and that the 1/2! factor is already taken care of. We still have two possibilities, as the final electrons emitted by $\overline{\psi}^-(x_1)$ or $\overline{\psi}^-(x_2)$ in Eq. (3.39), can be connected to electrons 3 and 4 in two different ways leading to the Feynman diagrams in Fig. 3.3. These two diagrams have a relative minus sign. We will do the

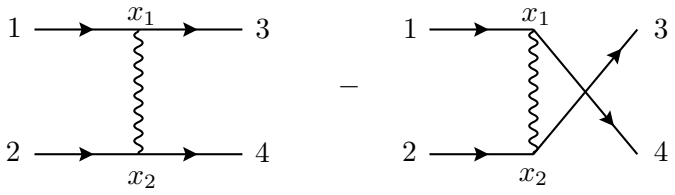


Figure 3.3: Feynman diagrams for Møller scattering.

full calculation in momenta space in the next section, but let us explain in simple terms the relative sign. The starting point is

$$X_{\mu\nu} = \langle 0| \, b(p_3) b(p_4) : \overline{\psi}_1 \gamma_\mu \psi_1 \overline{\psi}_2 \gamma_\nu \psi_2 : b^\dagger(p_2) b^\dagger(p_1) \,|0\rangle \ , \tag{3.51}$$

where we are not writing the photon contraction. Each of the fields can be separated in the positive and negative frequency parts,

$$\psi = \psi^+ + \psi^-, \quad \overline{\psi} = \overline{\psi}^+ + \overline{\psi}^- \,. \tag{3.52}$$

From Eqs. (2.66) and (2.67) we can see which particles each of these components absorb or emit. We collect this useful information in Table 3.1.

As we already have assigned that electron 1 is absorbed by $\psi^+(x_1)$, then electron 2 has to be absorbed by $\psi^+(x_2)$. Therefore we must have,

$$X_{\mu\nu} = \langle 0| \, b(p_3) b(p_4) : \overline{\psi}^-(x_1) \gamma_\mu \psi^+(x_1) \overline{\psi}^-(x_2) \gamma_\nu \psi^+(x_2) : b^\dagger(p_2) b^\dagger(p_1) \,|0\rangle$$

Field	Operator	Action
ψ^+	$b(p)$	Annihilates an electron
ψ^-	$d^\dagger(p)$	Creates a positron
$\overline{\psi}^+$	$d(p)$	Annihilates a positron
$\overline{\psi}^-$	$b^\dagger(p)$	Creates an electron

Table 3.1: Correspondence between positive and negative energies and annihilation and creation operators.

$$
\begin{aligned}
&= \langle 0|\, b_3 b_4 : \overline{\psi}^-(x_1)\gamma_\mu\psi^+(x_1)\overline{\psi}^-(x_2)\gamma_\nu\psi^+(x_2) : b_2^\dagger b_1^\dagger\, |0\rangle \\
&= \langle 0|\, b_3 b_4 (\overline{\psi}^-(x_1)\gamma_\mu)_\alpha(\overline{\psi}^-(x_2)\gamma_\nu)_\beta(\psi^+(x_2))_\beta(\psi^+(x_1))_\alpha b_2^\dagger b_1^\dagger\, |0\rangle\ , \qquad (3.53)
\end{aligned}
$$

where we have simplified the notation ($b_1 \equiv b(p_1), \ldots$), and removed the normal order symbol after moving all annihilation operators to the right and creation operators to the left. There is no sign, because we made an even number of permutations. Now, in the initial state we have to move $\psi^+(x_1)$ to annihilate $b_1^\dagger$. In the next section we will see the details, but here as we just want to count signs, we use the notation $\left[\psi^+(x_1)b_1^\dagger\right]$ for that operation. So we get, taking care of the signs for the anticommutation,

$$
X_{\mu\nu} = -\langle 0|\, b_3 b_4 (\overline{\psi}^-(x_1)\gamma_\mu)_\alpha(\overline{\psi}^-(x_2)\gamma_\nu)_\beta \left[(\psi^+(x_2))_\beta\, b_2^\dagger\right] \left[(\psi^+(x_1))_\alpha\, b_1^\dagger\right] |0\rangle\,, \quad (3.54)
$$

Now the creation operators in $\overline{\psi}^-(x_1)$ and $\overline{\psi}^-(x_2)$ have to hit the operators b_3 and b_4. This can be done in two different ways and taking care of the number of commutations we get,

$$
\begin{aligned}
X_{\mu\nu} =& \langle 0|\, \left[b_3(\overline{\psi}^-(x_1)\gamma_\mu)_\alpha\right] \left[b_4(\overline{\psi}^-(x_2)\gamma_\nu)_\beta\right] \left[(\psi^+(x_2))_\beta\, b_2^\dagger\right] \left[(\psi^+(x_1))_\alpha\, b_1^\dagger\right] |0\rangle \\
&- \langle 0|\, \left[b_3(\overline{\psi}^-(x_2)\gamma_\nu)_\beta\right] \left[b_4(\overline{\psi}^-(x_1)\gamma_\mu)_\alpha\right] \left[(\psi^+(x_2))_\beta\, b_2^\dagger\right] \left[(\psi^+(x_1))_\alpha\, b_1^\dagger\right] |0\rangle \\
\equiv& X^a_{\mu\nu} + X^b_{\mu\nu}\,, \qquad (3.55)
\end{aligned}
$$

corresponding, respectively, to the left and right diagrams in Fig. 3.3.

As another example with positrons, we consider the process $e^- + e^+ \to e^- + e^+$, known as Bhabha scattering. The starting point is now

$$
Y_{\mu\nu} = \langle 0|\, b_3 d_4 : \overline{\psi}_1\gamma_\mu\psi_1\overline{\psi}_2\gamma_\nu\psi_2 : d_2^\dagger b_1^\dagger\, |0\rangle\ , \qquad (3.56)
$$

where we consider that $1, 3$ are electrons and $2, 4$ positrons. Again to take care of the $1/2!$ we choose electron 1 to be connected to x_1. This means that we should

have $\psi^+(x_1)$. This also forces the creation of the final positron, particle 4, to be at x_2 through $\psi^-(x_2)$. We have therefore

$$Y_{\mu\nu} = \langle 0|\, b_3 d_4 : \overline{\psi}(x_1)\gamma_\mu \psi^+(x_1)\overline{\psi}(x_2)\gamma_\nu \psi^-(x_2) : d_2^\dagger b_1^\dagger \,|0\rangle \ . \tag{3.57}$$

Now for $\overline{\psi}(x_1)$ and $\overline{\psi}(x_2)$ we can have two possibilities, either $\overline{\psi}^+(x_1)\,\overline{\psi}^-(x_2)$, annihilation of a positron at x_1 and creation of an electron at x_2 or $\overline{\psi}^-(x_1)\,\overline{\psi}^+(x_2)$, creation of an electron at x_1 and annihilation of a positron at x_2, corresponding to the two diagrams of Fig. 3.4. We write therefore,

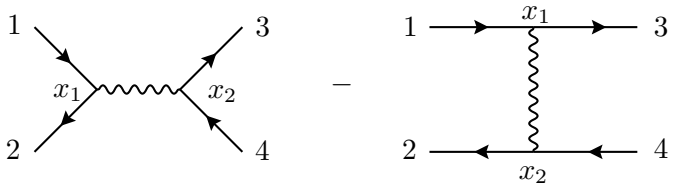


Figure 3.4: Feynman diagrams for Bhabha scattering.

$$Y_{\mu\nu} = Y^a_{\mu\nu} + Y^b_{\mu\nu}\,, \tag{3.58}$$

where

$$\begin{aligned}
Y^a_{\mu\nu} &= \langle 0|\, b_3 d_4 : \overline{\psi}^+(x_1)\gamma_\mu \psi^+(x_1)\overline{\psi}^-(x_2)\gamma_\nu \psi^-(x_2) : d_2^\dagger b_1^\dagger \,|0\rangle \\
&= \langle 0|\, b_3 d_4 \overline{\psi}^-(x_2)\gamma_\nu \psi^-(x_2)\overline{\psi}^+(x_1)\gamma_\mu \psi^+(x_1) d_2^\dagger b_1^\dagger \,|0\rangle \\
&= -\,\langle 0|\, b_3 d_4 \overline{\psi}^-(x_2)\gamma_\nu \psi^-(x_2)\left[(\overline{\psi}^+(x_1)\gamma_\mu)_\alpha\, d_2^\dagger\right]\left[\psi^+(x_1)_\alpha\, b_1^\dagger\right]|0\rangle \\
&= \langle 0|\left[b_3(\overline{\psi}^-(x_2)\gamma_\nu)_\beta\right]\left[d_4\psi^-(x_2)_\beta\,\right]\left[(\overline{\psi}^+(x_1)\gamma_\mu)_\alpha\, d_2^\dagger\right]\left[\psi^+(x_1)_\alpha\, b_1^\dagger\right]|0\rangle\,,
\end{aligned} \tag{3.59}$$

and

$$\begin{aligned}
Y^b_{\mu\nu} &= \langle 0|\, b_3 d_4 : \overline{\psi}^-(x_1)\gamma_\mu \psi^+(x_1)\overline{\psi}^+(x_2)\gamma_\nu \psi^-(x_2) : d_2^\dagger b_1^\dagger \,|0\rangle \\
&= \langle 0|\, b_3 d_4 (\overline{\psi}^-(x_1)\gamma_\mu)_\alpha\, \psi^-(x_2)_\beta\, \psi^+(x_1)_\alpha\, (\overline{\psi}^+(x_2)\gamma_\nu)_\beta\, d_2^\dagger b_1^\dagger \,|0\rangle \\
&= \langle 0|\, b_3 d_4 (\overline{\psi}^-(x_1)\gamma_\mu)_\alpha\, \psi^-(x_2)_\beta\, \left[(\overline{\psi}^+(x_2)\gamma_\nu)_\beta\, d_2^\dagger\right]\left[\psi^+(x_1)_\alpha\, b_1^\dagger\right]|0\rangle \\
&= -\,\langle 0|\left[b_3(\overline{\psi}^-(x_1)\gamma_\mu)_\alpha\right]\left[d_4\,\psi^-(x_2)_\beta\,\right]\left[(\overline{\psi}^+(x_2)\gamma_\nu)_\beta\, d_2^\dagger\right]\left[\psi^+(x_1)_\alpha\, b_1^\dagger\right]|0\rangle\,,
\end{aligned} \tag{3.60}$$

showing the relative minus sign between the two diagrams. The remaining processes in Table 3.2 can be worked along similar lines.

Processes in $S_D^{(2)}$

In $S_D^{(2)}$, Eq. (3.40) again we have two terms that are symmetric under the interchange $x_1 \leftrightarrow x_2$, taking care again of the 1/2! factor. This diagram corresponds to an electron being annihilated at x_1 and an electron being created at x_2 (we consider, as before this order), with an electron and photon being exchanged. After this choice we have,

$$S_D^{(2)} = -e^2 \int d^4x_1 d^4x_2 : \overline{\psi}_1 \gamma_\mu \psi_1 A_1^\mu \overline{\psi}_2 \gamma_\nu \psi_2 A_2^\nu :$$

$$= -e^2 \int d^4x_1 d^4x_2 \overline{\psi}^-(x_2) \gamma_\nu S_F(x_2 - x_1) \gamma_\mu \psi^+(x_1) D_F^{\mu\nu}(x_2 - x_1)\,, \qquad (3.61)$$

corresponding to the Feynman Diagram in Fig. 3.5. This diagram conducts to a

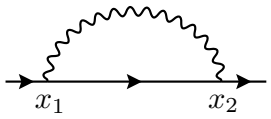


Figure 3.5: Feynman diagram for the electron self-energy.

divergent integral that will be studied in chapter 9.

Processes in $S_E^{(2)}$

Consider now the term $S_E^{(2)}$. We have two fermion propagators and two photon in the external lines. When connected to real external photons these can be connected in two ways that again compensates for the 1/2! factor. We choose the photon to be absorbed at x_1. Then the element of the S-matrix reads

$$S_E^{(2)} = -e^2 \int d^4x_1 d^4x_2 \overline{\psi}_1 \gamma_\mu \psi_1 \overline{\psi}_2 \gamma_\nu \psi_2 A_2^{-\,\nu} A_1^{+\,\mu}$$

$$= -e^2 \int d^4x_1 d^4x_2 (-1) \mathrm{Tr}\left[S_F(x_2 - x_1)\gamma^\mu S_F(x_1 - x_2)\gamma^\nu\right] A_\nu^-(x_2) A_\mu^+(x_1), \quad (3.62)$$

corresponding to the diagram of Fig. 3.6. The important point to notice here is

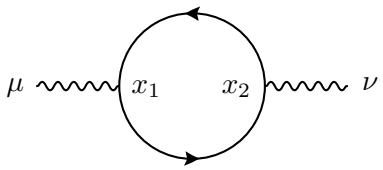


Figure 3.6: Feynman diagram for the vacuum polarization.

the extra minus sign due the anticommutation of fermionic fields. This happens to all closed fermion loops. This diagram is also divergent and will be discussed in chapter 9.

Processes in $S_F^{(2)}$

Finally, the processes in $S_F^{(2)}$ were already discussed in Eq. (3.32) and in Fig. 3.1. They correspond to the so-called vacuum-vacuum amplitudes or *bubbles*. We will discuss them in section 3.6.

Term in $S^{(2)}$	QED Process
$S_A^{(2)}$	No physical process
$S_B^{(2)}$	$e^- + \gamma \to e^-\gamma, e^+ + \gamma \to e^+\gamma, e^- + e^+ \to \gamma + \gamma, \gamma + \gamma \to e^- + e^+$
$S_C^{(2)}$	$e^- + e^- \to e^- + e^-, e^+ + e^+ \to e^+ + e^+, e^- + e^+ \to e^- + e^+$
$S_D^{(2)}$	$e^- \to e^-$ (1 loop) Electron Self-Energy
$S_E^{(2)}$	$\gamma \to \gamma$ (1 loop) Vacuum-Polarization
$S_F^{(2)}$	Vacuum-Vacuum Amplitude (Bubbles)

Table 3.2: QED processes contained in $S^{(2)}$, Eq. (3.36).

3.4 Feynman Diagrams in momentum space

In the last section we discussed the QED processes contained in the S-matrix at second order. We were able to see which processes corresponded to the different contractions and discussed the relative signs given by the Wick's theorem. As a final step in showing that we get the same Feynman rules by this second quantized approach as we did before, we must go to momentum space. We will leave the actual calculations of the QED processes to chapter 5, but will perform the calculation to the point where we can read the invariant amplitude $\mathcal{M}$ for each process and from there abstract the Feynman rules. We will do that just for a few processes, but before we have to be more specific about the normalization of states.

3.4.1 Normalizations and definitions

The subject of normalization of the states always brings some confusion, as many different possibilities appear in the literature. To explain our conventions, let us start by indicating the places where the different choices usually appear. Let us define,

$$\left[a(\vec{p}), a^\dagger(\vec{q})\right] \equiv N_{aa^\dagger}\delta^3(\vec{p} - \vec{q})\,, \tag{3.63}$$

$$\phi(x) \equiv \int \frac{d^3p}{N_\phi}\left[a(\vec{p})e^{-ip\cdot x} + a^\dagger(\vec{p})e^{ip\cdot x}\right]\,, \tag{3.64}$$

$$|\vec{p}\rangle \equiv N_p a^\dagger(\vec{p})\,|0\rangle\ , \tag{3.65}$$

$$S_{fi} \equiv \delta_{fi} + \left[(2\pi)^4\delta^4\left(\sum_i p_i - \sum_f p_f\right)\right] N_M\, i\mathcal{M}\,. \tag{3.66}$$

Using Eq. (3.63) and Eq. (3.65), we get by obviously

$$\langle\vec{p}|\vec{q}\rangle = N_p^2 N_{aa^\dagger}\delta^3(\vec{p}-\vec{q}) \tag{3.67}$$

Now different authors have different choices for these normalization factors. In Table 3.3 we collect the conventions of a few standard textbooks. So, with our

Author	$N_{aa^\dagger}$	N_ϕ	N_p	N_M
This Text	$(2\pi)^3 2p^0$	$(2\pi)^3 2p^0$	1	1
Peskin&Schroeder	$(2\pi)^3$	$(2\pi)^3\sqrt{2p^0}$	$\sqrt{2p^0}$	1
Schwartz	$(2\pi)^3$	$(2\pi)^3\sqrt{2p^0}$	$\sqrt{2p^0}$	1
Maggiori	$(2\pi)^3$	$(2\pi)^3\sqrt{2p^0}$	$\sqrt{2p^0}$	1
Tong	$(2\pi)^3$	$(2\pi)^3\sqrt{2p^0}$	$\sqrt{2p^0}$	1
Pokorski	$(2\pi)^3 2p^0$	$(2\pi)^3 2p^0$	1	1
Lancaster&Blundell	1	$\sqrt{(2\pi)^3 2p^0}$	$\sqrt{(2\pi)^3 2p^0}$	1

Table 3.3: Normalization constants for different textbooks.

conventions, a state of momentum p (we omit the arrow) is given by,

$$|p\rangle = a^\dagger(p)\,|0\rangle \tag{3.68}$$

with the normalization

$$\langle\vec{p}|\vec{q}\rangle = (2\pi)^3\,2p^0\,\delta^3(\vec{p}-\vec{q})\,. \tag{3.69}$$

We will use this normalization both for fermions as well as for bosons. [7]. Now we consider the effect of positive energy part of the quantum field, $\psi^+(x)$, applied to a state of momentum p. We have,

$$\begin{aligned}
\psi^+(x)\,|p\rangle &= \psi^+(x)b^\dagger(p,s)\,|0\rangle \\
&= \int \widetilde{dp'}\ \sum_s \left[b(p',s')u(p',s)e^{-ip'\cdot x}\right] b^\dagger(p,s)|\,|0\rangle \\
&= \int \frac{d^3p'}{(2\pi)^3 2p'^0}(2\pi)^3 2p^0\delta^3(\vec{p}'-\vec{p})\delta_{s,s'}u(p',s')e^{-ip'\cdot x}\,|0\rangle \\
&= |0\rangle\,u(p,s)e^{-ip\cdot x}\,,
\end{aligned} \tag{3.70}$$

[7] We did not include in our table older books, like Bjorken& Drell, Itzykson& Zuber and Mandl& Shaw that have different normalization for bosons and fermions.

where we have used Eq. (2.68). For future use we collect here the results for all cases. For the initial states,

$$\psi^+(x)\,|p\rangle = \psi^+(x)b^\dagger(p)\,|0\rangle = |0\rangle\,u(p)e^{-ip\cdot x}\,, \tag{3.71}$$

$$\overline{\psi}^+(x)\,|p\rangle = \overline{\psi}^+(x)d^\dagger(p)\,|0\rangle = |0\rangle\,\overline{v}(p)e^{-ip\cdot x}\,, \tag{3.72}$$

$$A^+_\mu(x)\,|k\rangle = A^+_\mu(x)a^\dagger(k)\,|0\rangle = |0\rangle\,\epsilon_\mu(k)e^{-ik\cdot x}\,, \tag{3.73}$$

and for the final sates,

$$\langle p|\,\overline{\psi}^-(x) = \langle 0|\,b(p)\overline{\psi}^-(x) = \overline{u}(p)\,e^{ip\cdot x}\,\langle 0|\,, \tag{3.74}$$

$$\langle p|\,\psi^-(x) = \langle 0|\,d(p)\psi^-(x) = v(p)e^{ip\cdot x}\,\langle 0|\,, \tag{3.75}$$

$$\langle k|\,A^-_\mu(x) = \langle 0|\,a(k)A^-_\mu(x) = \epsilon^*_\mu(k)\,e^{ik\cdot x}\,\langle 0|\,. \tag{3.76}$$

The other point that it is necessary to be precise, is the relation between the S-matrix element and the invariant amplitude $\mathcal{M}$. Here we will define,

$$S_{fi} \equiv \delta_{fi} + \left[(2\pi)^4\delta^4\left(\sum_i p_i - \sum_f p_f\right)\right] i\mathcal{M}\,, \tag{3.77}$$

where we incorporated an extra i in the definition of $\mathcal{M}$. Note that this convention is not universal in the literature, in some books there is a minus sign. However, that this will not change any physical result, as we will always need $|\mathcal{M}|$. To get $i\,\mathcal{M}$ we have to factor out the expression in square brackets. The definition of $|\mathcal{M}|$ in Eq. (3.77) corresponds to extract from the S-matrix all the kinematical factors. We will see that this is very useful when we will look at the cross sections in section 4.2.

With the previous definitions we are now in position of evaluating the invariant amplitude for various processes. We just calculate a few of them to be able to enunciate the Feynman rules for QED.

3.4.2 Compton scattering

To learn how to handle electrons and photons in the initial and final state, we consider as a first process the Compton scattering,

$$e^-(p) + \gamma(k) \to e^-(p') + \gamma(k')\,. \tag{3.78}$$

The S-matrix element is obtained from Eqs. (3.47) and (3.48). We use the convention for the momenta defined in Eq. (3.78). We obtain

$$S^a_{fi} = -\,e^2 \int d^4x_1 d^4x_2\,\langle 0|\,a(k')b(p')\overline{\psi}^-(x_1)\gamma^\mu S_F(x_1-x_2)\gamma^\nu A^-_\mu(x_1)A^+_\nu(x_2)\psi^+(x_2)$$

$$b^\dagger(p)a^\dagger(k)\,|0\rangle$$

$$
\begin{aligned}
&= - e^2 \int d^4x_1 d^4x_2 \, \langle 0| \, a(k') A^-_\mu(x_1) b(p') \overline{\psi}^-(x_1) \gamma^\mu S_F(x_1 - x_2) \gamma^\nu \psi^+(x_2) b^\dagger(p) \\
&\qquad\qquad A^+_\nu(x_2) a^\dagger(k) \, |0\rangle \\
&= - e^2 \int \frac{d^4q}{(2\pi)^4} \overline{u}(p') \gamma^\mu S_F(q) \gamma^\nu u(p) \epsilon^*_\mu \epsilon_\nu \\
&\qquad\qquad \int d^4x_1 d^4x_2 e^{-i(p\cdot x_2 + k\cdot x_2)} e^{-iq\cdot(x_1 - x_2)} e^{i(k'\cdot x_1 + p'\cdot x_1)} \, ,
\end{aligned} \tag{3.79}
$$

where we have used Eq. (2.80) to express the Feynman propagator in momentum space and Eqs. (3.71) and (3.74) to evaluate the amplitude. We are left with calculating the integral

$$
\begin{aligned}
&\int d^4x_1 d^4x_2 e^{-i(p\cdot x_2 + k\cdot x_2)} e^{-iq\cdot(x_1 - x_2)} e^{i(k'\cdot x_1 + p'\cdot x_1)} \\
&\qquad = \int d^4x_1 e^{ix_1\cdot(-q + k' + p')} \int d^4x_2 e^{-ix_2\cdot(p + k - q)} \\
&\qquad = (2\pi)^4 \delta^4(q - p - k)(2\pi)^4 \delta^4(q - p' - k') \, .
\end{aligned} \tag{3.80}
$$

Inserting in Eq. (3.79) we get

$$
\begin{aligned}
S^a_{fi} &= -e^2 \overline{u}(p') \gamma^\mu S_F(p + k) \gamma^\nu u(p) \epsilon^*_\mu \epsilon_\nu (2\pi)^4 \delta^4(p + k - p' - k') \\
&= \Big[(2\pi)^4 \delta^4(p + k - p' - k') \Big] \, \overline{u}(p') (ie\gamma^\mu) S_F(p + k) (ie\gamma^\nu) u(p) \epsilon^*_\mu \epsilon_\nu \, ,
\end{aligned} \tag{3.81}
$$

which gives

$$
i \, \mathcal{M}^a = \overline{u}(p') (ie\gamma^\nu) S_F(p + k) (ie\gamma^\mu) u(p) \epsilon^*_\nu(k') \epsilon_\mu(k) \, . \tag{3.82}
$$

In a similar way

$$
\begin{aligned}
S^b_{fi} &= - e^2 \int d^4x_1 d^4x_2 \;\; \langle 0| \, a(k') b(p') \overline{\psi}^-(x_1) \gamma^\mu S_F(x_1 - x_2) \gamma^\nu A^-_\nu(x_2) A^+_\mu(x_1) \psi^+(x_2) \\
&\qquad\qquad b^\dagger(p) a^\dagger(k) \, |0\rangle \\
&= - e^2 \int d^4x_1 d^4x_2 \;\; \langle 0| \, a(k') A^-_\nu(x_2) b(p') \overline{\psi}^-(x_1) \gamma^\mu S_F(x_1 - x_2) \gamma^\nu \psi^+(x_2) b^\dagger(p) \\
&\qquad\qquad A^+_\mu(x_1) a^\dagger(k) \, |0\rangle \\
&= - e^2 \overline{u}(p') \gamma^\mu S_F(p - k') \gamma^\nu u(p) \epsilon^*_\mu \epsilon_\nu (2\pi)^4 \delta^4(p + k - p' - k') \\
&= \Big[(2\pi)^4 \delta^4(p + k - p' - k') \Big] \, \overline{u}(p') (ie\gamma^\mu) S_F(p - k') (ie\gamma^\nu) u(p) \epsilon_\mu \epsilon^*_\nu \, ,
\end{aligned} \tag{3.83}
$$

which gives

$$
i \, \mathcal{M}^b = \overline{u}(p') (ie\gamma^\mu) S_F(p - k') (ie\gamma^\nu) u(p) \epsilon_\mu(k) \epsilon^*_\nu(k') \, . \tag{3.84}
$$

Now that we have the expressions for the amplitudes, Eq. (3.82) and Eq. (3.84) we want to learn how to extract the so-called Feynman rules for this process. These rules allow us to look at a diagram in momentum space and write the corresponding amplitude without all the previous calculations. We begin by showing the momentum space version of Fig. 3.2, which we show in Fig. 3.7,

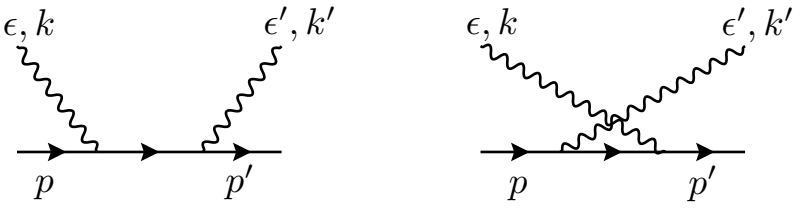


Figure 3.7: Diagrams for Compton scattering in momentum space.

Let us look first at the electron leaving the diagram. We have a spinor $\overline{u}(p')$[8]. This spinor has an index in Dirac space that should be contracted so that the final result is a 1×1 matrix in Dirac space. If we look at Eq. (3.82) we see that this corresponds to the fermion line

$$\overline{u}(p')(ie\gamma^{\nu})S_F(p+k)(ie\gamma^{\mu})u(p) \tag{3.85}$$

From here we immediately see that we should write this line starting from the tip of the arrow. In succession, we have $\overline{u}(p')$ for the outgoing electron, then a factor $(ie\gamma^{\nu})$ for the interaction (vertex) with the outgoing photon of momentum k'^{ν}, then an electron propagator $S_F(p+k)$ for the electron internal line, followed by another interaction $(ie\gamma^{\mu})$ with the initial photon of momentum k^{μ}. The line is then closed by a spinor $u(p)$ for the initial electron, resulting in a 1×1 matrix, in other words a complex number. There is momentum conservation at each vertex (the electron propagator has momentum $p+k$). Finally we should multiply by the polarization vectors $\epsilon^{\mu}(k)$ for the initial photon and $\epsilon^{\nu}(k')$ for the photon in the final state. Now if we apply these rules to the second diagram of Fig. 3.7 we immediately get the corresponding amplitude, Eq. (3.84). Notice that the momentum of the electron propagator is now $p-k'$ by momentum conservation.

It should be stressed that the understanding, by Feynman, that everything could be done with a small set of rules was an enormous breakthrough, just look at all the calculations that lead to Eq. (3.82) and Eq. (3.84).

3.4.3 Bhabha scattering

To have an example of positrons in external lines, we choose Bhabha scattering,

$$e^-(p_1) + e^+(p_2) \to e^-(p_3) + e^+(p_4)\,. \tag{3.86}$$

[8]For simplicity we are omitting the spin labels.

Again we have the two diagrams of Fig. 3.4 with a relative minus sign. From Eq. (3.59) we get

$$\begin{aligned}
S^a_{fi} =& -e^2 \int d^4x_1 d^4x_2 \, \langle 0| \left[b(p_3)(\overline{\psi}^-(x_2)\gamma_\nu)_\beta \right] \left[d(p_4)\psi^-(x_2)_\beta \right] \\
& \qquad \left[(\overline{\psi}^+(x_1)\gamma_\mu)_\alpha \, d^\dagger(p_2) \right] \left[\psi^+(x_1)_\alpha \, b^\dagger(p_1) \right] |0\rangle \, D_F^{\mu\nu}(x_1 - x_2) \\
=& -e^2 \overline{u}(p_3)\gamma_\nu v(p_4) \int \frac{d^4q}{(2\pi)^4} D_F^{\mu\nu}(q) \overline{v}(p_2)\gamma_\mu u(p_1) \\
& \qquad \int d^4x_1 d^4x_2 e^{-ix_1\cdot(p_1+p_2+q)} e^{ix_2\cdot(p_3+p_4+q)} \\
=& \left[(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \right] \\
& \qquad \overline{u}(p_3)(ie\gamma_\nu)v(p_4) D_F^{\mu\nu}(p_1+p_2) \overline{v}(p_2)(ie\gamma_\mu)u(p_1) \,, \qquad (3.87)
\end{aligned}$$

and therefore

$$i\,\mathcal{M}^a = \overline{u}(p_3)(ie\gamma_\nu)v(p_4) D_F^{\mu\nu}(p_1+p_2) \overline{v}(p_2)(ie\gamma_\mu)u(p_1) \,. \qquad (3.88)$$

In a similar way

$$\begin{aligned}
S^b_{fi} =& -e^2(-1) \int d^4x_1 d^4x_2 \, \langle 0| \left[b(p_3)(\overline{\psi}^-(x_1)\gamma_\mu)_\alpha \right] \left[d(p_4)\, \psi^-(x_2)_\beta \right] \\
& \qquad \left[(\overline{\psi}^+(x_2)\gamma_\nu)_\beta \, d^\dagger(p_2) \right] \left[\psi^+(x_1)_\alpha \, b^\dagger(p_1) \right] |0\rangle \, D_F^{\mu\nu}(x_1 - x_2) \\
=& \left[(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \right] \\
& \qquad (-1)\overline{u}(p_3)(ie\gamma_\mu)u(p_1) D_F^{\mu\nu}(p_3 - p_1) \overline{v}(p_2)(ie\gamma_\nu)v(p_4) \,, \qquad (3.89)
\end{aligned}$$

and

$$i\,\mathcal{M}^b = (-1)\overline{u}(p_3)(ie\gamma_\mu)u(p_1) D_F^{\mu\nu}(p_3 - p_1) \overline{v}(p_2)(ie\gamma_\nu)v(p_4) \,. \qquad (3.90)$$

To read the Feynman rules from Eq. (3.88) and Eq. (3.90), we start by showing the momentum space diagrams for Bhabha scattering in Fig. 3.8,

We verify that we obtain the same rule for the vertex, and for the initial electron. However there are two new rules. The first one has to do with the positrons. Their line is backwards in time, but like before, its fermion line always starts at the tip of the arrow. We have a spinor $\overline{v}(p_2)$ for the initial positron and a spinor $v(p_4)$ for the final positron. The second rule is that there is a minus sign between the diagrams. This minus sign comes form Wick's theorem and there is no easy way to spot it, unless you go back to the use of Wick's theorem. Luckily, nowadays there exist software that can automatically give the correct sign for you as we will see in chapter 7.

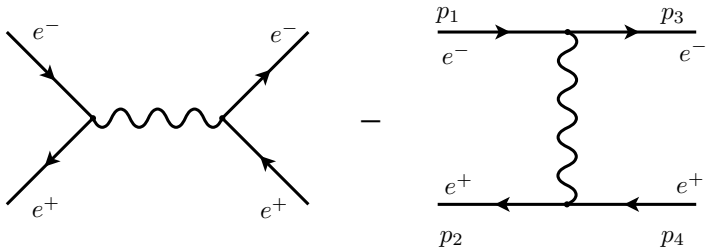


Figure 3.8: Difusão Bhabha

One can easily verify that all the processes in $S_B^{(2)}$ and $S_C^{(2)}$ are obtained using the same Feynman rules. We leave for the next chapters the evaluation of the cross section for these processes.

3.4.4 Closed loops

Before we close the section let us see an example of the rule for closed loops. Let us consider the vacuum polarization as an example. We start by the term $S_E^{(2)}$ in the S-matrix expansion, given in Eq. (3.62) and the contribution to the photon propagator is

$$\begin{aligned}
S_{fi} =& -e^2 \int d^4x_1 d^4x_2 (-1) \mathrm{Tr}\left[S_F(x_2-x_1)\gamma_\mu S_F(x_1-x_2)\gamma_\nu\right] \\
&\qquad \langle 0|\, a(k') A^{-\nu}(x_2) A^{+\mu}(x_1) a^\dagger(k)\, |0\rangle \\
=& -e^2(-1)\epsilon^\mu(k)\epsilon^{*\nu}(k') \int \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4} \mathrm{Tr}\left[S_F(p')\gamma_\mu S_F(p)\gamma_\nu\right] \\
&\qquad \int d^4x_1 d^4x_2 e^{-ip'\cdot(x_2-x_1)} e^{-ip\cdot(x_1-x_2)} e^{-ik\cdot x_1} e^{ik'\cdot x_2} \\
=& -e^2(-1)\epsilon^\mu(k)\epsilon^{*\nu}(k') \int \frac{d^4p}{(2\pi)^4}\frac{d^4p'}{(2\pi)^4} \mathrm{Tr}\left[S_F(p')\gamma_\mu S_F(p)\gamma_\nu\right] \\
&\qquad (2\pi)^4\delta^4(p'-p-k)(2\pi)^4\delta^4(p-p'+k') \\
=& \left[(2\pi)^4\delta^4(k-k')\right](-1)\epsilon^\mu(k)\epsilon^{*\nu}(k') \\
&\qquad \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[S_F(p+k)(ie\gamma_\mu)S_F(p)(ie\gamma_\nu)\right] , \qquad (3.91)
\end{aligned}$$

and therefore

$$i\mathcal{M} = (-1)\epsilon^\mu(k)\epsilon^{*\nu}(k) \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[S_F(p+k)(ie\gamma_\mu)S_F(p)(ie\gamma_\nu)\right] , \qquad (3.92)$$

In momentum space this corresponds to the diagram of Fig. 3.9.

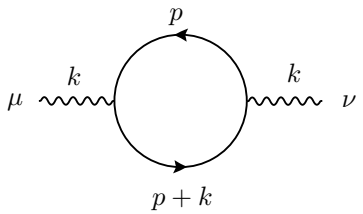


Figure 3.9: Vacuum Polarization.

This example gives us two extra Feynman rules. The first one tells us that when we have a closed loop, we have to choose an arbitrary momentum, say p, for one of the internal lines and multiply (integrate) by the factor

$$\int \frac{d^4p}{(2\pi)^4} \tag{3.93}$$

using energy momentum conservation at each vertex of the diagram. The second rule comes from Wick's theorem, there is a global minus sign for every closed loop of fermions. We will study this process in chapter 9.

3.4.5 Feynman Rules for QED

We have done enough examples to obtain the Feynman Rules for QED. These are obtained from the Dyson expansion of the S-matrix using second quantized fields. We emphasize that Wick's theorem is crucial in giving the correct signs for all cases. Also, it should be noted that second quantization is indeed necessary to understand the creation and annihilation of particles.

The Feynman rules to evaluate processes in this theory are the following:

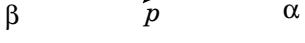
1. For a given process, draw all topologically distinct diagrams.

2. For each electron entering a diagram a factor $u(p,s)$. If it leaves the diagram a factor $\overline{u}(p,s)$.

3. For each positron leaving the diagram (final state) a factor $v(p,s)$. It it enters the diagram (initial state) then we have a factor $\overline{v}(p,s)$.

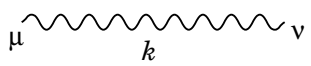
4. For each photon in the initial state we have the vector $\varepsilon^{\mu}(k)$ and in the final state $\varepsilon^{*\mu}(k)$.

5. For each internal fermionic line the propagator



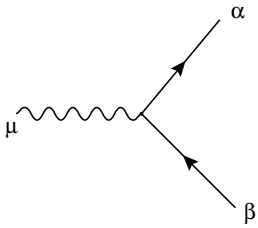
$$S_{F_{\alpha\beta}}(p) = i\frac{(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\varepsilon} \tag{3.94}$$

6. For each virtual photon the propagator (Feynman gauge, $\xi = 1$)



$$D_{F\mu\nu}(k) = -i\,\frac{g_{\mu\nu}}{k^2} \tag{3.95}$$

7. For each vertex the factor



$$ie\gamma^\mu \tag{3.96}$$

8. For each internal momentum, not fixed by conservation of momenta, as in the case of loops, a factor

$$\int \frac{d^4q}{(2\pi)^4} \tag{3.97}$$

9. For each loop of fermions, take the trace and multiply by -1.

10. A factor of -1 between diagrams that differ by exchange of fermionic lines. In doubt, revert to Wick's theorem.

11. In `QED` there are no symmetry factors, that is, they are always equal to 1.

12. The result of applying these rules gives $i\,\mathcal{M}$.

3.5 Scattering by an external field

In section 3.3 we said that there were no physical processes originating from the $n = 1$ term in the Dyson expansion. This was because the photon is massless and the processes were forbidden by energy-momentum conservation.

There is however an important situation where we have only conservation of energy, the elastic scattering off an heavy nucleus, where we assume that there is no recoil. Let us derive the Feynman rules for this case.

We start from the expression for $S^{(1)}$,

$$S^{(1)} = -\,i \int d^4x\, \langle f|\, (-e) : \overline{\psi}(x)\gamma^\mu\psi(x)A^c_\mu(x) : |i\rangle$$

$$=ie\int d^4x A^c_\mu(x)\,\langle f|:\overline{\psi}(x)\gamma^\mu\psi(x):|i\rangle \tag{3.98}$$

where the second line follows from the fact that $A^c_\mu(x)$ is a classical external field, that we define trough its Fourier transform

$$A^c_\mu(x)=A^c_\mu(\vec{x})\equiv\int\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{x}}A^c_\mu(\vec{q}) \tag{3.99}$$

To proceed we take the initial state to be an electron of momentum p and the final state an electron of momentum p' (spin indices omitted for simplicity), that is

$$|i\rangle=|p\rangle\,,\quad |f\rangle=|p'\rangle \tag{3.100}$$

which would correspond to the scattering of an electron from the electromagnetic field of a massive nucleus, the so-called Coulomb scattering. We have then

$$\begin{aligned}
S^{(1)}=&ie\int d^4x A^c_\mu(\vec{x})\,\langle p'|:\overline{\psi}(x)\gamma^\mu\psi(x):|p\rangle\\
=&ie\int d^4x A^c_\mu(\vec{x})\,\langle p'|\,\overline{\psi}^-(x)\gamma^\mu\psi^+(x)\,|p\rangle\\
=&ie\int d^4x\, A^c_\mu(\vec{x})\overline{u}(p')\gamma^\mu u(p)e^{i(p'-p)\cdot x}
\end{aligned} \tag{3.101}$$

where we have used Eq. (3.71) and Eq. (3.74). Now we introduce $A^c_\mu(\vec{x})$ through its Fourier transform in Eq. (3.99) to obtain

$$\begin{aligned}
S^{(1)}=&ie\,\overline{u}(p')\gamma^\mu u(p)\int dx^0 e^{i(E'-E)x^0}\int d^3x\int\frac{d^3q}{(2\pi)^3}A^c_\mu(\vec{q})e^{i(\vec{q}-\vec{p}'+\vec{p})\cdot\vec{x}}\\
=&(2\pi)\delta(E'-E)\,ie\,\overline{u}(p')\gamma^\mu u(p)\int d^3q\, A^c_\mu(\vec{q})\,\delta^3(\vec{q}-\vec{p}'+\vec{p})\\
=&(2\pi)\delta(E'-E)\,ie\,\overline{u}(p')\gamma^\mu u(p)A^c_\mu(\vec{q}=\vec{p}'-\vec{p})\\
\equiv&\Big[(2\pi)\delta(E'-E)\Big]\,i\mathcal{M}
\end{aligned} \tag{3.102}$$

where we have defined

$$i\mathcal{M}=\overline{u}(p')(ie\gamma^\mu)u(p)A^c_\mu(\vec{q}) \tag{3.103}$$

with the momentum transfer given by

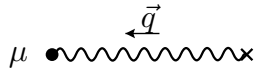
$$\vec{q}=\vec{p}'-\vec{p}. \tag{3.104}$$

So, to account for these cases we have to modify the Feynman rules with two additional rules,

13. In Eq. (3.77) relating the S-matrix element with the invariant amplitude, $\mathcal{M}$, we have to make the substitution

$$(2\pi)^4\delta^4(\sum_f p_f - \sum_i p_i) \to (2\pi)\delta(E' - E) \tag{3.105}$$

14. For each interaction of a charged particle with the external field $A^c_\mu(\vec{x})$ we write a factor



$$A^c_\mu(\vec{q}) = \int d^3x\, e^{-i\vec{q}\cdot\vec{x}} A^c_\mu(\vec{x}) \tag{3.106}$$

where $\vec{q}$ is the transfer momentum and the cross marks the external source. Energy, but not momentum, is conserved at the vertex. For the case of Coulomb scattering we have,

$$A^0_c(\vec{x}) = \frac{Ze}{|4\pi\vec{x}|}, \quad \vec{A}_c(\vec{x}) = 0, \tag{3.107}$$

and taking the Fourier transform (see Complement 3.1) we get,

$$A^0_c(\vec{q}) = \frac{Ze}{|\vec{q}|^2}\,. \tag{3.108}$$

3.6 Some points we swept under the rug

In this section we will address some questions that we avoided discussing so far. The reason for that is that the final result for the Feynman rules is correct despite the points that we will see below. Therefore we can be less technical and proceed with the calculations.

3.6.1 Initial state being a free particle

The first point is our definition of the initial and final states. We consider them as free particles. However, we know that this is not true because quantum fluctuations will change these states, even if initially they were free particles.

To address correctly this question it is necessary to use the formalism of Lehmann, Symanzik e Zimmermann (known as the LSZ formalism [31]) for *in* and *out* states. This is explained in many books in QFT, for instance in my text [12] but it is beyond the level of this introductory course and we will not go into this any further, except in connection with the next question.

3.6.2 What happens to the bubble diagrams?

We saw that in the Dyson expansion of the S-matrix there were terms fully contracted. These terms cannot connect to external particles and are known as vacuum-vacuum amplitudes and sometimes as *bubbles*, for obvious reasons, see Fig. 3.1. They can appear just like in Fig. 3.1, but also in higher order processes like those of Fig. 3.10. These diagrams are called disconnected, because two parts of the diagram

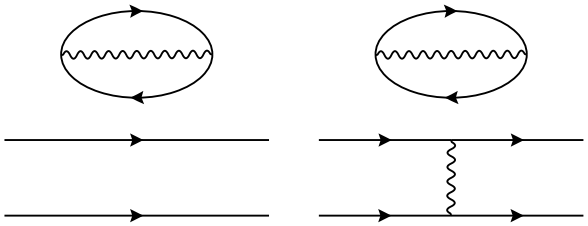


Figure 3.10: Disconnected diagrams.

are not connected by any interaction. What should we do with these disconnected diagrams? It turns out that the proper definition of asymptotic states in the LSZ formalism also solves this problem. In fact the corrections to the vacuum exactly cancel these contributions from the S-matrix. So we can safely consider only connected diagrams as we have been discussing up to now. For the details of this procedure see my text, Ref. [12].

3.6.3 And what about interactions with derivatives?

There is a final point in this discussion. Remember that when we discussed the IP, we said that we were considering interactions without derivatives, like in QED. The reason for this was that in this way we could be sure that the conjugate momentum would be the same for the free and interacting fields and we could use the free field expansion and (anti)commutation relations.

It turns out that this is just a technical complication and that the Feynman rules can be worked out giving similar results. This is not much discussed in the literature, but you find a discussion in the books of Itzykson and Zuber [32] and of Weinberg [33]. The interactions with derivatives are important in the standard model, because they are present in the non-abelian gauge interactions. The reason why they are not much discussed in the canonical formalism, is that to quantize those theories you need the Feynman path integral formalism, instead of second quantization, and there the problem does not appear. It is very interesting that the path integral formalism solves both difficulties. However its study is outside the level of this course, the interested reader can see my text *Advanced Quantum Field Theory* [12].

Complements

Complement 3.1 Fourier transform of the Coulomb potential

The evaluation of the Fourier transform of the Coulomb potential has some subtleties. We show here the steps that lead to Eq. (3.108). We have

$$\begin{aligned}\mathrm{FT}[\frac{1}{|\vec{x}|}] = \int d^3x \frac{e^{-i\vec{q}\cdot\vec{x}}}{|\ \vec{x}\ |} &= 2\pi \int_{-1}^{1} d\cos\theta \int_0^\infty d|\vec{x}||\vec{x}| e^{-i|\vec{q}||\vec{x}|\cos\theta} \\ &= i\frac{2\pi}{|\vec{q}|}\int_0^\infty d|\vec{x}| \left[e^{-i|\vec{q}||\vec{x}|} - e^{i|\vec{q}||\vec{x}|}\right] . \end{aligned} \tag{3.109}$$

The problem comes from the oscillatory behavior of the exponentials in the previous expression. To handle this problem we substitute the Coulomb potential by another that is well behaved and gives back the Coulomb potential in some limit. We take,

$$\frac{1}{|\vec{x}|} = \lim_{\mu\to 0} \frac{e^{-\mu|\vec{x}|}}{|\vec{x}|} \tag{3.110}$$

where the parameter μ has dimensions of mass. We then get,

$$\begin{aligned}\mathrm{FT}[\frac{1}{|\vec{x}|}] &= \lim_{\mu\to 0} i\frac{2\pi}{|\vec{q}|}\int_0^\infty d|\vec{x}| \left[e^{-(\mu+i|\vec{q}|)|\vec{x}|} - e^{-(\mu-i|\vec{q}|)|\vec{x}|}\right] \\ &= \lim_{\mu\to 0} i\frac{2\pi}{|\vec{q}|}\left[\frac{1}{\mu+i|\vec{q}|} - \frac{1}{\mu-i|\vec{q}|}\right] \\ &= \lim_{\mu\to 0} i\frac{2\pi}{|\vec{q}|}\frac{-2i|\vec{q}|}{\mu^2+|\vec{q}|^2} \\ &= \lim_{\mu\to 0} \frac{4\pi}{\mu^2+|\vec{q}|^2} = \frac{4\pi}{|\vec{q}|^2}\,, \end{aligned} \tag{3.111}$$

in agreement with Eq. (3.108).

Problems

3.1 Using

$$|\Phi(\infty)\rangle = \sum_f |f\rangle \langle f|\Phi\infty\rangle = \sum_f |f\rangle S_{fi}\,, \tag{3.112}$$

prove Eq. (3.10).

3.2 Show that

$$\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 H_{\rm int}(t_1)H_{\rm int}(t_2) = \frac{1}{2}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 T\left(H_{\rm int}(t_1)H_{\rm int}(t_2)\right)\,. \tag{3.113}$$

3.3 Show explicitly that the states are normalized as in Eq. (3.69),

$$\langle p|q\rangle = 2E\,(2\pi)^3\delta^3(\vec{p}-\vec{q})\,. \tag{3.114}$$

Show that this normalization is Lorentz covariant, that is

$$2E\,(2\pi)^3\delta^3(\vec{p}-\vec{q}) = 2E'\,(2\pi)^3\delta^3(\vec{p}'-\vec{q}') \tag{3.115}$$

where $E, \vec{p}, \vec{q}$ and $E', \vec{p}', \vec{q}'$ are the energy and momentum in frame S (S'), respectively.

3.4 Consider a real scalar field with the interaction Lagrangian

$$\mathcal{L} = -\frac{\lambda}{4!} : \phi^4 : \,. \tag{3.116}$$

Evaluate the S-matrix up to second order in the coupling λ.

3.5 Calculate the invariant amplitude $\mathcal{M}$ for the Møller scattering, $e^- + e^- \to e^- + e^-$, and show that it coincides with what one would get from the application of the Feynman rules.

3.6 Calculate the invariant amplitude $\mathcal{M}$ for pair production, $\gamma + \gamma \to e^- + e^+$, and show that it coincides with what one would get from the application of the Feynman rules.

3.7 Calculate the invariant amplitude $\mathcal{M}$ for pair annihilation, $e^- + e^+ \to \gamma + \gamma$, and show that it coincides with what one would get from the application of the Feynman rules.

Chapter 4

Cross Sections and Spin Sums

4.1 Introduction

In this chapter we want to go from the theoretical expression for the S-matrix amplitude, given in Eq. (3.77), to the physical measured quantities, the cross sections. We will explain this below, taking special care about the conventions for the normalization of the states.

In the relevant cases particles have different polarization states, like the spin of the electron or the polarization of the photon. In practice, in most of the experimental situations, we do not measure those polarization states, so to compare with experiment we also have to average over the initial state and sum over the final state polarizations. We will explain in detail in this chapter how this can be done.

Finally, the procedure described above, although straightforward can become quite tedious and time consuming. Nowadays this difficulty is solved with dedicated software. In the final part of the chapter we review a few of these software packages that we will use in the following chapters.

4.2 Cross Section

We start with the expression for the S-matrix amplitude, Eq. (3.77), that we recall here.

$$S_{fi} \equiv \delta_{fi} + \left[(2\pi)^4 \delta^4 \left(\sum_i p_i - \sum_f p_f \right) \right] i\mathcal{M} \,. \tag{4.1}$$

Now we go through the various steps in the construction of the cross section.

4.2.1 Transition rates

We start by defining the transition rate per unit time, that is,

$$w_{fi} \equiv \lim_{T\to\infty} \frac{1}{T} \frac{|\,\langle f|S|i\rangle\,|^2}{\langle f|f\rangle\,\langle i|i\rangle} \tag{4.2}$$

There are two points in this definition that deserve explanation. The first is why divide by T. The reason for this definition lies in the fact that S_{fi} stands for all transitions for all time, that is in the limit $T \to \infty$. As the experiments take a finite time, that can be different between different experiments, we want to have a normalized definition that allows to compare those different experiments. We will see below that V, T will cancel out in the final result. The second point has to do with the normalization of the states. Our states are not normalized to unity, in fact we saw in Eq. (3.69) that we have,

$$\langle \vec{p}|\vec{q}\rangle = (2\pi)^3 2p^0 \delta^3(\vec{p}-\vec{q})\,, \tag{4.3}$$

and this is the reason why we have to have to include the normalization in Eq. (4.2).

4.2.2 Delta function

In the expression in Eq. (4.2) appears the square of a delta function. As this might appear a difficulty we will go over it carefully.

We start with the case of just one delta function, for the energy conservation and then generalize the result. We consider transitions in the time interval $(-T/2, T/2)$. We have then

$$\begin{aligned}(2\pi)\delta(E_f - E_i) &= \lim_{T\to\infty}\int_{-T/2}^{T/2} dt e^{i(E_f-E_i)t} \\ &= \lim_{T\to\infty} 2\frac{\sin\left[T/2(E_f-E_i)\right]}{E_f - E_i}\end{aligned} \tag{4.4}$$

For fixed T, this curve is shown in Fig. 4.1, and the area limited by the curve is

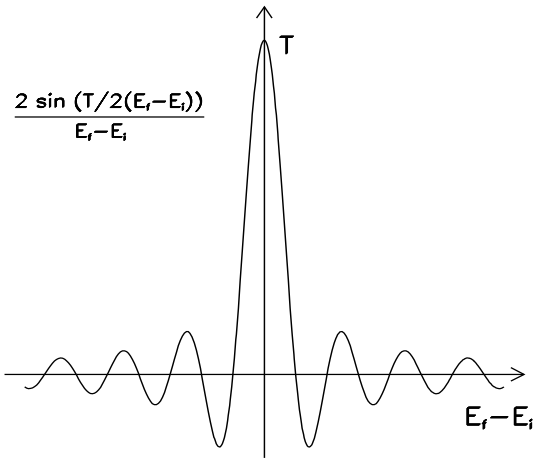


Figure 4.1: Representation of the approximation for the Dirac delta function.

finite and equal to 2π, as it should be if we look at the left-handed side of Eq. (4.4). To see this we define $x = E_f - E_i$ and $\xi = \frac{T}{2}x$. We have then

$$\int_{-\infty}^{\infty} dx \frac{2\sin\left(\frac{T}{2}x\right)}{x} = \int_{-\infty}^{\infty} d\xi \frac{2\sin\xi}{\xi} = 4\int_{0}^{\infty} d\xi \frac{\sin\xi}{\xi} = 2\pi \tag{4.5}$$

where we have used the result,

$$\int_{0}^{\infty} d\xi \frac{\sin\xi}{\xi} = \frac{\pi}{2} \tag{4.6}$$

Noticing that from Eq. (4.4) we have

$$2\pi\delta(0) = \lim_{T\to\infty} \int_{-T/2}^{T/2} dt = \lim_{T\to\infty} T = T \tag{4.7}$$

we finally arrive at the correct expression for the square of the delta function,

$$[2\pi\delta(E_f - E_i)]^2 = 2\pi\delta(0)2\pi\delta(E_f - E_i) = 2\pi T\delta(E_f - E_i). \tag{4.8}$$

Now we are in position to generalize this result to the case of the delta function in Eq. (4.1). We have four delta functions giving energy-momentum conservation, so we must have

$$\left[(2\pi)^4\delta^4\left(\sum p_f - \sum p_i\right)\right]^2 \Rightarrow VT(2\pi)^4\delta^4\left(\sum p_f - \sum p_i\right) \tag{4.9}$$

4.2.3 The phase space

Up to now we have evaluated the transition rate per unit time and volume to a given final state. To evaluate the cross section we have to sum over all final states allowed by energy-momentum conservation. To be more precise we use the following convention, p_1, p_2 are the four momenta of the initial state and $p_3, p_4, \ldots, p_n$ are those of the final state with $n-2$ particles.

We have then to sum (integrate) over the number of states with momenta in the intervals $\vec{p}_3$ to $\vec{p}_3 + d\vec{p}_3$, $\cdots$, $\vec{p}_n$ and $\vec{p}_n + d\vec{p}_n$ given by

$$V\frac{d^3p_3}{(2\pi)^3}V\frac{d^3p_4}{(2\pi)^3}\cdots V\frac{d^3p_n}{(2\pi)^3} = \prod_{i=3}^{n} V\frac{d^3p_i}{(2\pi)^3}. \tag{4.10}$$

4.2.4 The incident flux

The last ingredient going into the cross section is the incident flux. When we compare two experiments we want to have results that do not depend on number of incident particles, so we have to normalize dividing by the incident flux. For the normalization that we are using for the states, the volume V contains one scattering center and the incident flux is

$$\mid \vec{J}_{inc} \mid = \frac{1}{V} \mid \vec{v}_{\rm rel} \mid \tag{4.11}$$

where $v_{\rm rel}$ is the relative velocity of the two incident particles.

4.2.5 The cross section

We have now all the ingredients to evaluate the cross section. Using the previous results we get for the transition rate,

$$w_{fi} = V(2\pi)^4\delta^4(\sum p_i - \sum p_f)\,\frac{|\mathcal{M}|^2}{\langle f|f\rangle\,\langle i|i\rangle} \tag{4.12}$$

Now we have to discuss the normalization of the states. For one particle of momentum p we have,

$$\langle p|p\rangle = (2\pi)^3 2p^0\delta^3(0) \tag{4.13}$$

From our discussion on the delta function should be clear that we have

$$(2\pi)^3\delta^3(0) = V \tag{4.14}$$

therefore, for one particle state we get,

$$\langle p|p\rangle = 2p^0 V \tag{4.15}$$

Using this result we get for the initial and final state normalizations,

$$\langle i|i\rangle = \prod_{i=1}^{2}(V2E_i), \quad \langle f|f\rangle = \prod_{f}(V2E_f) \tag{4.16}$$

where the sum in f runs over all the final state particles.

To obtain the expression for the cross section we divide by the flux and sum over all the final states to get,

$$\begin{aligned} d\sigma =& w_{fi}\frac{V}{v_{\rm rel}}\prod_{j=3}^{n}\frac{Vd^3p_j}{(2\pi)^3} \\ =&(2\pi)^4\delta^4(\sum p_i - \sum p_f)\;\frac{1}{4E_1E_2v_{\rm rel}}\;\prod_{j=3}^{n}\frac{d^3p_j}{2E_j(2\pi)^3} \end{aligned} \tag{4.17}$$

where we notice that all factors of V exactly cancel. For future use we note that, for head-on collisions we have,

$$\begin{aligned} 4E_1E_2\;v_{\rm rel} =&4p_1^0p_2^0\left|\frac{\vec{p}_2}{p_2^0} - \frac{\vec{p}_1}{p_1^0}\right| = 4\left|p_1^0\vec{p}_2 - p_2^0\vec{p}_1\right| \\ =&4\sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2} \end{aligned} \tag{4.18}$$

where the last expression shows that it is a Lorentz invariant quantity. To arrive at Eq. (4.18) it is necessary to assume that $\vec{p}_1$ e $\vec{p}_2$ are collinear, which the normal

situation, either for fixed target or collider experiments. Now we use Eq. (4.18) to rewrite the cross section in a more suggestive form,

$$\sigma = \int \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} |\mathcal{M}_{fi}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - \sum_{j=3}^{n} p_j) \prod_{j=3}^{n} \frac{d^3 p_j}{(2\pi)^3 2p_j^0} \tag{4.19}$$

which our master formula. In Eq. (4.19) there are three distinct parts:

- **Initial State**
  The factor
  $$\frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \tag{4.20}$$
  has to do only with the incident flux and the target, which is completely known.

- **Final State**
  The factor
  $$(2\pi)^4 \delta^4(p_1 + p_2 - \sum_{j=3}^{n} p_j) \prod_{j=3}^{n} \frac{d^3 p_j}{(2\pi)^3 2p_j^0} \tag{4.21}$$
  corresponds to the final state. Either of these two factors are Lorentz invariant[1], which is particularly useful in calculations. The invariance of the final state phase space results from,
  $$\int \frac{d^3 p}{2E} = \int d^4 p \; \delta(p^2 - m^2)\theta(p^0) \tag{4.22}$$
  Another, more explicit way of showing this, is given in Complement 4.2.

- **The matrix element**
  Finally the physics is in the matrix element $\mid \mathcal{M}_{fi} \mid^2$ that, has we have seen, we associate to Feynman diagrams and are written using the Feynman rules for the theory.

4.2.6 The cross section for scattering from an external field

There is a final case that we have to address, that is the case of scattering from an external field like in Coulomb scattering. We have seen the modified Feynman Rules in Eqs. (3.105) and (3.106). If we go through the previous procedure we get for the transition rate

$$w_{fi} = (2\pi)\delta(E_f - E_i) \; \frac{|\mathcal{M}|^2}{\langle f|f\rangle \, \langle i|i\rangle}$$

[1]More precisely for Lorentz transformations along the collision direction.

$$= (2\pi)\delta(E_f - E_i)\frac{1}{V2E_i}\frac{1}{V2E_f}\ |\mathcal{M}|^2 \tag{4.23}$$

To get the cross section we have to divide by the incident flux v_i/V and multiply by the final states available. We get

$$\begin{aligned} d\sigma =& w_{fi}\frac{V}{v_i}|\mathcal{M}|^2\frac{Vd^3p_f}{(2\pi)^3} \\ =& (2\pi)\delta(E_f - E_i)\ \frac{1}{2v_iE_i}|\mathcal{M}|^2\frac{d^3p_f}{2E_f(2\pi)^3} \end{aligned} \tag{4.24}$$

Notice that the dependance on V cancelled out. Now we observe that $v_iE_i = |\vec{p}_i| = |\vec{p}_f| \equiv |\vec{p}|$ and that $d^3p_f = |\vec{p}|E_f dE_f d\Omega$ to finally obtain

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2}|\mathcal{M}|^2. \tag{4.25}$$

To proceed we need to know $\mathcal{M}$, that we have already discussed in section 3.5. The cross section will be discussed in section 5.1.

4.3 Kinematics

In this section we will explain how to evaluate the cross sections for the case of two particles in the final state. We can evaluate the cross section in two reference frames, the so-called laboratory frame where one of the particles is at rest, and the center of mass frame where the total 3-momentum vanishes for the initial and final state. As this two frames are connected by a Lorentz boost along the direction of the incident particle, the area perpendicular is not affected and the same happens to the cross section. However the explicit formulas are very different and normally much more complicated in the Lab frame.

4.3.1 Laboratory frame

In this frame one of the particles is a rest, $\vec{p}_2 = 0$ and the kinematics is indicated in Fig. 4.2, where

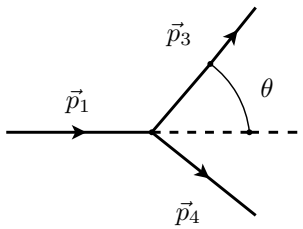


Figure 4.2: Kinematics of the Lab frame.

$$\begin{array}{ll} p_1 = (E_1, \vec{p}_1) & p_3 = (E_3, \vec{p}_3) \\ p_2 = (m_2, \vec{0}) & p_4 = (E_1 + m_2 - E_3, \vec{p}_4) \end{array} . \tag{4.26}$$

We get

$$\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} = \sqrt{m_2^2 E_1^2 - m_1^2 m_2^2} = p_{1\text{Lab}} m_2 \tag{4.27}$$

and therefore

$$\sigma = \int \frac{1}{4 m_2 p_{1\text{Lab}}} \mid \mathcal{M}_{fi} \mid^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \frac{d^3 p_3}{(2\pi)^3 2 p_3^0} \frac{d^3 p_4}{(2\pi)^3 2 p_4^0} \tag{4.28}$$

$$= \frac{1}{64 m_2 \pi^2 p_{1\text{Lab}}} \int \frac{d^3 p_3}{p_3^0 \, p_4^0} \, \delta(p_1^0 + p_2^0 - p_3^0 - p_4^0) \mid \mathcal{M}_{fi} \mid^2 \tag{4.29}$$

$$= \frac{1}{64 \pi^2 m_2 p_{1\text{Lab}}} \int \frac{d|\vec{p}_3||\vec{p}_3|^2 d\Omega}{p_3^0 \, p_4^0} \, \delta(p_1^0 + p_2^0 - p_3^0 - p_4^0) \mid \mathcal{M}_{fi} \mid^2 \tag{4.30}$$

Finally we integrate in $|\vec{p}_3|$ leaving only two independent variables, the scattering angles of particle 3.

$$\begin{aligned} \frac{d\sigma}{d\Omega} =& \frac{1}{64 \pi^2 m_2 p_{1\text{Lab}}} \int \frac{d|\vec{p}_3||\vec{p}_3|^2}{p_3^0 \, p_4^0} \delta\Big(\sqrt{\mid \vec{p}_1 \mid^2 + \mid \vec{p}_3 \mid^2 - 2 \mid \vec{p}_1 \mid \mid \vec{p}_3 \mid \cos\theta + m_e^2} \\ & + \sqrt{\mid \vec{p}_3 \mid^2 + m_\mu^2} - m_2 - E_1 \Big) \mid \mathcal{M}_{fi} \mid^2 \\ =& \frac{1}{64 \pi^2 m_2 \mid \vec{p}_1 \mid} \int d|\vec{p}_3||\vec{p}_3|^2 \frac{\delta(\mid \vec{p}_3 \mid - \cdots)}{p_4^0 \mid \vec{p}_3 \mid + p_3^0 \mid \vec{p}_3 \mid - {p_3}^0 \mid \vec{p}_1 \mid \cos\theta} \mid \mathcal{M}_{fi} \mid^2 \end{aligned} \tag{4.31}$$

where we used the properties of the delta function in Eq. (4.31) (see Complement 4.1) to obtain,

$$\frac{d\sigma}{d\Omega} = \frac{1}{64 \pi^2 m_2} \, \frac{|\vec{p}_3|}{|\vec{p}_1|} \, \frac{\mid \mathcal{M}_{fi} \mid^2}{m_2 + E_1 - \frac{|\vec{p}_1| E_3}{|\vec{p}_3|} \cos\theta} \tag{4.32}$$

where E_3 e $p_{3\text{Lab}}$ are implicitly defined in terms of the scattering angles θ. For each θ we can obtain them. We get in the Lab frame

$$\begin{aligned} p_3 &= (E_3, 0, p_{3\text{Lab}} \sin\theta, p_{3\text{Lab}} \cos\theta) \\ p_4 &= (E_1 + m_2 - E_3, 0, -p_{3\text{Lab}} \sin\theta, p_{1\text{Lab}} - p_{3\text{Lab}} \cos\theta) \end{aligned} \tag{4.33}$$

Using $p_4^2 = m_4^2$, it is possible to obtain $p_{3\text{Lab}}$

$$p_{3\text{Lab}} = \frac{B \pm \sqrt{B^2 - AC}}{A} \tag{4.34}$$

with

$$A = 4(E_1 + m_2)^2 - 4 p_{1\text{Lab}}^2 \cos^2\theta$$

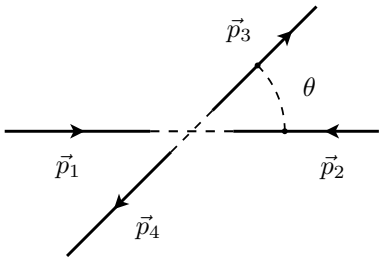


Figure 4.3: CM kinematics.

$$\begin{aligned} B &= 2p_{1\mathrm{Lab}}\cos\theta\left[(E_1+m_2)^2 - m_4^2 + m_3^2 - p_{1\mathrm{Lab}}^2\right] \\ C &= 4m_3^2(E_1+m_2)^2 - \left[(E_1+m_2)^2 - m_4^2 + m_3^2 - p_{1\mathrm{Lab}}^2\right]^2 \end{aligned} \tag{4.35}$$

To find the final result we would have to introduce $\mid \mathcal{M}_{fi} \mid^2$ in Eq. (4.32) and perform the angular integrations. This will depend on the process. We will come back to this later.

4.3.2 Center of mass frame

In the center of mass frame (CM), we can define,

$$P_{\mathrm{CM}} = (\sqrt{s}, \vec{0}) = p_{1\mathrm{CM}} + p_{2\mathrm{CM}} = p_{3\mathrm{CM}} + p_{4\mathrm{CM}} \tag{4.36}$$

where $\sqrt{s}$ is the total energy in the CM frame. Using Eq. (4.36) we can than show that,

$$\begin{aligned} p^0_{1\mathrm{CM}} &= \frac{s+m_1^2-m_2^2}{2\sqrt{s}}, \qquad p^0_{2\mathrm{CM}} = \frac{s+m_2^2-m_1^2}{2\sqrt{s}} \\ p^0_{3\mathrm{CM}} &= \frac{s+m_3^2-m_4^2}{2\sqrt{s}}, \qquad p^0_{4\mathrm{CM}} = \frac{s+m_4^2-m_3^2}{2\sqrt{s}} \\ |\vec{p}_1|_{\mathrm{CM}} &= \frac{\lambda(\sqrt{s},m_1,m_2)}{2\sqrt{s}}, \quad |\vec{p}_3|_{\mathrm{CM}} = \frac{\lambda(\sqrt{s},m_3,m_4)}{2\sqrt{s}} \end{aligned} \tag{4.37}$$

where

$$\lambda(x,y,z) = \sqrt{(x^2-y^2-z^2)^2 - 4y^2z^2} \tag{4.38}$$

Let us specialize for the important case of elastic scattering when $m_1 = m_3$ e $m_2 = m_4$. Then $|\vec{p}_1|_{\mathrm{CM}} = |\vec{p}_3|_{\mathrm{CM}}$. The kinematics is shown in Fig. 4.3. It can be useful to relate the Lab reference frame with the CM frame. Noting that

$$P_{\mathrm{Lab}} = (E_1 + m_2, \vec{p}_{1\mathrm{Lab}}), \quad P_{\mathrm{CM}} = (\sqrt{s}, \vec{0}) \tag{4.39}$$

we get

$$\sqrt{s} = \gamma\left(E_1 + m_2 - \vec{\beta}\cdot\vec{p}_{1\mathrm{Lab}}\right)$$

$$0 \;=\; \gamma\left(\vec{p}_{1\mathrm{Lab}} - \vec{\beta}(E_1 + m_2)\right) \tag{4.40}$$

and finally

$$\vec{\beta} = \frac{\vec{p}_{1\mathrm{Lab}}}{E_1 + m_2}, \quad \gamma = \frac{E_1 + m_2}{\sqrt{s}}, \quad s = m_1^2 + m_2^2 + 2E_1 m_2 \ . \tag{4.41}$$

With these relations we can get

$$\begin{aligned} \vec{p}_{1\mathrm{CM}} &= \gamma(\vec{p}_{1\mathrm{Lab}} - \vec{\beta}E_1) = \frac{m_2}{\sqrt{s}}\vec{p}_{1\mathrm{Lab}} \\ \vec{p}_{2\mathrm{CM}} &= -\gamma\vec{\beta}\ m_2 = -\frac{m_2}{\sqrt{s}}\vec{p}_{1\mathrm{Lab}} \end{aligned} \tag{4.42}$$

and therefore verify

$$\vec{p}_{1\mathrm{CM}} + \vec{p}_{2\mathrm{CM}} = 0 \ . \tag{4.43}$$

We will now turn to the differential scattering cross section in `CM` frame. The general expression is again Eq. (4.19). To calculate the flux term we note that from the definition $p_1 + p_2 = (\sqrt{s}, \vec{0})$ we get

$$p_1 \cdot p_2 = \frac{s - m_1^2 - m_2^2}{2} \tag{4.44}$$

Then

$$4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} = 2\,\lambda(\sqrt{s}, m_1, m_2) = 4\sqrt{s}\ |\vec{p}_{1\mathrm{CM}}| \tag{4.45}$$

and therefore

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2\sqrt{s}|\vec{p}_{1\mathrm{CM}}|}\int \frac{d|\vec{p}_3|\,|\vec{p}_3|^2}{p_3^0\,p_4^0}\ \delta\left(\sqrt{s} - p_3^0 - \sqrt{(p_3^0)^2 - m_3^2 + m_4^2}\right)|\mathcal{M}_{fi}|^2 \\ &= \frac{1}{64\pi^2\sqrt{s}|\vec{p}_{1\mathrm{CM}}|}\int \frac{dp_3^0\,|\vec{p}_{3\mathrm{CM}}|}{p_4^0}\ \frac{\delta(p_3^0 - \cdots)}{1 + \dfrac{p_3^0}{p_4^0}}\ |\mathcal{M}_{fi}|^2 \end{aligned} \tag{4.46}$$

and finally

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\vec{p}_{3\mathrm{CM}}|}{|\vec{p}_{1\mathrm{CM}}|}|\mathcal{M}_{fi}|^2 \tag{4.47}$$

For elastic scattering we have $|\vec{p}_{1\mathrm{CM}}| = |\vec{p}_{3\mathrm{CM}}|$ and we get,

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}|\mathcal{M}_{fi}|^2 \tag{4.48}$$

To proceed we will need to know the matrix element $\mathcal{M}_{fi}$. This is where the dynamics is and it is given by the Feynman rules of the theory. We will see examples in QED in the next Chapter.

4.3.3 Mandelstam variables

Before we leave this section, let us discuss, for the important case of the scattering $1 + 2 \to 3 + 4$, a set of very useful invariants, the Mandelstam variables. For this case we have

$$p_1 + p_2 \to p_3 + p_4 \tag{4.49}$$

and we define the Mandelstam variables by,

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2. \tag{4.50}$$

We will see that when we apply the Feynman rules for the Feynman diagrams these are named after the Mandelstam variable the corresponds to the squared of the transferred momentum in that diagram. The variables s, t, u are not independent. In fact

$$\begin{aligned} s + t + u &= (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2 \\ &= m_1^2 + m_2^2 + 2p_1 \cdot p_2 + m_1^2 + m_3^2 - 2p_1 \cdot p_3 + m_1^2 + m_4^2 - 2p_1 \cdot p_4 \\ &= 3m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2p_1 \cdot (p_2 - p_3 - p_4) \\ &= 3m_1^2 + m_2^2 + m_3^2 + m_4^2 - 2m_1^2 \\ &= m_1^2 + m_2^2 + m_3^2 + m_4^2 \end{aligned} \tag{4.51}$$

where we have used Eq. (4.49). Notice that t and u are normally negative while s is always positive.

4.4 How to evaluate spin sums for fermions

In most situations the spin of the initial and final state particles is not known, and to compare with experiment one should sum over all polarizations and divide by the initial state number of different polarization combinations. To have a concrete example consider the scattering $e^- + e^+ \to \mu^- + \mu^+$ in QED, that is just with the photon exchange diagram of Fig. 4.4. Using the Feynman rules one get for the

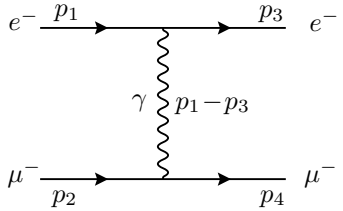


Figure 4.4: Scattering $e^-\mu^- \to e^-\mu^-$.

invariant amplitude

$$\begin{aligned} i\,\mathcal{M} =& \overline{u}(p_3)(ie\gamma^\mu)u(p_1)\ \frac{-i\ g_{\mu\nu}}{(p_3-p_1)^2}\ \overline{u}(p_4)(ie\gamma^\nu)u(p_2) \\ =& i\,\frac{e^2}{t}\ \overline{u}(p_3)\gamma^\mu u(p_1)\ \overline{u}(p_4)\gamma_\mu u(p_2) \end{aligned} \tag{4.52}$$

or

$$\mathcal{M} = \frac{e^2}{t}\ \overline{u}(p_3,s_e')\gamma^\mu u(p_1,s_e)\overline{u}(p_4,s_\mu')\gamma_\mu u(p_2,s_\mu) \tag{4.53}$$

Now the spin averaged squared matrix element is

$$\overline{|\mathcal{M}_{fi}|^2} = \frac{1}{4}\sum_{\text{spins}} |\ \mathcal{M}_{fi}\ |^2 = \frac{1}{4}\sum_{\text{spins}} \left|\overline{u}(p_4,s_e')\gamma^\mu u(p_2,s_e)\overline{u}(p_3,s_\mu')\gamma_\mu u(p_1,s_\mu)\right|^2 \frac{e^4}{t^2} \tag{4.54}$$

4.4.1 The Casimir trick: traces of γ matrices

Let us evaluate the spin sums in Eq. (4.54). We have

$$\begin{aligned} &\sum_{\text{spins}} \left|\overline{u}(p_3,s_e')\gamma^\mu u(p_1,s_e)\overline{u}(p_4,s_\mu')\gamma_\mu u(p_2,s_\mu)\right|^2 = \\ &= \sum_{\text{spins}} \left[\overline{u}(p_3,s_e')\gamma^\mu u(p_1,s_e)\overline{u}(p_4,s_\mu')\gamma_\mu u(p_2,s_\mu)\right]\left[\overline{u}(p_3,s_e')\gamma^\nu u(p_1,s_e)\overline{u}(p_4,s_\mu')\gamma_\nu u(p_2,s_\mu)\right]^\dagger \\ &= \sum_{\text{spins } e} \left[\overline{u}(p_3,s_e')\gamma^\mu u(p_1,s_e)\overline{u}(p_1,s_e)\gamma^0(\gamma^\nu)^\dagger\gamma^0 u(p_3,s_e')\right]\times \\ &\quad \sum_{\text{spins } \mu} \left[\overline{u}(p_4,s_\mu')\gamma_\mu u(p_2,s_\mu)\overline{u}(p_2,s_\mu)\gamma^0(\gamma_\nu)^\dagger\gamma^0 u(p_4,s_\mu')\right] \\ &\equiv \sum_{\text{spins } e} \left[\overline{u}(p_3,s_e')\gamma^\mu u(p_1,s_e)\overline{u}(p_1,s_e)\,\overline{\gamma^\nu}\,u(p_3,s_e')\right]\times \\ &\quad \sum_{\text{spins } \mu} \left[\overline{u}(p_4,s_\mu')\gamma_\mu u(p_2,s_\mu)\overline{u}(p_2,s_\mu)\,\overline{\gamma_\nu}\,u(p_4,s_\mu')\right] \end{aligned} \tag{4.55}$$

where we have defined, for any combination of γ matrices, Γ, the quantity

$$\overline{\Gamma} \equiv \gamma^0\Gamma^\dagger\gamma^0 \tag{4.56}$$

With this in mind we can get a general result. Consider the electron line but use a general combination Γ of γ matrices as will appear in other problems.

$$\begin{aligned} &\sum_{\text{spins } e} \left[\overline{u}(p_3,s_e')\Gamma u(p_1,s_e)\overline{u}(p_1,s_e)\,\overline{\Gamma}\,u(p_3,s_e')\right] \\ &\qquad = \sum_{\text{spins } e} \left[\overline{u}_\alpha(p_3,s_e')\Gamma_{\alpha\beta}u_\beta(p_1,s_e)\overline{u}_\delta(p_1,s_e)\,\overline{\Gamma}_{\delta\tau}\,u_\tau(p_3,s_e')\right] \end{aligned}$$

$$= \sum_{\text{spins } e} \left[u_\tau(p_3, s'_e)\overline{u}_\alpha(p_3, s'_e)\Gamma_{\alpha\beta}u_\beta(p_1, s_e)\overline{u}_\delta(p_1, s_e)\,\overline{\Gamma}_{\delta\tau} \right]$$

$$= \text{Tr}\left[(\not{p}_3 + m_e)\Gamma(\not{p}_1 + m_e)\overline{\Gamma}\right] \tag{4.57}$$

where we have used the result,

$$\sum_{\text{spins}} u_\alpha(p, s_e)\overline{u}_\beta(p, s) = (\not{p} + m)_{\alpha\beta}. \tag{4.58}$$

This is the important result known as Casimir's trick [34]. The sum of spins of a fermion line can be transformed in a trace of γ matrices. This is an huge simplification, as we do not have to use the explicit form for the spinors and the traces can be automatized as we will explain below. To be complete we have now for our problem in Eq. (4.54),

$$\overline{|\mathcal{M}_{fi}|^2} = \frac{e^2}{4t^2}\text{Tr}\left[(\not{p}_3 + m_e)\gamma^\mu(\not{p}_1 + m_e)\overline{\gamma}^\nu\right]\text{Tr}\left[(\not{p}_4 + m_\mu)\gamma_\mu(\not{p}_2 + m_\mu)\overline{\gamma}_\nu\right] \tag{4.59}$$

4.4.2 Theorems on traces of γ matrices

We just saw that the sums of spins of fermions can be transformed in traces of γ matrices. To use this powerful result one needs to be able to evaluate these traces. The first observation is that the traces are independent of the representation. This results from the general rule connecting different representations of the Dirac matrices, known as *similarity transformation*,

$$\gamma'^\mu = U^{-1}\gamma^\mu U \tag{4.60}$$

and from the cyclic property of the traces. We are going to present the relevant results in the form of theorems, some of which we will leave as exercise for the reader.

Theorem 4.1 *The trace of an odd number of γ matrices is zero.*

Proof:

$$\begin{aligned}
\text{Tr}\left[\not{a}_1\not{a}_2\cdots\not{a}_n\right] &= \text{Tr}\left[\not{a}_1\cdots\not{a}_n\gamma_5\gamma_5\right] \\
&= \text{Tr}\left[\gamma_5\not{a}_1\cdots\not{a}_n\gamma_5\right] \\
&= (-1)^n\text{Tr}\left[\not{a}_1\cdots\not{a}_n\right]
\end{aligned} \tag{4.61}$$

Then for n odd the trace vanishes.

Theorem 4.2 *The traces of 0 and 2 γ matrices are*

$$\begin{aligned} \mathrm{Tr}1 &= 4 \\ \mathrm{Tr}[\not{a}\not{b}] &= \mathrm{Tr}[(\not{b}\not{a})] = \tfrac{1}{2}\mathrm{Tr}[(\not{a}\not{b}+\not{b}\not{a})] = a\cdot b\ \mathrm{Tr}1 \\ &= 4a\cdot b \end{aligned} \tag{4.62}$$

Theorem 4.3 *The trace of n (even) γ matrices can be obtained by recurrence from the traces of $n-2$ (even) γ matrices.*

$$\begin{aligned} \mathrm{Tr}\left[\not{a}_1\cdots\not{a}_n\right] &= a_1\cdot a_2\ \mathrm{Tr}\left[\not{a}_3\cdots\not{a}_n\right] - a_1\cdot a_3\ \mathrm{Tr}\left[\not{a}_2\not{a}_4\cdots\not{a}_n\right] \\ &\quad + \cdots + a_1\cdot a_n\ \mathrm{Tr}\left[\not{a}_2\cdots\not{a}_{n-1}\right] \end{aligned} \tag{4.63}$$

To prove this theorem one uses

$$\not{a}_1\not{a}_k = -\not{a}_k\not{a}_1 + 2a_1\cdot a_k \tag{4.64}$$

to bring $\not{a}_1$ to the end of the string and then the cyclic property to bring it back to the beginning again. This recurrence relation can be used to make an automatic evaluation of any trace.

This theorem has an important corollary,

Corollary: *For the case of 4 matrices γ we have:*

$$\begin{aligned} \mathrm{Tr}\left[\not{a}_1\not{a}_2\not{a}_3\not{a}_4\right] &= a_1\cdot a_2\ \mathrm{Tr}\left[\not{a}_3\not{a}_4\right] - a_1\cdot a_3\ \mathrm{Tr}\left[\not{a}_2\not{a}_4\right] + a_1\cdot a_4\ \mathrm{Tr}\left[\not{a}_2\not{a}_3\right] \\ &= 4\left[a_1\cdot a_2\ a_3\cdot a_4 - a_1\cdot a_3\ a_2\cdot a_4 + a_1\cdot a_4\ a_2\cdot a_3\right] \end{aligned} \tag{4.65}$$

Theorem 4.4 *The traces with γ_5 are obtained from the following results*[2]

$$\begin{aligned} &\mathrm{Tr}\left[\gamma_5\right] = 0 \\ &\mathrm{Tr}\left[\gamma_5\not{a}\not{b}\right] = 0 \\ &\mathrm{Tr}\left[\gamma_5\not{a}\not{b}\not{c}\not{d}\right] = -4i\varepsilon_{\mu\nu\rho\sigma}a^\mu b^\nu c^\rho d^\sigma\ . \end{aligned} \tag{4.66}$$

The next theorem is not on traces, but it is important because it can be used to reduce the number of γ matrices in the traces to be evaluated.

[2]Our definition of the Levi Civita tensor has $\varepsilon^{0123} = +1$.

Theorem 4.5

$$\begin{aligned}
&\gamma_\mu\gamma^\mu = 4\\
&\gamma_\mu \not{a}\gamma^\mu = -2\not{a}\\
&\gamma_\mu \not{a}\not{b}\gamma^\mu = 4a.b\\
&\gamma_\mu \not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a}\\
&\gamma_\mu \not{a}\not{b}\not{c}\not{d}\gamma^\mu = 2\left[\not{d}\not{a}\not{b}\not{c} + \not{c}\not{b}\not{a}\not{d}\right]
\end{aligned} \tag{4.67}$$

and finally a very useful result,

Theorem 4.6

$$\mathrm{Tr}\left[\not{a}_1 \cdots \not{a}_{2n}\right] = \mathrm{Tr}\left[\not{a}_{2n} \cdots \not{a}_1\right] \ , \tag{4.68}$$

that can be obtained from the fact that, for any matrix M, one has

$$\mathrm{Tr}[M^T] = \mathrm{Tr}[M] \tag{4.69}$$

4.5 How to evaluate polarization sums for gauge fields

This was already explained in the previous sections. We just collect here the results for completeness.

Photons

For photons we can show that (see Complement 4.3)) that we have

$$\sum_\lambda \epsilon^\mu(k,\lambda)\epsilon^{*\mu}(k,\lambda) = -g^{\mu\nu} - \frac{k^\mu k^\nu}{(k\cdot\eta)^2} + \frac{\eta^\mu k^\nu + \eta^\nu k^\mu}{k\cdot\eta} \tag{4.70}$$

where η^μ is a time-like unit vector not proportional to k^μ, otherwise arbitrary. Different choices correspond to different gauges. Therefore the polarization sum has the form,

$$\sum_\lambda \epsilon^\mu(k,\lambda)\epsilon^{*\mu}(k,\lambda) = -g^{\mu\nu} + \text{terms proportional to } k. \tag{4.71}$$

The terms proportional to k do not contribute due to gauge invariance (see also the discussion in Complement 5.1). Therefore, when we have photons we simply take, without loss of generality,

$$\sum_\lambda \epsilon^\mu(k,\lambda)\epsilon^{*\mu}(k,\lambda) = -g^{\mu\nu} \tag{4.72}$$

Massive gauge fields

In this case the gauge fields have three polarizations and the sum over polarizations gives

$$\sum_{\lambda} \epsilon^{\mu}(q,\lambda)\epsilon^{*\nu}(q,\lambda) = -g^{\mu\nu} + \frac{q^{\mu}q^{\nu}}{M_V^2} \tag{4.73}$$

We will see specific examples in chapter 8.

Complements

Complement 4.1 Properties of Dirac δ function

In the derivation of Eq. (4.32) we used the following property of the Dirac delta function,

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|} \tag{4.74}$$

where x_i are the zeros of $f(x)$. To understand the origin of this expression let us consider a simple case of a function just with one zero. In the neighborhood of this zero one should have,

$$f(x) = f'(x_0)(x - x_0) + \cdots \ . \tag{4.75}$$

Therefore we obtain

$$\begin{aligned} \int dx\, g(x)\delta(f(x)) &= \int dx\, g(x)\delta(f'(x_0)(x - x_0)) \\ &= \int dy\, g(x)\frac{\delta(y - y_0)}{|f'(x_0)|} \\ &= \frac{g(x_0)}{|f'(x_0)|} \end{aligned} \tag{4.76}$$

where we have made the change of variable $y = |f'(x_0)|x$, and the absolute value comes from the fact that the delta function is an even function of its argument. The generalization to several zeros is immediate.

Complement 4.2 Lorentz invariance of phase space factors

Here we show in another way the Lorentz invariance of the phase space factor,

$$\frac{d^3p}{(2\pi)^3 2E} \tag{4.77}$$

For this we consider a Lorentz transformation along the z axis

$$d^3p' = dp'_x dp'_y dp'_z = dp_x dp_y \frac{dp'_z}{dp_z} dp_z = d^3p\ \frac{dp'_z}{dp_z}\ . \tag{4.78}$$

Using

$$p'_z = \gamma(p_z - \beta E),\ E' = \gamma(E - \beta p_z),\ \frac{\partial E}{\partial p_z} = \frac{p_z}{E}\ , \tag{4.79}$$

we get

$$\frac{dp'_z}{dp_z} = \gamma\left(1 - \beta\frac{\partial E}{\partial p_z}\right) = \gamma\left(1 - \beta\frac{p_z}{E}\right) = \frac{1}{E}\gamma(E - \beta p_z) = \frac{E'}{E}\ , \tag{4.80}$$

implying

$$\frac{d^3p'}{E'} = \frac{d^3p}{E}\ , \tag{4.81}$$

as we wanted to show.

Complement 4.3 Polarizations Sum for Photons

We are going to show here Eq. (4.70). First of all the photon is described by a real field and we can always choose its polarization vector real. However it can be useful to consider it being complex, like in classical electromagnetism we do for left and right polarized light. Note that the field defined in Eq. (2.91) is always a real field. The polarization vector should obey (remember a plane wave propagating in the z direction with transversal degrees of freedom),

$$k_\mu k^\mu = 0, \qquad ; \qquad \varepsilon_\mu k^\mu = 0 \qquad ; \qquad \varepsilon^*_\mu \varepsilon^\mu = -1 \tag{4.82}$$

Consider now a reference frame where the photon moves along the z axis. In this frame one can choose the transversal polarization along x and y, that is,

$$k^\mu = (k,0,0,k), \quad \varepsilon^\mu(k,1) = (0,1,0,0), \quad \varepsilon^\mu(k,2) = (0,0,1,0) \tag{4.83}$$

If we define

$$P^{\mu\nu} \equiv \sum_\lambda \varepsilon^\mu(k,\lambda)\varepsilon^{*\nu}(k,\lambda) \tag{4.84}$$

we get in this frame,

$$P^{11} = P^{22} = 1, \quad P^{\mu\nu} = 0, \text{ for all other cases.} \tag{4.85}$$

The problem now is how to write this result in a covariant form. As a first try one could suggest that $P^{\mu\nu} = -g^{\mu\nu}$. This assignment works for $\mu,\nu = 1,2$ but would give $P^{00} = -1$ and $P^{33} = 1$ in disagreement with Eq. (4.85). We could think of adding a term of the form $b\ k^\mu k^\nu$. The problem is that we can choose the value of b to make P^{00} or P^{33} to vanish, but not both at the same time. This leads us to choose another 4-vector not proportional to k. The choice here is arbitrary (would lead to different gauge choices). We choose in the above frame;

$$\eta^\mu = (1,0,0,0), \quad \text{with } k\cdot\eta \neq 0, \quad \eta\cdot\varepsilon(k,\lambda) = 0, \quad \eta\cdot\eta = 1 \tag{4.86}$$

Now we have enough freedom to be in agreement with Eq. (4.85). Using the fact that $P^{\mu\nu}$ is a second rank symmetric tensor, we can write,

$$P^{\mu\nu} = a g^{\mu\nu} + b k^\mu k^\nu + c\,(\eta^\mu k^\nu + \eta^\nu k^\mu) + d\eta^\mu\eta^\nu \tag{4.87}$$

and determine the coefficients a, b, c e d. As ε^μ is orthogonal to both k^μ and η^μ and has the normalization of Eq. (4.82), we multiply Eq. (4.87) by ε_ν to get

$$-\varepsilon^\mu = a\ \varepsilon^\mu \tag{4.88}$$

which gives $a = -1$. Using now that $k_\mu P^{\mu\nu} = \eta_\mu P^{\mu\nu} = 0$ we get two equations

$$0 = -k^\nu + c(k\cdot\eta)k^\nu + d(k\cdot\eta)\eta^\nu \tag{4.89}$$

$$0 = -\eta^\nu + b(k\cdot\eta)k^\nu + c(k^\nu + (k\cdot\eta)\eta^\nu) + d\eta^\nu \tag{4.90}$$

that have as solution

$$b = -\frac{1}{(k\cdot\eta)^2}, \quad c = \frac{1}{k\cdot\eta}, \quad d = 0 \tag{4.91}$$

and therefore

$$P^{\mu\nu} = -g^{\mu\nu} - \frac{k^\mu k^\nu}{(k\cdot\eta)^2} + \frac{\eta^\mu k^\nu + \eta^\nu k^\mu}{k\cdot\eta} \tag{4.92}$$

in agreement with Eq. (4.70). Note that in our frame, Eq. (4.86), we have $P^{11} = P^{22} = 1$ e $P^{00} = P^{33} = P^{03} = P^{30} = 0$ as required.

Problems

4.1 Show that scattering cross section for $p_1 + p_2 \to p_3 + p_4$ can be written in the CM frame as

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_{3\mathrm{CM}}|}{|\vec{p}_{1\mathrm{CM}}|} \overline{|\mathcal{M}_{fi}|^2} \tag{4.93}$$

where $|\vec{p}_{1\mathrm{CM}}|$ and $|\vec{p}_{3\mathrm{CM}}|$ are the 3-momenta of particles 1 and 3 in the CM frame. Simplify the expression for the case of massless particles.

4.2 Consider the decay of an unstable particle of mass M and 4-momentum P, in n particles ($n \geq 2$) with 4-momenta q_i. Show that the expression for the decay width, defined as the transition rate per unit time and unit of volume and for one decaying particle is given by,

$$d\Gamma = \frac{1}{2M} \overline{|\mathcal{M}_{fi}|^2} \, (2\pi)^4 \delta^4 \left(P - \sum_i^n q_i \right) \prod_i^n \frac{d^3 q_i}{(2\pi)^3 2q_i^0} \tag{4.94}$$

4.3 Using the results of problem 4.2, show that for the decay $P \to q_1 + q_2$ the expression for the decay width is the rest frame of the decaying particle is,

$$\frac{d\Gamma}{d\Omega} = \frac{1}{32\pi^2} \frac{|\vec{q}_{1\mathrm{CM}}|}{M^2} \overline{|\mathcal{M}_{fi}|^2} \, , \tag{4.95}$$

where $P^2 = M^2$, and $\vec{q}_{1\mathrm{CM}}$ is the 3-momentum in the CM frame.

4.4 Consider the definition $\overline{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0$ for any combination Γ of Dirac matrices.

a) For the matrices in the basis Γ^A (see section 1.4.4) evaluate $\overline{\Gamma^A}$.

b) Consider the matrix Γ defined by

$$\Gamma = \gamma^\mu (g_V - g_A \gamma_5) \tag{4.96}$$

where g_V and g_A are constants. Show that

$$\overline{\Gamma} = \Gamma \tag{4.97}$$

c) Evaluate $\overline{\gamma^\mu P_L}$ e $\overline{\gamma^\mu P_R}$.

4.5 Prove theorems 4.3, 4.4, 4.5 e 4.6.

4.6 In the space of Dirac matrices we can define a basis of 16 matrices as follows,

$$\Gamma_0 = 1, \Gamma_\mu = \gamma_\mu, \Gamma_{\mu\nu} = \gamma_{[\mu}\gamma_{\nu]}, \Gamma_{\mu\nu\rho} = \gamma_{[\mu}\gamma_\nu\gamma_{\rho]}, \Gamma_{\mu\nu\rho\sigma} = \gamma_{[\mu}\gamma_\nu\gamma_\rho\gamma_{\sigma]}$$

where $\Gamma_{\mu_1\mu_2\ldots\mu_n}$ are the completely antisymmetric products of, n, γ matrices defined by,

$$\Gamma_{\mu_1\mu_2\ldots\mu_n} = \frac{1}{n!}\sum_{\text{permP}}(-1)^P\gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_n}$$

One can show that any matrix in Dirac space can be written as

$$M = \frac{1}{4}\sum_{n=0}^{4}\frac{1}{n!}\text{Tr}\left[M\Gamma^{\mu_1\mu_2\ldots\mu_n}\right]\Gamma_{\mu_n\ldots\mu_2\mu_1}\ .$$

Show that the relation with the more familiar basis of section 1.4.4 is,

$$\begin{aligned}\Gamma_{\mu\nu} &= -i\ \sigma_{\mu\nu}\\ \Gamma_{\mu\nu\rho} &= -i\ \epsilon_{\mu\nu\rho\alpha}\gamma_5\gamma^\alpha\\ \Gamma_{\mu\nu\rho\sigma} &= -i\ \epsilon_{\mu\nu\rho\sigma}\gamma_5.\end{aligned}$$

4.7 When one uses the Mandelstam variables it is also important to express the angles in an invariant form, as the angles are frame dependent. In the scattering of two particles into two particles with $p_1 + p_2 = p_3 + p_4$ that scattering angle θ is usually defined by the angle made by $\vec{p}_3$ and $\vec{p}_1$. Now this angle is related with the Mandelstam variable $t = (p_1 - p_3)^2$. Use this fact and the formula in Eq. (4.93) to show that we have,

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s}\frac{1}{|p_{1\text{CM}}|^2}\overline{|\mathcal{M}|^2}, \tag{4.98}$$

where $\mathcal{M} = \mathcal{M}(s,t,u)$ and

$$\sqrt{s}|p_{1\text{CM}}| = \sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2} = \frac{1}{2}\lambda(\sqrt{s}, m_1, m_2), \tag{4.99}$$

and (see Eq. (4.38))

$$\lambda(x,y,z) = \sqrt{(x^2-y^2-z^2)^2 - 4y^2z^2}. \tag{4.100}$$

4.8 When $m_1 > m_2$ there is in the lab frame a value for $\theta_{\text{Lab}}^{\min}$ and another for $\theta_{\text{Lab}}^{\max}$. Let us determine these angles.

a) Show that the relation between the scattering angle in the lab and CM frames is,

$$\tan\theta_{\rm Lab} = \frac{p_{3\rm CM}\sin\theta_{\rm CM}}{\gamma\left(p_{3\rm CM}\cos\theta_{\rm CM} + \beta E_{3\rm CM}\right)} \tag{4.101}$$

with $\beta = p_{1\rm CM}/E_{2\rm CM}$ and $\gamma = 1/\sqrt{1-\beta^2}$.

b) In our case we consider elastic scattering so $p_{1\rm CM} = p_{3\rm CM}$ and Eq. (4.101) simplifies to

$$\tan\theta_{\rm Lab} = \frac{\sin\theta_{\rm CM}}{\gamma\left(\cos\theta_{\rm CM} + \frac{E_{3\rm CM}}{E_{2\rm CM}}\right)} \tag{4.102}$$

c) This relation allows us to determine $\theta^{\rm min}_{\rm Lab}$. It is to see that this angle is much smaller then in the `CM`, due to the Lorentz *boost* factor $1/\gamma$. To determine $\theta_{\rm Lab}$ we consider the derivative of $\tan\theta_{\rm Lab}$ in order to $\theta_{\rm Lab}$ and find where it vanishes. We get

$$(\tan\theta_{\rm Lab})' = \frac{1 + \frac{E_{3\rm CM}}{E_{2\rm CM}}\cos\theta_{\rm CM}}{\gamma\left(\cos\theta_{\rm CM} + \frac{E_{3\rm CM}}{E_{2\rm CM}}\right)^2} \tag{4.103}$$

which show that it vanishes for

$$\cos\theta_{\rm CM} = -\frac{E_{3\rm CM}}{E_{2\rm CM}} \tag{4.104}$$

Substituting in Eq. (4.101) we get after some algebra,

$$\tan^2\theta_{\rm Lab} = \frac{m_2^2}{E_{1\rm CM}^2 - E_{2\rm CM}^2} \tag{4.105}$$

where we have used $E_{2\rm CM} = \gamma m_2$. Finally we get from Eq. (4.105)

$$\sin\theta^{\rm max}_{\rm Lab} = \frac{m_2}{m_1} \tag{4.106}$$

If $m_2 > m_1$ there are no limitations and the maximum angle would be π. However if $m_1 > m_2$ there is a limitation given by Eq. (4.106). Notice that this relativistic result is the same as in non-relativistic mechanics.

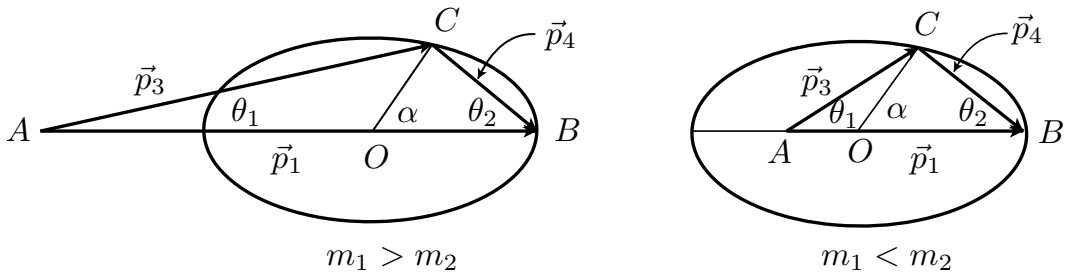
4.9 Consider the elastic collision of two particles of masses m_1 and m_2.

a) Show that the components of the momentum of the scattered particle, $\vec{p}_3$, in the lab frame obey the equation,

$$\frac{(p_3^y)^2}{p_{\rm CM}^2} + \frac{\left(p_3^z - \gamma p_{\rm CM}\frac{E_{3\rm CM}}{E_{2\rm CM}}\right)^2}{\left(\gamma p_{\rm CM}\right)^2} = 1 \tag{4.107}$$

where $\gamma^{-1} = \sqrt{1-\beta^2}$ with $\beta = p_{\rm CM}/E_{2\rm CM}$.

b) Use the previous result to justify the following construction for the momenta in the lab frame:



The point C runs along the ellipse with smaller semi-axis $p_{\rm CM}$ and larger semi-axis $\gamma p_{\rm CM}$.

c) Show that when $m_1 > m_2$ the angle $\theta_{\rm Lab}^{\rm max}$ given by Eq. (4.106) can also be obtained from Eq. (4.34) with the condition $B^2 = AC$. Interpret graphically this result.

d) Show that the relation between the angle α in the figure and the angle $\theta_{\rm CM}$ is,

$$\sin\alpha = \frac{\sin\theta_{\rm CM}}{\sqrt{\sin^2\theta_{\rm CM} + \gamma^2\cos^2\theta_{\rm CM}}}$$

Use this relation to show that when $\theta_{\rm CM}$ runs its domain of variation, $\theta_{\rm CM} \in [0, 2\pi]$, then also $\alpha \in [0, 2\pi]$. Interpret graphically this result.

e) Discuss what happens in the previous item and figure in the non-relativistic limit, that is, $\gamma \simeq 1$.

Chapter 5

Simple Examples in QED

We are going to study simple processes in `QED` using the techniques that we have learned. If we limit ourselves to two particles in the final state the number of processes is quite reduced. In table 5.1 there is a summary, where we also indicate where the process is studied.

Process	Comment	Section
e^-+ Nucleus(Z) $\to e^-$+ Nucleus(Z)	Coulomb scattering	5.1
$\gamma + e^- \to \gamma + e^-$	Compton scattering	5.2
$e^- + e^+ \to \mu^- + \mu^+$	In `QED`	5.3
$e^- + e^+ \to e^- + e^+$	Bhabha scattering	5.4 + Problems
$e^- + e^+ \to \gamma + \gamma$	Pair annihilation	5.5 + Problems
e^-+ Nucleus(Z) $\to e^-$+ Nucleus(Z) $+\gamma$	Bremsstrahlung	5.6
$e^- + e^- \to e^- + e^-$	Møller scattering	Problems
$\gamma + \gamma \to e^- + e^+$	Pair creation	Problems
γ+ Nucleus(Z) $\to$ Nucleus(Z) $+e^- + e^+$	Pair creation	Problems

Table 5.1: Simple processes in `QED`.

5.1 Coulomb Scattering

As a first example we are going to evaluate the cross section for Coulomb scattering. This is the scattering of an electron off the an heavy nucleus, represented by an external classical field. We have seen in Eq. (4.25) that the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2} |\mathcal{M}|^2 \tag{5.1}$$

where $\mathcal{M}$ is given by Eq. (3.103),

$$i\mathcal{M} = \overline{u}(p')(ie\gamma_\mu)u(p)A_c^\mu(\vec{q}) \tag{5.2}$$

and

$$A_c^\mu(\vec{q}) = \int d^3x\, e^{-i\vec{q}\cdot\vec{x}} A_c^\mu(\vec{x}). \tag{5.3}$$

The solid angle in Eq. (5.1) corresponds to the solid angle of the scattered electron, as shown in Fig. 5.1,

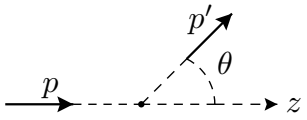


Figure 5.1: Kinematics for Coulomb scattering.

To proceed we consider the Coulomb field of the nucleus,

$$A_c^\mu(\vec{x}) = \left\{ \frac{Ze}{|4\pi\vec{x}|}, 0, 0, 0 \right\}. \tag{5.4}$$

Taking the Fourier transform of the Coulomb field (see Eq. (3.108)) we get,

$$A_c^0(\vec{q}) = \frac{Ze}{|\vec{q}|^2} \tag{5.5}$$

As we discussed in section 4.4 we normally do not measure the polarizations of the particles and therefore we are interested in the unpolarized cross section, summing over the spins of the final state and taking the average for the initial state.

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{1}{16\pi^2}\overline{|\mathcal{M}|^2} \tag{5.6}$$

For this problem we have two polarizations for the initial state and thus

$$\begin{aligned}
\overline{|\mathcal{M}|^2} =& \frac{1}{2}\sum_{\text{spins}} |\mathcal{M}|^2 \\
=& \frac{1}{2}\left(\frac{Ze}{|\vec{q}|^2}\right)^2 \sum_{\text{spins}} |\overline{u}(p')(ie\gamma^0)u(p)|^2 \qquad (5.7) \\
=& \frac{1}{2}\frac{Z^2e^4}{|\vec{q}|^4}\text{Tr}\left[(\not{p}' + m)\gamma^0(\not{p} + m)\gamma^0\right] \qquad (5.8)
\end{aligned}$$

Now using the results from the theorems on traces of Dirac γ matrices we get

$$\text{Tr}\left[(\not{p}' + m)\gamma^0(\not{p} + m)\gamma^0\right] = \text{Tr}\left[\not{p}'\gamma^0\not{p}\gamma^0\right] + m^2\text{Tr}\left[\gamma^0\gamma^0\right]$$

$$=8EE' - 4p\cdot p' + 4m^2 \tag{5.9}$$

It is usual to write the cross section in terms of the scattering angle θ. Then ($E = E'$),

$$p\cdot p' = E^2 - \mid \vec{p} \mid^2 \cos\theta = m^2 + 2\beta^2 E^2 \sin^2(\theta/2)\ , \tag{5.10}$$

and we get therefore

$$\frac{1}{2}\mathrm{Tr}\left[(\not{p}' + m)\gamma^0(\not{p} + m)\gamma^0\right] = 4E^2\left(1 - \beta^2\sin^2(\theta/2)\right) \tag{5.11}$$

and

$$\mid \vec{q} \mid^2 = 4 \mid \vec{p} \mid^2 \sin^2(\theta/2) \tag{5.12}$$

Putting everything together we obtain ($\beta = |\vec{p}|/E$),

$$\frac{d\overline{\sigma}}{d\Omega} = \frac{Z^2\alpha^2}{4 \mid \vec{p} \mid^2 \beta^2 \sin^4(\theta/2)}\left[1 - \beta^2\sin^2(\theta/2)\right] \tag{5.13}$$

which the so-called Mott [35] scattering cross section. In the limit $\beta \to 0$ it reduces to the Rutherford formula,

$$\frac{d\overline{\sigma}}{d\Omega} = \frac{Z^2\alpha^2 m^2}{4 \mid \vec{p} \mid^4 \sin^4(\theta/2)} \tag{5.14}$$

where we used $\beta \to 0$ in the parenthesis and $\beta^2 \to |\vec{p}|^2/m^2$, as in the non-relativistic limit $E \to m$.

5.2 Compton scattering

As a second and more difficult problem we are going to evaluate the differential cross section for Compton scattering.

5.2.1 The amplitudes

As we saw in section 3.4, we have two diagrams for Compton scattering, shown in Fig. 5.2.

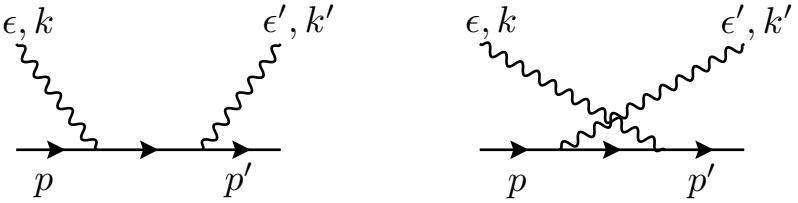


Figure 5.2: Diagrams for Compton scattering.

The total amplitude is

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 \tag{5.15}$$

where ($Q_e = -1$, $e > 0$)

$$i\,\mathcal{M}_1 = (ie)^2 \frac{i}{(p+k)^2 - m^2} \overline{u}(p')\gamma_\nu(\not{p} + \not{k} + m)\gamma_\mu u(p)\varepsilon^\mu(k)\varepsilon'^{\nu *}(k') \tag{5.16}$$

$$i\,\mathcal{M}_2 = (ie)^2 \frac{i}{(p-k')^2 - m^2} \overline{u}(p')\gamma_\mu(\not{p} - \not{k}' + m)\gamma_\nu u(p)\varepsilon^\mu(k)\varepsilon'^{\nu *}(k') \tag{5.17}$$

Therefore we can write, in a compact form,

$$\mathcal{M}_i \equiv -\overline{u}(p', s')\Gamma_i u(p, s) \tag{5.18}$$

where

$$\Gamma_1 = \frac{e^2}{2p\cdot k}\gamma_\nu(\not{p} + \not{k} + m)\gamma_\mu \varepsilon^\mu(k, \lambda)\varepsilon'^{\nu *}(k', \lambda') \tag{5.19}$$

$$\Gamma_2 = \frac{-e^2}{2p\cdot k'}\gamma_\mu(\not{p} - \not{k}' + m)\gamma_\nu \varepsilon^\mu(k, \lambda)\varepsilon'^{\nu *}(k', \lambda') \tag{5.20}$$

To evaluate the cross section we have calculate $|\mathcal{M}|^2$. Besides this, normally in most experimental situations the beams are not polarized and we do not measure the final state polarization. So we have to sum over the final state polarizations and take an average over the initial state to be able to compare with the experiment[1]. Therefore the quantity of interest is

$$\frac{1}{4}\sum_{s,s'}\sum_{\lambda,\lambda'} |\mathcal{M}|^2 \tag{5.21}$$

Let us first concentrate in the spin sums for the electrons. First we note that,

$$|\mathcal{M}|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + \mathcal{M}_1^\dagger \mathcal{M}_2 + \mathcal{M}_1 \mathcal{M}_2^\dagger \tag{5.22}$$

We start with the first term,

$$\begin{aligned} \sum_{s,s'} |\mathcal{M}_1|^2 &= \sum_{s,s'} \overline{u}(p', s')\Gamma_1 u(p, s) u^\dagger(p, s)\Gamma_1^\dagger \gamma^0 u(p', s') \\ &= \sum_{s,s'} \overline{u}(p', s')\Gamma_1 u(p, s)\overline{u}(p, s)\overline{\Gamma}_1 u(p', s') \end{aligned} \tag{5.23}$$

where, like in Eq. (4.56), we have defined,

$$\overline{\Gamma}_1 \equiv \gamma^0 \Gamma_1^\dagger \gamma^0 \tag{5.24}$$

[1] We will explain later how to handle polarized beams.

As we saw in the section 4.4.2 fermion spin sums can be transformed in traces of the γ involved. For this, as we have seen one uses the relations,

$$\sum_s u_\alpha(p,s)\overline{u}_\beta(p,s) = (\not{p}+m)_{\alpha\beta} \tag{5.25}$$

and

$$\sum_s v_\alpha(p,s)\overline{v}_\beta(p,s) = (\not{p}-m)_{\alpha\beta} \tag{5.26}$$

We get then for Eq. (5.23)

$$\sum_{s,s'} |\mathcal{M}_1|^2 = \mathrm{Tr}\left[(\not{p}'+m)\Gamma_1(\not{p}+m)\overline{\Gamma}_1\right] \tag{5.27}$$

For the other terms we collect here the results easily obtained,

$$\sum_{s,s'} |\mathcal{M}_2|^2 = \mathrm{Tr}\left[(\not{p}'+m)\Gamma_2(\not{p}+m)\overline{\Gamma}_2\right] \tag{5.28}$$

and

$$\begin{aligned}\sum_{s,s'}(\mathcal{M}_1\mathcal{M}_2^\dagger + \mathcal{M}_1^\dagger\mathcal{M}_2) &= \mathrm{Tr}\left[(\not{p}'+m)\Gamma_1(\not{p}+m)\overline{\Gamma}_2\right] \\ &\quad +\mathrm{Tr}\left[(\not{p}'+m)\Gamma_2(\not{p}+m)\overline{\Gamma}_1\right]\end{aligned} \tag{5.29}$$

For the photons the polarization sums are done using the relation (see Complement 4.3)

$$\sum_\lambda \varepsilon^\mu(k,\lambda)\varepsilon^{*\nu}(k,\lambda) = -g^{\mu\nu} + \text{terms proportional to } k \tag{5.30}$$

However gauge invariance as the consequence that the terms proportional to the photon momentum k do not contribute to the amplitude (see Complement 5.1), therefore in the following we always use the simplified relation

$$\sum_\lambda \varepsilon^\mu(k,\lambda)\varepsilon^{*\nu}(k,\lambda) = -g^{\mu\nu} \tag{5.31}$$

Relation in Eq. (5.31), together with the trace technique allows us to easily compute $\sum_{\text{spins}} |\mathcal{M}|^2$ for any process in QED. At this point one can ask the question what happens if we do have the beams polarized or measure the final state polarizations. For the case of the photon one have to write explicitly the expressions for $\varepsilon^\mu(k,\lambda)$ in the frame where we are doing the calculation. For the case of the electron, the best way to use the trace technique is to introduce a spin projector. Then we substitute,

$$u(p,s) \to \frac{1+\gamma_5\not{s}}{2}u(p,s) \tag{5.32}$$

where s^μ specifies the chosen polarization. After this we can reduce to traces. The only difference is that now the spin projector appears inside the trace, but as it is also a 4×4 matrix in Dirac space this really not a problem.

5.2.2 The Compton cross section

In the historic experiment of Compton, the electron is at rest in the lab frame. Therefore we have the kinematics,

$$\begin{aligned} p^\alpha &= (m, \vec{0}) \qquad p'^\alpha = (E', \vec{p}') \\ k^\alpha &= (k, 0, 0, k) \quad k'^\alpha = (k', k'\sin\theta, 0, k'\cos\theta) \end{aligned} \tag{5.33}$$

The formula for the differential cross section is then, Eq. (4.19)

$$d\sigma = \frac{1}{4mk}\,(2\pi)^4\delta^4(p+k-p'-k')\overline{|\mathcal{M}|^2}\frac{d^3p'}{(2\pi)^3 2p'^0}\frac{d^3k'}{(2\pi)^3 2k'^0} \tag{5.34}$$

Using the delta function we can make four of the six integrations. We get,

$$\frac{d\sigma}{d\Omega_{k'}} = \frac{1}{4mk}\frac{1}{(2\pi)^2}\int dk'\frac{k'^2}{2k'2E'}\delta(m+k-E'-k')\overline{|\mathcal{M}|^2} \tag{5.35}$$

To use the last delta function we have to realize that E' is constrained and related to k'. In fact from the $\delta^3(\vec{p}+\vec{k}-\vec{p}\,'-\vec{k}')$ function we have the constraint,

$$\vec{p}\,' = \vec{k} - \vec{k}' \tag{5.36}$$

therefore

$$E' = \sqrt{\vec{p}\,'^2 + m^2} = \sqrt{k^2 + k'^2 - 2kk'\cos\theta + m^2} \tag{5.37}$$

Then (see Complement 4.1)

$$\delta(m+k-E'-k') = \frac{\delta\left(k' - \frac{k}{1+\frac{k}{m}(1-\cos\theta)}\right)}{\left|1+\frac{dE'}{dk'}\right|} \tag{5.38}$$

and

$$\frac{dE'}{dk'} = \frac{k'-k\cos\theta}{E'} \tag{5.39}$$

or

$$\begin{aligned} \left|1+\frac{dE'}{dk'}\right| &= \frac{|E'+k'-k\cos\theta|}{E'} = \frac{m+k(1-\cos\theta)}{E'} \\ &= \frac{m}{E'}\frac{k}{k'} \end{aligned} \tag{5.40}$$

Putting everything together we get,

$$\frac{d\sigma}{d\Omega_{k'}} = \frac{1}{64\pi^2}\frac{1}{m^2}\left(\frac{k'}{k}\right)^2\overline{|\mathcal{M}|^2} \tag{5.41}$$

where

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\sum_{s,s'}\sum_{\lambda,\lambda'}|\mathcal{M}|^2 \tag{5.42}$$

for the unpolarized cross section. To proceed we have to evaluate the traces. These are not trivial as they involve a maximum of eight γ matrices. We will learn in chapter 7 ways to this automatically with dedicated software, but here we do it *by hand*. We get

$$\begin{aligned}
\overline{|\mathcal{M}_1|^2} &= \frac{1}{4}\mathrm{Tr}\left[(\not{p}'+m)\gamma_\nu(\not{p}+\not{k}+m)\gamma_\mu(\not{p}+m)\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu\right]\frac{e^4}{(2p\cdot k)^2}\\
&= \frac{1}{4}\mathrm{Tr}\left[(-2\not{p}'+4m)(\not{p}+\not{k}+m)(-2\not{p}+4m)(\not{p}+\not{k}+m)\right]\frac{e^4}{(2p\cdot k)^2}\\
&= \{\mathrm{Tr}\left[\not{p}'(\not{p}+\not{k}+m)\not{p}(\not{p}+\not{k}+m)\right]-2\ m\mathrm{Tr}\left[\not{p}'(\not{p}+\not{k}+m)(\not{p}+\not{k}+m)\right]\\
&-2m\mathrm{Tr}\left[(\not{p}+\not{k}+m)\not{p}(\not{p}+\not{k}+m)\right]+4m^2\mathrm{Tr}\left[(\not{p}+\not{k}+m)(\not{p}+\not{k}+m)\right]\}\frac{e^4}{(2p\cdot k)^2}\\
&= \left\{4m^2p\cdot p'+\mathrm{Tr}\left[\not{p}'(\not{p}+\not{k})\not{p}(\not{p}+\not{k})\right]-4m^2\mathrm{Tr}\left[\not{p}'(\not{p}+\not{k})\right]\right.\\
&\left.-4m^2\mathrm{Tr}\left[\not{p}(\not{p}+\not{k})\right]+16m^2(2m^2+2p\cdot k)\right\}\frac{e^4}{(2p\cdot k)^2}\\
&= \left\{4m^2p\cdot p'+8(p'\cdot p+p'\cdot k)(m^2+p\cdot k)-4p\cdot p'(2m^2+2p\cdot k)\right.\\
&\left.-16m^2(p'\cdot p+p'\cdot k)-16m^2(m^2+p\cdot k)+16m^2(2m^2+2p\cdot k)\right\}\frac{e^4}{(2p\cdot k)^2}\\
&= 8\left[2\ m^4+m^2(-p\cdot p'-p'\cdot k+2p\cdot k)+(p\cdot k)(p'\cdot k)\right]\frac{e^4}{(2p\cdot k)^2}
\end{aligned}\tag{5.43}$$

In a similar way

$$\overline{|\mathcal{M}_2|^2} = 8\left[2m^4+m^2(-p\cdot p'+p'\cdot k'-2p\cdot k')+(p\cdot k')(p'\cdot k')\right]\frac{e^4}{(2p\cdot k')^2} \tag{5.44}$$

and

$$\begin{aligned}
\overline{[\mathcal{M}_1\mathcal{M}_2^\dagger+\mathcal{M}_1^\dagger\mathcal{M}_2]} &= \frac{8e^4}{4(k\cdot p)(k'\cdot p)}\left[2(k\cdot p)(p\cdot p')-2(k\cdot k')(p\cdot p')-2(p\cdot p')(p\cdot k')\right.\\
&\left.+m^2(-2k\cdot p-k\cdot p'+k\cdot k'-p\cdot p'+2p\cdot k'+p'\cdot k')-m^4\right]
\end{aligned}\tag{5.45}$$

Finally summing everything and using the lab kinematics,

$$p' = p+k-k' \qquad p\cdot k = mk \tag{5.46}$$

$$p\cdot k' = mk' \qquad k\cdot k' = kk'(1-\cos\theta) = m(k-k') \tag{5.47}$$

we can write,

$$\frac{1}{4}\sum_{s,s'}\sum_{\lambda,\lambda'}\{|\mathcal{M}_1|^2+|\mathcal{M}_2|^2+\mathcal{M}_1\mathcal{M}_2^\dagger+\mathcal{M}_1^\dagger\mathcal{M}_2\}=2e^4\left[\left(\frac{k}{k'}\right)+\left(\frac{k'}{k}\right)-\sin^2\theta\right] \tag{5.48}$$

and we get the famous Klein-Nishina [36] formula for the differential cross section of the Compton scattering[2]

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2}{2\ m^2}\left(\frac{k'}{k}\right)^2\left[\left(\frac{k'}{k}\right)+\left(\frac{k}{k'}\right)-\sin^2\theta\right]. \tag{5.49}$$

In practice there are algebraic programs that are very useful to evaluate the traces automatically. Nowadays it is quite common to use the program `FeynCalc` [37–39] which is a software package for `Mathematica`. To give an example of the use of this package we give in section 7.10 the code necessary to evaluate the following quantity,

$$\texttt{ANS}=\frac{1}{4}\sum_{s,s'}\sum_{\lambda,\lambda'}\left[|\mathcal{M}_1|^2+|\mathcal{M}_2|^2+\mathcal{M}_1\mathcal{M}_2^\dagger+\mathcal{M}_1^\dagger\mathcal{M}_2^2\right]\ . \tag{5.50}$$

which is relevant for Compton effect. We recommend the reader to compare the time needed doing by hand and with the computer. All software codes described in this book can be obtained in my web page [37].

5.3 Scattering $e^-e^+\to\mu^-\mu^+$

Let us consider now the process $e^-(p_1)+e^+(p_2)\to\mu^-(p_3)+\mu^+(p_4)$ in QED (more precisely QED for charged leptons). We will see in Chapter 8 that when we consider the Standard Model of the electroweak interactions this process will be more complicated and that QED is only a part of the whole picture. We will see there that the QED result is a good approximation for CM energies much below the mass of the Z^0 boson. In QED we have only one diagram shown in Fig. 5.3 contributing to the process.

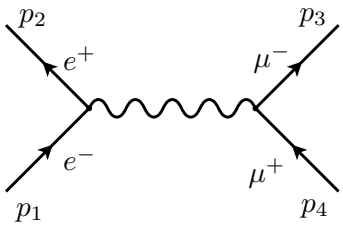


Figure 5.3: Scattering $e^-e^+\to\mu^-\mu^+$ in QED.

[2] In fact the Klein-Nishina formula is for polarized photons. Eq. (5.49) is the non polarized limit (see Problem 5.11).

The Feynman rules lead us to the following amplitude,

$$\begin{aligned} i\,\mathcal{M} &= \overline{v}(p_2)(ie\gamma^\mu)u(p_1)\ \frac{-i\ g_{\mu\nu}}{(p_1+p_2)^2+i\varepsilon}\ \overline{u}(p_3)(ie\gamma^\nu)v(p_4) \\ &= ie^2\frac{1}{(p_1+p_2)^2+i\varepsilon}\ \overline{v}(p_2)\gamma^\mu u(p_1)\ \overline{u}(p_3)\gamma_\mu v(p_4) \end{aligned} \tag{5.51}$$

5.3.1 Evaluation using traces

We are going to evaluate the average over the initial and sum over final polarizations (spins) using the trace technique, the so-called Casimir's trick. We get,

$$\begin{aligned} \frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 =& \frac{e^4}{4(p_1+p_2)^4}\mathrm{Tr}\left[(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1+m_e)\gamma^\nu\right]\mathrm{Tr}\left[(\not{p}_3+m_\mu)\gamma_\mu(\not{p}_4-m_\mu)\gamma_\nu\right] \\ =& \frac{8e^4}{(p_1+p_2)^4}\left[(p_1\cdot p_2)m_\mu^2 + (p_1\cdot p_3)(p_2\cdot p_4)\right. \\ & \left. +(p_1\cdot p_4)(p_2\cdot p_3) + (p_3\cdot p_4)m_e^2 + 2m_e^2m_\mu^2\right] \end{aligned} \tag{5.52}$$

and we get for the scattering cross section,

$$\sigma = \int \frac{1}{4\sqrt{(p_1\cdot p_2)^2 - m_e^4}}\overline{|\mathcal{M}|^2}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\ \prod_{i=3}^{4}\frac{d^3p_i}{(2\pi)^3 2p_i^0}. \tag{5.53}$$

To continue, we are going to neglect the electron mass but not the mass of the muon. In this way the final result can be also applied to the produced of pairs of any charged fermion in the scattering e^+e^- in QED[3]. We also consider the scattering to take place in the CM frame. With these conventions the kinematics is

$$\begin{aligned} p_1 =& \frac{\sqrt{s}}{2}\ (1,0,0,1) & p_2 =& \frac{\sqrt{s}}{2}\ (1,0,0,-1) \\ p_3 =& \frac{\sqrt{s}}{2}\ (1,\beta\sin\theta,0,\beta\cos\theta) & p_4 =& \frac{\sqrt{s}}{2}\ (1,-\beta\sin\theta,0,-\beta\cos\theta) \end{aligned} \tag{5.54}$$

where $s = (p_1+p_2)^2$ is the square of the CM energy, θ is the scattering angle of the μ^- with respect to the direction of the incident e^- and β is the velocity of the produced muons in that frame.

$$\beta = \sqrt{1-\frac{4m_\mu^2}{s}}. \tag{5.55}$$

To write Eq. (5.54) we chose the scattering plane to be the xz plane. This can be done without loss of generality as there is azimuthal symmetry with respect to the collision axis.

[3]With the exception of the production of an e^+e^- pair, as in that case we have two diagrams. We will study this process separately in section 5.4.

With these conventions we get for the differential cross section for the production of a μ^- in the solid angle Ω (see Problem 4.1),

$$\begin{aligned}\frac{d\sigma}{d\Omega} =& \frac{1}{64\pi^2 s}\,\frac{|\vec{p}_{3\mathrm{CM}}|}{|\vec{p}_{1\mathrm{CM}}|}\overline{|\mathcal{M}|^2} \\ =& \frac{1}{32\pi^2 s}\,\frac{|\vec{p}_3|}{\sqrt{s}}\overline{|\mathcal{M}|^2} = \frac{1}{64\pi^2 s}\,\beta\overline{|\mathcal{M}|^2} \\ =& \frac{\alpha^2}{4s}\,\beta\left(\beta^2\cos^2\theta + 1 + \frac{4m_\mu^2}{s}\right) \\ =& \frac{\alpha^2}{4s}\,\beta\left[1+\cos^2\theta + (1-\beta^2)\sin^2\theta\right]\,. \end{aligned} \tag{5.56}$$

Note that in the limit $m_\mu \to 0$, the differential cross section as a behavior as $1+\cos^2\theta$. Integrating over the solid angle we get the total cross section,

$$\sigma = \frac{2\pi\alpha^2}{3s}\,\beta(3-\beta^2) \tag{5.57}$$

and in the relativistic limit, $\beta \to 1$,

$$\sigma = \frac{4\pi\alpha^2}{3s}\,. \tag{5.58}$$

5.3.2 Helicity using helicity spinors

To understand better the angular dependence of the differential cross section we are now going to use the helicity spinors of section 1.8.2 in the limit of massless fermions.

Using the results of Eq. (1.239) we get for the initial state,

$$u_\uparrow(p_1) = \sqrt{E}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, u_\downarrow(p_1) = \sqrt{E}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}, v_\uparrow(p_2) = \sqrt{E}\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, v_\downarrow(p_2) = \sqrt{E}\begin{bmatrix}0\\-1\\0\\-1\end{bmatrix} \tag{5.59}$$

where we have used $p_1(0,0)$ and $p_2(\pi,\pi)$ with the notation $p_i(\theta,\phi)$. In a similar way for the final state we have, $p_3(\theta,0), p_4(\pi-\theta,\pi)$ and we get,

$$u_\uparrow(p_3) = \sqrt{E}\begin{bmatrix}c\\s\\c\\s\end{bmatrix}, u_\downarrow(p_3) = \sqrt{E}\begin{bmatrix}-s\\c\\s\\-c\end{bmatrix}, v_\uparrow(p_4) = \sqrt{E}\begin{bmatrix}c\\s\\-c\\-s\end{bmatrix}, v_\downarrow(p_4) = \sqrt{E}\begin{bmatrix}s\\-c\\s\\-c\end{bmatrix} \tag{5.60}$$

with the simplified notation $c = \cos(\theta/2), s = \sin(\theta/2)$. Let us start by the muon current,

$$J^\alpha_{\mathrm{muon}} = \overline{u}(p_3)\gamma^\alpha v(p_4) \Rightarrow J^0_{\mathrm{muon}} = u^\dagger(p_3)v(p_4), J^i_{\mathrm{muon}} = u^\dagger(p_3)\alpha^i v(p_4) \tag{5.61}$$

where

$$\alpha^i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} \tag{5.62}$$

We then get,

$$J^0_{\text{muon}}(\uparrow,\uparrow) = E \begin{bmatrix} c & s & c & s \end{bmatrix} \begin{bmatrix} c \\ s \\ -c \\ s \end{bmatrix} = 0 \tag{5.63}$$

$$J^1_{\text{muon}}(\uparrow,\uparrow) = E \begin{bmatrix} c & s & c & s \end{bmatrix} \begin{bmatrix} -s \\ -c \\ s \\ c \end{bmatrix} = 0 \tag{5.64}$$

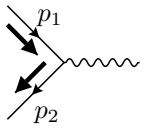
$$J^2_{\text{muon}}(\uparrow,\uparrow) = E \begin{bmatrix} c & s & c & s \end{bmatrix} \begin{bmatrix} is \\ -ic \\ -is \\ ic \end{bmatrix} = 0 \tag{5.65}$$

$$J^3_{\text{muon}}(\uparrow,\uparrow) = E \begin{bmatrix} c & s & c & s \end{bmatrix} \begin{bmatrix} -c \\ s \\ c \\ -s \end{bmatrix} = 0 \tag{5.66}$$

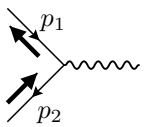
and therefore $J^\alpha_{\text{muon}}(\uparrow,\uparrow) = 0$. In a similar way we can calculate the other combinations. Using a suggestive notation we obtain,

$$\begin{aligned} J_{u_3v_4}(\uparrow,\uparrow) =& 0 \\ J_{u_3v_4}(\uparrow,\downarrow) =& \sqrt{s}\ (0, -\cos\theta, i, \sin\theta) \\ J_{u_3v_4}(\downarrow,\uparrow) =& \sqrt{s}\ (0, -\cos\theta, -i, \sin\theta) \\ J_{u_3v_4}(\downarrow,\downarrow) =& 0 \end{aligned} \tag{5.67}$$

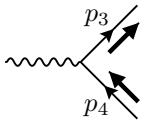
where $\sqrt{s} = 2E$. In a similar way we can get the electron current. The results can be summarized as follows. The non vanishing currents are (see Problem 5.5)



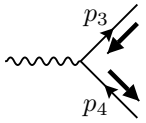
$$J_{u_1v_2}(\uparrow,\downarrow) = \sqrt{s}\ (0, -1, -i, 0) \tag{5.68}$$



$$J_{u_1v_2}(\downarrow,\uparrow) = \sqrt{s}\ (0, -1, i, 0) \tag{5.69}$$



$$J_{u_3v_4}(\uparrow,\downarrow) = \sqrt{s}\ (0, -\cos\theta, i, \sin\theta) \tag{5.70}$$



$$J_{u_3v_4}(\downarrow,\uparrow) = \sqrt{s}\ (0, -\cos\theta, -i, \sin\theta) \tag{5.71}$$

Therefore we get from Eq. (5.51),

$$\begin{aligned}\mathcal{M}(\uparrow\downarrow;\uparrow\downarrow) =& \frac{e^2}{s}\left[\sqrt{s}(0,-1,-i,0)\right]\cdot\left[\sqrt{s}(0,-\cos\theta,i,\sin\theta)\right] \\ =& -\frac{e^2}{s}s\,(1+\cos\theta) \equiv -4\pi\alpha\,(1+\cos\theta)\end{aligned} \tag{5.72}$$

and in a similar way,

$$|\mathcal{M}(\uparrow\downarrow;\uparrow\downarrow)|^2 = |\mathcal{M}(\downarrow\uparrow;\downarrow\uparrow)|^2 = (4\pi\alpha)^2\,(1+\cos\theta)^2 \tag{5.73}$$

$$|\mathcal{M}(\uparrow\downarrow;\downarrow\uparrow)|^2 = |\mathcal{M}(\downarrow\uparrow;\uparrow\downarrow)|^2 = (4\pi\alpha)^2\,(1-\cos\theta)^2 \tag{5.74}$$

Then

$$\overline{|\mathcal{M}_{fi}|^2} = \frac{1}{4}\,(4\pi\alpha)^2\left[2(1+\cos\theta)^2 + 2(1-\cos\theta)^2\right] \tag{5.75}$$

$$= (4\pi\alpha)^2\,(1+\cos^2\theta) \tag{5.76}$$

and the differential cross section will be,

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\overline{|\mathcal{M}|^2} = \frac{\alpha^2}{4s}(1+\cos^2\theta) \tag{5.77}$$

in agreement with Eq. (5.56) in the relativistic limit $\beta \to 1$. The result is shown in Fig. 5.4 where we also show the experimental results from the JADE experiment,

To understand the result we note that the only non-zero amplitudes are those for which the projection adds to ± 1, as shown in Fig. 5.5. Now for spin 1 in the θ direction we have (see Problem 5.7)

$$|1,+1\rangle_\theta = \frac{1}{2}(1-\cos\theta)\,|1,-1\rangle_z + \frac{1}{\sqrt{2}}\sin\theta)\,|1,0\rangle_z + \frac{1}{2}(1+\cos\theta)\,|1,1\rangle_z \tag{5.78}$$

and therefore

$$\mathcal{M}(\uparrow\downarrow;\uparrow\downarrow) \propto {}_\theta\langle 1,+1|1,+1\rangle_z = \frac{1}{2}(1+\cos\theta) \tag{5.79}$$

$$\mathcal{M}(\downarrow\uparrow;\uparrow\downarrow) \propto {}_\theta\langle 1,+1|1,-1\rangle_z = \frac{1}{2}(1-\cos\theta) \tag{5.80}$$

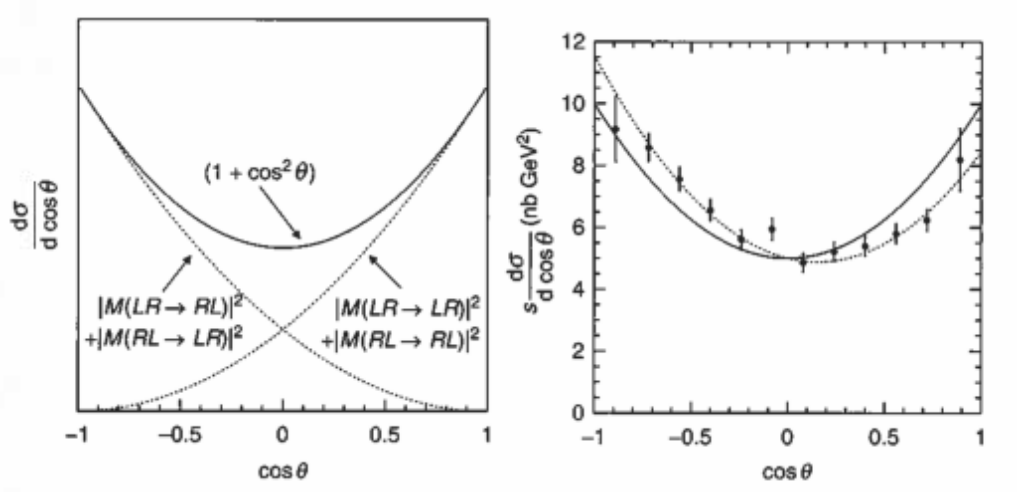


Figure 5.4: Behavior the differential cross section with the scattering angle and results from the JADE experiment (Bartel et al. (1985). The solid curve is the QED prediction while the dotted line also includes the electroweak contributions.

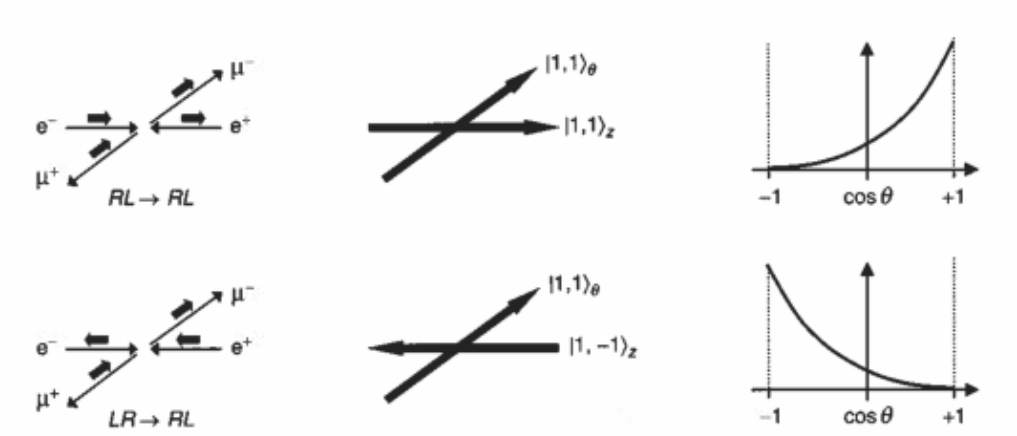


Figure 5.5: Spin projections.

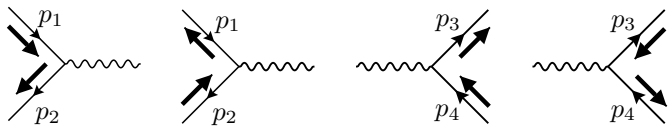


Figure 5.6: Non-zero amplitudes and the directions of the spin arrows.

We realize that the non-vanishing amplitudes do not change the direction of the spin arrow. This is due to the fact that the QED interaction preserver chirality as we saw in the Chapter 1, and in the limit of massless fermions chirality is equal to helicity.

5.4 Bhabha scattering ($e^-e^+ \to e^-e^+$)

5.4.1 Evaluation using traces

Let us now consider the process $e^-e^+ \to e^-e^+$ known as *Bhabha scattering* [40]. In QED there are two diagrams contributing to this process, and there is a relative

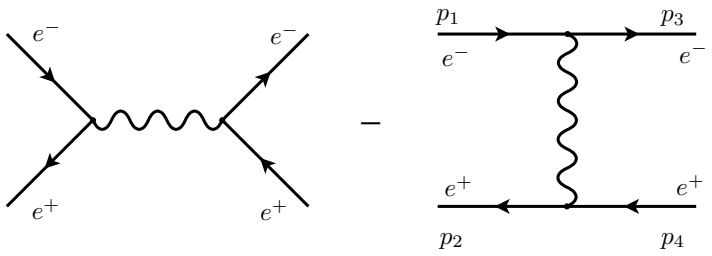


Figure 5.7: Bhabha scattering.

minus sign between the two diagrams. This minus sign can be understood with the help of Wick's theorem. We will explain in Chapter 7 how to use dedicated software, like QGRAF [41] to get the diagrams and the relative sign. In section 7.9 we can look at the input and output code for this case and verify the relative sign. Using the Feynman rules for QED we can write the expression for the amplitudes,

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 \tag{5.81}$$

where

$$i\,\mathcal{M}_1 = i\,\frac{e^2}{s}\overline{v}(p_2)\gamma^\mu u(p_1)\overline{u}(p_3)\gamma_\mu v(p_4), \quad i\,\mathcal{M}_2 = -i\,\frac{e^2}{t}\overline{u}(p_3)\gamma^\mu u(p_1)\overline{v}(p_2)\gamma_\mu v(p_4) \tag{5.82}$$

and

$$s = (p_1+p_2)^2, \qquad t = (p_1-p_3)^2 \tag{5.83}$$

The variables s, t are two of the Mandelstam variables which are very important in processes $1+2 \to 3+4$ in the `CM`, and that were discussed in section 4.3.3.

We are going to do the calculation in the limit where $\sqrt{s} \gg m_e$, and therefore we can neglect the mass of the electron. The average over initial and sum over the final state polarizations gives[4],

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_1+\mathcal{M}_2|^2 &= \frac{e^4}{4}\left\{\frac{1}{t^2}\mathrm{Tr}\left[\not{p}_3\gamma^\mu\not{p}_1\gamma^\nu\right]\mathrm{Tr}\left[\not{p}_2\gamma_\mu\not{p}_4\gamma_\nu\right]\right. \\ &\left. +\frac{1}{s^2}\mathrm{Tr}\left[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu\right]\mathrm{Tr}\left[\not{p}_3\gamma_\mu\not{p}_4\gamma_\nu\right] - \frac{2}{st}\mathrm{Tr}\left[\not{p}_3\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2\gamma_\mu\not{p}_4\gamma_\nu\right]\right\}\end{aligned} \tag{5.84}$$

[4]The two traces coming from the interference are equal, and therefore the factor of 2 in Eq. (5.84).

After some algebra we get

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_1+\mathcal{M}_2|^2 = 2e^4\left[\frac{t^2+(s+t)^2}{s^2}+\frac{s^2+(s+t)^2}{t^2}+2\frac{(s+t)^2}{st}\right] \tag{5.85}$$

which can also be obtained, more easily, with the help of dedicated software as explained in section 7.9.2. We the get for the differential cross section,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\left[\frac{t^2+(s+t)^2}{s^2}+\frac{s^2+(s+t)^2}{t^2}+2\frac{(s+t)^2}{st}\right] . \tag{5.86}$$

5.5 Pair annihilation

The next process we consider is the so-called pair annihilation, $e^-e^+ \to \gamma\gamma$ in QED, in the limit of massless fermions. We have the two diagrams of Fig. 5.8 with a relative plus sign, because the particles that are exchanged between the two diagrams are boson (the photons). The amplitudes are

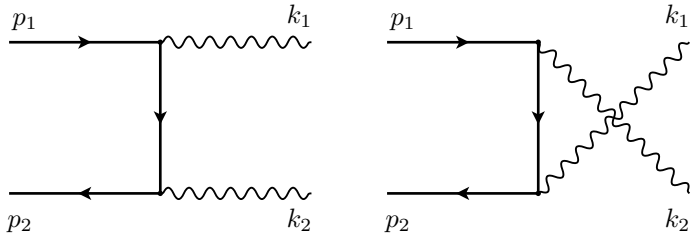


Figure 5.8: Diagrams for $e^-e^+ \to \gamma\gamma$

$$\begin{aligned}\mathcal{M}_1 &= -e^2\,\overline{v}(p_2)\not{\epsilon}^*(k_2)\,(\not{p}_1-\not{k}_1)\not{\epsilon}^*(k_1)u(p_1)\,\frac{1}{t}\\ \mathcal{M}_2 &= -e^2\,\overline{v}(p_2)\not{\epsilon}^*(k_1)\,(\not{p}_1-\not{k}_2)\not{\epsilon}^*(k_2)u(p_1)\,\frac{1}{u}\end{aligned} \tag{5.87}$$

where $t=(p_1-k_1)^2$ e $u=(p_1-k_2)^2$. Using the trace technique we get,

$$\begin{aligned}\sum_{\text{spins}}\sum_{\lambda_1,\lambda_2}|\mathcal{M}_1|^2 =& \frac{e^4}{t^2}(-g_{\mu\mu'})(-g_{\nu\nu'})\text{Tr}\left[\not{p}_2\gamma^\nu(\not{p}_1-\not{k}_1)\gamma^\mu\not{p}_1\gamma^{\mu'}(\not{p}_1-\not{k}_1)\gamma^{\nu'}\right]\\ =& 8e^4\frac{u}{t}\end{aligned} \tag{5.88}$$

$$\sum_{\text{spins}}\sum_{\lambda_1,\lambda_2}|\mathcal{M}_2|^2 = \frac{e^4}{u^2}(-g_{\mu\mu'})(-g_{\nu\nu'})\text{Tr}\left[\not{p}_2\gamma^\mu(\not{p}_1-\not{k}_2)\gamma^\nu\not{p}_1\gamma^{\nu'}(\not{p}_1-\not{k}_2)\gamma^{\mu'}\right]$$

$$= 8e^4 \frac{t}{u} \tag{5.89}$$

$$\sum_{\text{spins}} \sum_{\lambda_1,\lambda_2} \mathcal{M}_1 \mathcal{M}_2^\dagger = \frac{e^4}{ut}(-g_{\mu\mu'})(-g_{\nu\nu'}) \text{Tr}\left[\not{p}_2 \gamma^\nu (\not{p}_1 - \not{k}_1)\gamma^\mu \not{p}_1 \gamma^{\nu'} (\not{p}_1 - \not{k}_2)\gamma^{\mu'}\right]$$

$$= -8e^4 \frac{s}{ut}(s+t+u) = 0 \tag{5.90}$$

$$\sum_{\text{spins}} \sum_{\lambda_1,\lambda_2} \mathcal{M}_2 \mathcal{M}_1^\dagger = \frac{e^4}{ut}(-g_{\mu\mu'})(-g_{\nu\nu'}) \text{Tr}\left[\not{p}_2 \gamma^\mu (\not{p}_1 - \not{k}_2)\gamma^\nu \not{p}_1 \gamma^{\mu'} (\not{p}_1 - \not{k}_1)\gamma^{\nu'}\right]$$

$$= -8e^4 \frac{s}{ut}(s+t+u) = 0 \tag{5.91}$$

where in the last two equations we have used Eq. (4.51) for massless particles. Putting everything together we finally obtain,

$$\sum_{\text{spins}} \sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 = 8e^4 \frac{u}{t} + 8e^4 \frac{t}{u} = 8e^4 \frac{u^2+t^2}{ut} \ . \tag{5.92}$$

5.6 Bremsstrahlung

Bremsstrahlung (*braking radiation*) corresponds to the emission of one or more photons associated with the charged fermions of a given process. The interest of this process is that the cross section diverges as $\frac{dk}{k}$ when $k \to 0$ which means that it is more and more probable to emit photons as the energy of these photons becomes lower and lower. To understand how this divergence, known as *infrared divergence*, disappears in physical processes, we will consider the simplest case that is bremsstrahlung in Coulomb scattering. The diagrams are represented in Fig. 5.9.

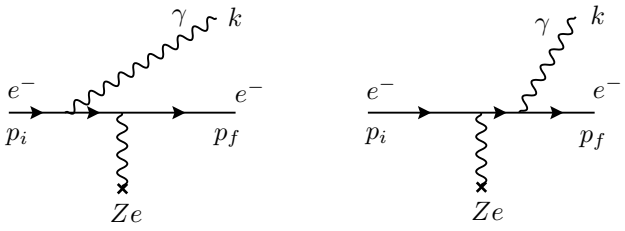


Figure 5.9: Bremsstrahlung in the Coulomb field.

The amplitude $\mathcal{M}$ is then given by,

$$i\,\mathcal{M}_{fi} \;=\; \frac{Ze}{|\vec{q}|^2}\overline{u}(p_f, s_f)\left[(ie\gamma^0)i\frac{(\not{p}_i - \not{k} + m)}{-2p_i \cdot k}(ie\not{\epsilon}^*)\right.$$

$$+(ie\not{\epsilon}^*)i\frac{(\not{p}_f+\not{k}+m)}{2p_f\cdot k}(ie\gamma^0)\Big]\,u(p_i,s_i) \qquad (5.93)$$

or

$$\mathcal{M}_{fi}=-\frac{Ze^3}{|\vec{q}|^2}\overline{u}(p_f)\left[\gamma^0\frac{\not{p}_i-\not{k}+m}{-2p_i\cdot k}\not{\epsilon}^*+\not{\epsilon}^*\frac{(\not{p}_f+\not{k}+m)}{2p_f\cdot k}\gamma^0\right]u(p_i) \qquad (5.94)$$

5.6.1 The Bethe-Heitler formula.

Let us first evaluate the cross section without assuming that the energy of the photon is small. We will get what is called the Bethe-Heitler formula. For this it is convenient to write,

$$\mathcal{M}_{fi}=\frac{Ze^3}{|\vec{q}|^2}\overline{u}(p_f,s_f)\Gamma u(p_i,s_i) \qquad (5.95)$$

Then

$$\frac{1}{2}\sum_{s_f,s_i}|\mathcal{M}|^2=\frac{64\pi^3(Z^2\alpha^3)}{2|\vec{q}|^4}\mathrm{Tr}\left[(\not{p}_f+m)\Gamma(\not{p}_i+m)\overline{\Gamma}\ \right] \qquad (5.96)$$

Now we sum over the polarizations of the photon to get,

$$\begin{aligned}
&\sum_{\lambda,\lambda'}\mathrm{Tr}\left[(\not{p}_f+m)\Gamma(\not{p}_i+m)\overline{\Gamma}\right]\\
&=\frac{4}{\omega^2}\Bigg\{-\omega^2\frac{D_i}{D_f}-\omega^2\frac{D_f}{D_i}+m^2\omega\frac{D_i}{D_f^2}-m^2\omega\frac{D_f}{D_i^2}+\frac{m^2}{D_f^2}(p_i\cdot p_f-m^2-2\omega E_i-2E_iE_f)\\
&\quad+\frac{m^2}{D_i^2}(p_i\cdot p_f-m^2+2E_f\omega-2E_iE_f)+\frac{\omega}{D_f}(-2p_i\cdot p_f-m^2+2\omega E_i+2E_f^2+2E_fE_i)\\
&\quad+\frac{\omega}{D_i}(2p_i\cdot p_f+m^2+2\omega E_f-2E_fE_i-2E_i^2) \qquad (5.97)\\
&\quad+\frac{2}{D_iD_f}\left[-(p_i\cdot p_f)^2+m^2(p_i\cdot p_f)+\omega(E_i-E_f)p_i\cdot p_f+2p_i\cdot p_fE_fE_i-m^2\omega^2\right]\Bigg\}
\end{aligned}$$

where ω is the photon energy and D_i, D_f are

$$\begin{aligned}
D_i &= E_i-p_i\cos\theta_i\\
D_f &= E_f-p_f\cos\theta_f\,, \qquad (5.98)
\end{aligned}$$

where the angles are defined in Fig. 5.10. Bethe and Heitler [42] evaluated this process for the first time and were able to express the result in a very compact form,

$$\begin{aligned}
&\sum_{\lambda,\lambda'}\mathrm{Tr}\left[(\not{p}_f+m)\Gamma(\not{p}_i+m)\overline{\Gamma}\right]=\frac{2}{\omega^2}\Bigg\{\frac{p_f^2\sin^2\theta_f}{D_f^2}(4E_i^2-q^2)+\frac{p_i^2\sin^2\theta_i}{D_i^2}(4E_f^2-q^2)\\
&+\frac{2\omega^2}{D_iD_f}(p_i^2\sin^2\theta_j+p_f^2\sin^2\theta_f)-\frac{2p_ip_f}{D_iD_f}\sin\theta_i\sin\theta_f\cos\varphi(2E_i^2+2E_f^2-q^2)\Bigg\} \qquad (5.99)
\end{aligned}$$

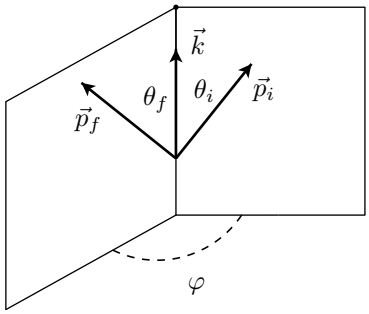


Figure 5.10: Definition of the geometry of Eq. (5.97).

where

$$q^2 = \left(\vec{p}_i - \vec{p}_f - \vec{k}\right)^2 \ . \tag{5.100}$$

Then the expression for the differential cross section is (see Complement 5.2)

$$\frac{d\sigma}{d\Omega_\gamma d\Omega_e} = \frac{Z^2\alpha^3}{(2\pi)^2}\frac{p_f}{p_i q^4}\frac{d\omega}{\omega}\left\{\cdots\right\} \ . \tag{5.101}$$

where the expression inside the parenthesis $\{\cdots\}$ does not vanish in the limit $\omega \to 0$.

This expression show the logarithmic divergence $\dfrac{d\omega}{\omega}$ when $\omega \to 0$. To arrive from Eq. (5.97) to Eq. (5.99) is not trivial although it is stated in many book [32, 43]. This is due to the fact that the variables in Eq. (5.99) are not all independent. In my web site [37] you can find a program where we evaluate Eq. (5.97) and show its equivalence to Eq. (5.99).

5.6.2 The *soft photon* limit

To analyze better this divergence, that can be understood by the fact that the photon is massless, we are going to study the limit $k \to 0$ of Eq. (5.94). In this limit we have,

$$\begin{aligned}
\lim_{k\to 0} &= \overline{u}(p_f)\gamma^0(\not{p}_i - \not{k} + m)\not{\varepsilon}^* u(p_i) \\
&= \overline{u}(p_f)\gamma^0(\not{p}_i + m)\not{\varepsilon}^* u(p_i) \\
&= \overline{u}(p_f)\gamma^0[2\varepsilon^*\cdot p_i - \not{\varepsilon}^*(\not{p}_i - m)]u(p_i) \\
&= \overline{u}(p_f)\gamma^0 u(p_i)2\varepsilon^*\cdot p_i
\end{aligned} \tag{5.102}$$

where we have used the Dirac equation $(\not{p}_i - m)u(p_i) = 0$. In the same way and in the same limit,

$$\lim_{k\to 0}\overline{u}(p_f)\not{\varepsilon}^*(\not{p}_f + \not{k} + m)\gamma^0 u(p_i) = \overline{u}(p_f)\gamma^0 u(p_i)2\varepsilon^*\cdot p_f \tag{5.103}$$

Using these two expression we get,

$$\mathcal{M} = -\frac{Ze^2}{|\vec{q}|^2}\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)\left(\frac{\varepsilon^* \cdot p_f}{k \cdot p_f} - \frac{\varepsilon^* \cdot p_i}{k \cdot p_i}\right). \tag{5.104}$$

This shows that, in this limit, the bremsstrahlung cross section is proportional to the elastic cross section that we derived in section 5.1. In fact we get easily,

$$\lim_{k\to 0}\left(\frac{d\sigma}{d\Omega_e}\right)_{\rm BR} \simeq \left(\frac{d\sigma}{d\Omega_e}\right)_{\rm elastic} \frac{e^2}{2\omega(2\pi)^3}\omega^2 d\omega d\Omega_\gamma \left|\frac{\varepsilon^* \cdot p_f}{k \cdot p_f} - \frac{\varepsilon^* \cdot p_i}{k \cdot p_i}\right|^2 \tag{5.105}$$

As $k \cdot p_f$ and $k \cdot p_i$ are proportional to ω the behavior is indeed as $\frac{d\omega}{\omega}$. This divergence is not real, it has never been observed in the lab. What is then the explanation? What happens is that the detectors can not detect photons with energy arbitrarily small. Therefore when we perform an experiment in the laboratory we are in fact observing two distinct processes: the elastic scattering and the inelastic scattering where one photon is not detected because its energy is below the detector threshold. Now if we look at the next order corrections to the elastic scattering (see Chapter 9) we find that the interference between the lowest order (LO) diagram and the next order (NLO) diagram is exactly of the form $\left(\frac{d\sigma}{d\Omega}\right)_{\rm elástica} \times O(e^2)$ and it is also infrared divergent. When we sum the two processes this infrared divergence exactly cancels. After studying the radiative corrections we will come back, in section 9.3.4, to prove this result.

5.7 Crossing Symmetry

We have seen a few examples where the result for a given process seems to be related to the so-called *crossed process*. These correspond to processes where particles change into antiparticles crossing the arrow of the reaction and vice-versa. Examples are $e^- + \mu^- \to e^- + \mu^-$ in Eq. (4.59) and $e^- + e^+ \to \mu^- + \mu^+$ in Eq. (5.52), and $e^- + \gamma \to e^- + \gamma$ and $e^- + e^+ \to \gamma + \gamma$. In this last case we have not calculated the averaged squared amplitude with the same conditions, but the comparison of the results of Prob. 5.12 and Prob. 5.13 show that there is indeed some relation between the results. The same is true for Prob. 5.14 and Prob. 5.15 for the Møller and Bhabha scattering, respectively. In this section we will study this crossing symmetry in some detail.

To illustrate let us start with the simple processes $e^- + \mu^- \to e^- + \mu^-$ in Eq. (4.59) and $e^- + e^+ \to \mu^- + \mu^+$ in Eq. (5.52). These are described by the two diagrams shown in Fig. 5.7. The amplitude for the scattering process (t-channel) is

$$\mathcal{M}^{\rm scatt} = \frac{e^2}{t}\overline{u}(p_3)\gamma^\mu u(p_1)\overline{u}(p_4)\gamma_\mu u(p_2) \tag{5.106}$$

while the amplitude for the pair annihilation is

$$\mathcal{M}^{\rm pair} = \frac{e^2}{s'}\overline{v}(p_2')\gamma^\mu u(p_1')\overline{u}(p_3')\gamma_\mu v(p_4') \tag{5.107}$$

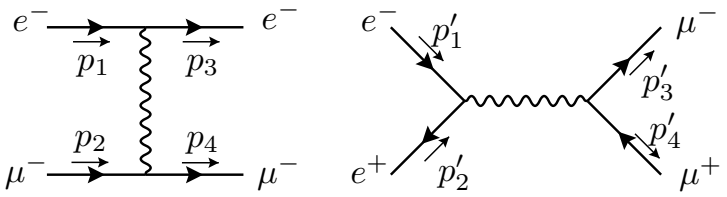


Figure 5.11: Diagrams for $e^- + \mu^- \to e^- + \mu^-$ and $e^- + e^+ \to \mu^- + \mu^+$.

where we have defined $t = (p_1 - p_3)^2$ and $s' = (p_1' + p_2')^2$. We use the prime to distinguish the two processes. A trivial calculation for the spin sum of the squared amplitudes gives (not neglecting the masses),

$$\begin{aligned}\sum_{\text{spins}} |\mathcal{M}^{\text{scatt}}|^2 =& \frac{8\,e^4}{t^2}\left[2m_e^4 + 2m_\mu^4 + 4m_e^2 m_\mu^2 - 2(m_e^2 + m_\mu^2)(s - t + u) + s^2 + u^2\right] \\ \equiv& F(s,t,u) = 4\,e^2\,\frac{f(s,u)}{t^2}\end{aligned} \tag{5.108}$$

where $f(x,y)$ is the function,

$$f(x,y) = 2(x-h)^2 + 2(y-h)^2 - h^2, \quad h = \sum_{i=1}^{4} m_i^2\ . \tag{5.109}$$

For the pair annihilation we get,

$$\begin{aligned}\sum_{\text{spins}} |\mathcal{M}^{\text{pair}}|^2 =& \frac{8\,e^4}{s^2}\left[2m_e^4 + 2m_\mu^2 + 4m_e^2 m_\mu^2 - 2(m_e^2 + m_\mu^2)(u' - s' + t') + t'^2 + u'^2\right] \\ =& F(u',s',t') = 4\,e^4\frac{f(u',t')}{s'^2}\end{aligned} \tag{5.110}$$

where $F(s,t,u)$ and $f(x,y)$ where defined Eq. (5.108) and Eq. (5.109). Notice that F is not symmetrical in its entries but f is. It is clear that it should be some relation. Let us explain how this could have been obtained without doing the calculations.

The idea goes back to the Feynman-Stuckelberg interpretation of antiparticles as negative energy particles going back in time. We start by writing the reaction for pair annihilation

$$e^-(p_1') + e^+(p_2') \to \mu^-(p_3') + \mu^+(p_4') \tag{5.111}$$

and then cross the positron to the right-hand side and the anti-muon to the left-hand side. Doing this we reverse the momenta and interchange particles with antiparticles. We get

$$e^-(p_1') + \mu^-(-p_4') \to e^-(-p_2') + \mu^-(p_3') \tag{5.112}$$

We now compare with the scattering process in the notation of the left panel of Fig. 5.7,

$$e^-(p_1) + \mu^-(p_2) \to e^-(p_3) + \mu^-(p_4) \tag{5.113}$$

They should correspond to the same process if we make the identification,

$$p_1 \to p_1' \quad p_2 \to -p_4' \quad p_3 \to -p_2' \quad p_4 \to p_3' \tag{5.114}$$

For the Mandelstam variables this implies

$$s = (p_1 + p_2)^2 \to (p_1' - p_4')^2 = u' \tag{5.115}$$
$$t = (p_1 - p_3)^2 \to (p_1' + p_2')^2 = s' \tag{5.116}$$
$$u = (p_1 - p_4)^2 \to (p_1' - p_3')^2 = t' \tag{5.117}$$

Therefore we should have

$$\begin{aligned}\sum_{\text{spins}} |\mathcal{M}^{\text{pair}}|^2(s', t', u') &= \sum_{\text{spins}} |\mathcal{M}^{\text{scatt}}|^2(s \to u', t \to s', u \to t') \\ &= F(u', s', t') = 4\, e^4 \frac{f(u', t')}{s'^2}\end{aligned} \tag{5.118}$$

in agreement with Eq. (5.110). This is true if the same number of fermions that go to the left also go to the right. The general result valid for all the cases can be stated as follows:
Take any process and define

$$\sum_{\text{spins}} |\mathcal{M}|^2 \equiv \mathcal{F}(s, t, u) \ . \tag{5.119}$$

Then for the crossed process we get

$$\sum_{\text{spins}} |\mathcal{M}|^2_{\text{crossed}} = \mathcal{F}(\text{crossed } s, t, u) \times (-1)^{\#\text{FC}} \tag{5.120}$$

where the crossed s, t, u are obtained as above and #FC is the number of crossed fermionic lines. In the above example this was two and therefore there was no extra sign. You can check that for $e^- + \gamma \to e^- + \gamma$ and $e^- + e^+ \to \gamma + \gamma$ there is indeed one minus sign, see Prob. 5.12 and Prob. 5.13.

To illustrate this sign in a simpler context consider a theory just with one fermion f and a neutral scalar ϕ that couple through the interaction Lagrangian

$$\mathcal{L}_{\text{int}} = \lambda\, \overline{f} f\, \phi - \frac{\mu}{3!}\phi^3 \tag{5.121}$$

and consider again the scattering, $f(p_1) + \phi(p_2) \to f(p_3) + \phi(p_4)$, and annihilation, $f(p_1') + \overline{f}(p_2') \to \phi(p_3') + \phi(p_4')$, process as shown in Fig. 5.12

The amplitudes can be easily calculated, and one obtains

$$\mathcal{M}^{\text{scatt}} = \frac{\lambda\, \mu}{t}\, \overline{u}(p_3) u(p_1) \ , \quad \mathcal{M}^{\text{pair}} = \frac{\lambda\, \mu}{s'}\, \overline{v}(p_2') u(p_1') \tag{5.122}$$

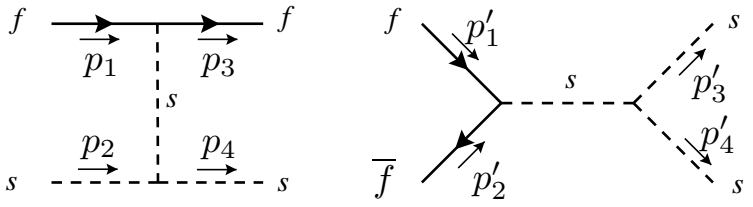


Figure 5.12: Diagrams for scattering and annihilation.

For the spin summed squared amplitudes a trivial exercise gives

$$\sum_{\text{spins}} |\mathcal{M}^{\text{scatt}}|^2 = \frac{2(\lambda\mu)^2}{t^2}\left(4m_f^2 - t\right) \tag{5.123}$$

$$\sum_{\text{spins}} |\mathcal{M}^{\text{pair}}|^2 = \frac{2(\lambda\mu)^2}{s'^2}\left(s' - 4m_f^2\right) \tag{5.124}$$

Now if we apply the crossing rules

$$p_1 \to p_1' \quad p_2 \to -p_3' \quad p_3 \to -p_2' \quad p_4 \to p_4' \tag{5.125}$$

or

$$s \to t', \quad t \to s', \quad u \to u' \tag{5.126}$$

Now if we apply the rules in Eq. (5.126) to Eq. (5.123) we obtain Eq. (5.124) but with the opposite sign. This is corrected by the factor $(-1)^{\#\text{FC}}$. As a final note we should emphasize that the result in Eq. (5.120) in valid for the spin sums and not for the spin averaged squared amplitude. This is clear in the last example where the number of initial polarizations is two in the scattering process and four in the pair annihilation case.

Complements

Complement 5.1 Gauge invariance in Compton scattering

In Eq. (5.31) we neglected the terms proportional to the momentum of the photon invoking gauge invariance. Let us look at this in more detail. From classical electromagnetism we know that the theory is invariant for gauge transformations of the form,

$$A'_\mu = A_\mu + \partial_\mu \Lambda \tag{5.127}$$

where Λ is an arbitrary function of spatial coordinates and time. Maxwell equations are invariant for the transformations of Eq. (5.127) because (see Complements 1.2 and 1.7),

$$F'_{\mu\nu} = F_{\mu\nu} \tag{5.128}$$

In quantum field theory we describe the photon by the plane wave expansion of Eq. (2.91). For a photon of momentum k^μ the transformation in Eq. (5.127) leads to

$$\varepsilon'_\mu(k) = \varepsilon_\mu(k) + c\ k_\mu \tag{5.129}$$

where c is an arbitrary constant. If we have an amplitude with one photon with momentum k in one external line we can write the matrix element as,

$$\mathcal{M} = \mathcal{M}_\mu \varepsilon^\mu(k) \tag{5.130}$$

Then the gauge invariance of the theory means in quantum field theory that,

$$\mathcal{M}_\mu k^\mu = 0 \tag{5.131}$$

and this shows that the additional terms in Eq. (4.92) will give a vanishing contribution and therefore can be neglected from the start. Let us show explicitly that Eq. (5.131) is satisfied for Compton scattering. We write,

$$\mathcal{M} = \mathcal{M}_{\mu\nu}\ \varepsilon^\mu(k)\varepsilon'^{\nu *}(k') \tag{5.132}$$

where

$$i\,\mathcal{M}_{\mu\nu} = -ie^2\overline{u}(p')\left[\gamma_\nu \frac{\not{p}+\not{k}+m}{2p\cdot k}\gamma_\mu - \gamma_\mu \frac{\not{p}'-\not{k}+m}{2p'\cdot k}\gamma_\nu\right] u(p) \tag{5.133}$$

and where we have used Eq. (5.18) and momentum conservation, $p - k' = p' - k$. We get then,

$$\begin{aligned}
k^\mu \mathcal{M}_{\mu\nu} &= -e^2\overline{u}(p')\left[\gamma_\nu \frac{\not{p}+\not{k}+m}{2p\cdot k}\not{k} - \not{k}\frac{\not{p}'-\not{k}+m}{2p'\cdot k}\gamma_\nu\right] u(p) \\
&= -e^2\overline{u}(p')\left[\gamma_\nu \frac{\not{p}+m}{2p\cdot k}\not{k} - \not{k}\frac{\not{p}'+m}{2p'\cdot k}\gamma_\nu\right] u(p) \\
&= -e^2\overline{u}(p')\left[\frac{\gamma_\nu\not{k}(-\not{p}+m)+\gamma_\nu 2p\cdot k}{2p\cdot k} - \frac{2p'\cdot k\gamma_\nu + (-\not{p}'+m)\not{k}\gamma_\nu}{2p'\cdot k}\right] u(p) \\
&= -e^2\overline{u}(p')\left[\gamma_\nu - \gamma_\nu\right] u(p)
\end{aligned}$$

$$= \quad 0 \tag{5.134}$$

where we have used $\not{k}\not{k} = k \cdot k = 0$ and the Dirac equation, $(\not{p} - m)u(p) = 0$ and $\overline{u}(p')(\not{p}' - m) = 0$. In a similar way we can show that,

$$k'^{\nu} \mathcal{M}_{\mu\nu} = 0 \ . \tag{5.135}$$

Complement 5.2 Bethe-Heitler differential cross section

In this complement we want to derive the Bethe-Heitler differential cross section, Eq. (5.101). We start by reviewing the formula for the Coulomb scattering, Eq. (3.103) and Eq. (4.24). We have shown that it can be written has

$$d\bar{\sigma} = \frac{1}{2|\vec{p}_i|} \overline{|\mathcal{M}|^2} 2\pi\delta(E_f - E_i) \frac{d^3p_f}{(2\pi)^3 2E_f} \tag{5.136}$$

where

$$\mathcal{M} = \frac{Ze^2}{|\vec{q}|^2} \overline{u}(p_f)\gamma^0 u(p_i) \tag{5.137}$$

Now we write

$$d^3p_f = d\Omega_e \, p_f^2 dp_f = d\Omega_e \, p_f E_f dE_f \tag{5.138}$$

and use the delta function to perform the E_f integration to obtain

$$\frac{d\bar{\sigma}}{d\Omega_e} = \frac{p_f}{p_i} \frac{1}{4(2\pi)^2} \overline{|\mathcal{M}|^2} = \frac{1}{16\pi^2} \overline{|\mathcal{M}|^2} \, , \tag{5.139}$$

as in the last step $p_i = p_f = |\vec{p}|$ and we have used the usual convention of denoting the absolute value of the 3-vector by the same symbol as the 4-vector. Now we evaluate $\overline{|\mathcal{M}|^2}$. We get

$$\overline{|\mathcal{M}|^2} = \frac{16\pi^2 Z^2\alpha^2}{2|\vec{q}|^4} \mathrm{Tr}\left[(\not{p}_f + m)\gamma^0(\not{p}_i + m)\gamma^0\right] \tag{5.140}$$

Putting everything together we obtain

$$\begin{aligned} \frac{d\bar{\sigma}}{d\Omega_e} =& \frac{p_f}{p_i} \frac{Z^2\alpha^2}{2|\vec{q}|^4} \mathrm{Tr}\left[(\not{p}_f + m)\gamma^0(\not{p}_i + m)\gamma^0\right] \\ =& \frac{Z^2\alpha^2}{2|\vec{q}|^4} \mathrm{Tr}\left[(\not{p}_f + m)\gamma^0(\not{p}_i + m)\gamma^0\right] \end{aligned} \tag{5.141}$$

where we have used $p_f = p_i$. Now this is precisely the result of Eqs. (5.1) and Eq. (5.7), for the Coulomb scattering.

Now we move the the bremsstrahlung in Coulomb scattering. This continues to be a fixed target process with only conservation of energy. However we have now a photon in the final state. So our formula for the differential cross section gets modified to

$$d\bar{\sigma} = \frac{1}{2|\vec{p}_i|} \overline{|\mathcal{M}|^2} 2\pi\delta(E_f + \omega - E_i) \frac{d^3p_f}{(2\pi)^3 2E_f} \frac{d^3k}{(2\pi)^3 2\omega} \tag{5.142}$$

Again we use $d^3p_f = p_f E_f dE_f d\Omega_e$ and $d^3k = \omega^2 d\omega d\Omega_\gamma$, to obtain after the E_f integration,

$$\frac{d\bar{\sigma}}{d\Omega_e d\Omega_\gamma} = \frac{p_f}{p_i}\frac{1}{8(2\pi)^5}\omega d\omega\,\overline{|\mathcal{M}|^2} \tag{5.143}$$

Now, in section 5.6, we have shown that

$$\overline{|\mathcal{M}|^2} = \frac{64\pi^3 Z^2\alpha^3}{2|\vec{q}|^4}\frac{2}{\omega^2}\left\{\ldots\right\} \tag{5.144}$$

Therefore putting everything together we get

$$\frac{d\bar{\sigma}}{d\Omega_e d\Omega_\gamma} = \frac{Z^2\alpha^3}{(2\pi)^2|\vec{q}|^4}\frac{p_f}{p_i}\frac{d\omega}{\omega}\left\{\ldots\right\} \tag{5.145}$$

in agreement with Eq. (5.101). Note that in this case $p_f \neq p_i$.

Problems

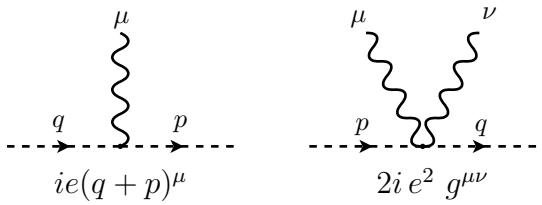
5.1 Consider the process $\gamma\gamma \to e^+e^-$ (pair creation) in QED.

a) Write the amplitude for the process.

b) Show that the amplitude is gauge invariant (see Complement 5.1).

c) Evaluate the total cross section for this process.

5.2 Consider the interaction of the photon with a charged particle with negative charge ϕ^- (this theory is known as *Scalar Electrodynamics*). The vertices are



Within this model consider the process equivalent to the Compton scattering,

$$\gamma(k) + \phi^-(p) \to \gamma(k') + \phi^-(p')$$

where k, p, k' e p' are the momenta of the particles.

a) Draw the diagrams that contribute to the process in lowest order.

b) Write the amplitude for the process.

c) Show that the amplitude is gauge invariant, that is, if $\mathcal{M} \equiv \epsilon^\mu(k)\,\epsilon^\nu(k')\,\mathcal{M}_{\mu\nu}$, then we have $k^\mu \mathcal{M}_{\mu\nu} = 0$ and $k'^\nu \mathcal{M}_{\mu\nu} = 0$. Just do for one case.

5.3 Consider the elastic scattering $e^-e^- \to e^-e^-$.

a) Write the amplitudes for the diagrams that contribute to the process in lowest order, not forgetting the relative minus sign.

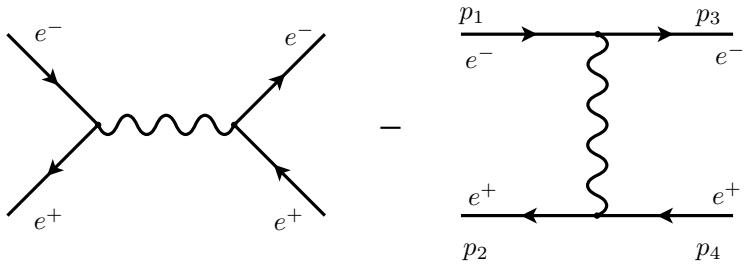
b) Show that in the high energy limit, that is, when $\sqrt{s} \gg m$, we get the following expression for the differential cross section,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\left[\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{2}{\sin^2(\theta/2)\ \cos^2(\theta/2)} + \frac{1+\sin^4(\theta/2)}{\cos^4(\theta/2)}\right] \tag{5.146}$$

where θ is the scattering angle in the CM frame. This process was calculated for the first time by Møller [44].

c) Show that in the forward direction, that is for small angles, the expression reduces to the Mott's differential cross section for relativistic electrons.

5.4 Consider the process $e^-e^+ \to e^-e^+$ known as *Bhabha scattering* . In QED, in lowest order, there are two diagrams contributing to this process,



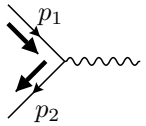
and, as we have seen in Chapter 3, there is a relative minus sign between the two diagrams. Show that in the high energy limit, $\sqrt{s} \gg m$, where $\sqrt{s}$ is the CM total energy, we get,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\left[\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)} - \frac{2\cos^4(\theta/2)}{\sin^2(\theta/2)} + \frac{1+\cos^2\theta}{2}\right] \tag{5.147}$$

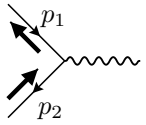
where θ is the scattering angle in the CM frame.

5.5 Show that for processes $1+2 \to 3+4$ in the `CM` frame the non-vanishing currents defined in section 5.3.2 are the following,

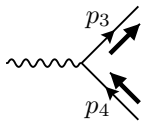
•canal s



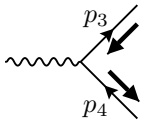
$$J_{u_1 v_2}(\uparrow,\downarrow) = \sqrt{s}\ (0,-1,-i,0) \tag{5.148}$$



$$J_{u_1 v_2}(\downarrow,\uparrow) = \sqrt{s}\ (0,-1,i,0) \tag{5.149}$$



$$J_{u_3 v_4}(\uparrow,\downarrow) = \sqrt{s}\ (0,-\cos\theta,i,\sin\theta) \tag{5.150}$$



$$J_{u3v4}(\downarrow,\uparrow) = \sqrt{s}\ (0, -\cos\theta, -i, \sin\theta) \tag{5.151}$$

•canal t

$$J_{u_1u_3}(\uparrow,\uparrow) = \sqrt{s}\ \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}, i\sin\frac{\theta}{2}, \cos\frac{\theta}{2}\right) \tag{5.152}$$

$$J_{u_1u_3}(\downarrow,\downarrow) = \sqrt{s}\ \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}, -i\sin\frac{\theta}{2}, \cos\frac{\theta}{2}\right) \tag{5.153}$$

$$J_{v_1v_3}(\uparrow,\uparrow) = \sqrt{s}\ \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}, i\sin\frac{\theta}{2}, \cos\frac{\theta}{2}\right) \tag{5.154}$$

$$J_{v_1v_3}(\downarrow,\downarrow) = \sqrt{s}\ \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}, -i\sin\frac{\theta}{2}, \cos\frac{\theta}{2}\right) \tag{5.155}$$

$$J_{u_2u_4}(\uparrow,\uparrow) = \sqrt{s}\ \left(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}, i\sin\frac{\theta}{2}, -\cos\frac{\theta}{2}\right) \tag{5.156}$$

$$J_{u_2u_4}(\downarrow,\downarrow) = \sqrt{s}\ \left(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}, -i\sin\frac{\theta}{2}, -\cos\frac{\theta}{2}\right) \tag{5.157}$$

$$J_{v_2v_4}(\uparrow,\uparrow) = \sqrt{s}\ \left(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}, i\sin\frac{\theta}{2}, -\cos\frac{\theta}{2}\right) \tag{5.158}$$

$$J_{v_2v_4}(\downarrow,\downarrow) = \sqrt{s}\ \left(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}, -i\sin\frac{\theta}{2}, -\cos\frac{\theta}{2}\right) \tag{5.159}$$

•u-channel

$$J_{u_1u_4}(\uparrow,\uparrow) = \sqrt{s}\ \left(\sin\frac{\theta}{2}, -\cos\frac{\theta}{2}, -i\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right) \tag{5.160}$$

$$J_{u_1u_4}(\downarrow,\downarrow) = \sqrt{s}\ \left(-\sin\frac{\theta}{2}, \cos\frac{\theta}{2}, -i\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\right) \tag{5.161}$$

$$J_{u_2u_3}(\uparrow,\uparrow) = \sqrt{s}\ \left(-\sin\frac{\theta}{2}, -\cos\frac{\theta}{2}, i\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right) \tag{5.162}$$

$$J_{u_2u_3}(\downarrow,\downarrow) = \sqrt{s}\ \left(\sin\frac{\theta}{2}, \cos\frac{\theta}{2}, i\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\right) \tag{5.163}$$

$$J_{v_1v_4}(\uparrow,\uparrow) = \sqrt{s}\ \left(-\sin\frac{\theta}{2}, \cos\frac{\theta}{2}, i\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\right) \tag{5.164}$$

$$J_{v_1v_4}(\downarrow,\downarrow) = \sqrt{s}\ \left(\sin\frac{\theta}{2}, -\cos\frac{\theta}{2}, i\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right) \tag{5.165}$$

$$J_{v_2v_3}(\uparrow,\uparrow) = \sqrt{s}\ \left(\sin\frac{\theta}{2}, \cos\frac{\theta}{2}, -i\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\right) \tag{5.166}$$

$$J_{v_2v_3}(\downarrow,\downarrow) = \sqrt{s}\ \left(-\sin\frac{\theta}{2}, -\cos\frac{\theta}{2}, -i\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right) \tag{5.167}$$

5.6 Use the results from Prob. 5.5 to evaluate the Bhabha scattering in the limit in which we neglect the masses. Compare the results with those of Prob. 5.4 and those Chapter 6 for this process.

5.7 This problem has the goal of showing Eq. (5.78). To that end follow the steps:

a) Consider without loss of generality (why?) $\phi = 0$. Define the spin along direction $\vec{n} = (\sin\theta, 0, \cos\theta)$, as follows,

$$\vec{S}\cdot\vec{n} = \sin\theta S_x + \cos\theta S_z = \frac{1}{2}\sin\theta(S_+ + S_-) + \cos\theta S_z \tag{5.168}$$

Show that in the S_z basis, that is where

$$S_z\left|11\right\rangle_z = \left|11\right\rangle_z, S_z\left|10\right\rangle_z = 0, S_z\left|1,-1\right\rangle_z = -\left|1,-1\right\rangle_z \tag{5.169}$$

we get

$$\begin{aligned}(\vec{S}\cdot\vec{n})\left|11\right\rangle_z &= \cos\theta\left|1,1\right\rangle_z + \frac{1}{\sqrt{2}}\sin\theta\left|10\right\rangle_z \\ (\vec{S}\cdot\vec{n})\left|10\right\rangle_z &= \frac{1}{\sqrt{2}}\sin\theta\left|1,1\right\rangle_z + \frac{1}{\sqrt{2}}\sin\theta\left|1,-1\right\rangle_z \\ (\vec{S}\cdot\vec{n})\left|1,-1\right\rangle_z &= -\cos\theta\left|1,-1\right\rangle_z + \frac{1}{\sqrt{2}}\sin\theta\left|10\right\rangle_z\end{aligned} \tag{5.170}$$

b) Define now $\left|11\right\rangle_\theta$ such that

$$(\vec{S}\cdot\vec{n})\left|11\right\rangle_\theta = \left|11\right\rangle_\theta \tag{5.171}$$

and expand in the basis $\left|1m\right\rangle_z$

$$\left|11\right\rangle_\theta = \alpha\left|1,-1\right\rangle_z + \beta\left|10\right\rangle_z + \gamma\left|1,1\right\rangle_z \tag{5.172}$$

Apply the operator $(\vec{S}\cdot\vec{n})$ and determine α, β, γ verifying in this way Eq. (5.78).

5.8 Consider the process $e^- + e^+ \to \mu^- + \mu^+$ in QED. Not neglecting the masses and using these explicit spinors and the explicit form for the Dirac γ matrices, we can then obtain the helicity amplitudes that we write as

$$\mathcal{M}(h_1,h_2;h_3,h_4) = \frac{4\pi\alpha}{s}\ \overline{v}(p_2,h_2)\gamma^\mu u(p_1,h_1)\ \overline{u}(p_3,h_3)\gamma_\mu v(p_4,h_4)\ . \tag{5.173}$$

where $h_i = \uparrow, \downarrow$ for each particle. This is a straightforward but tedious calculation, that can be best done with a `mathematica` program [37]. Show that the result is

$$\mathcal{M}(\uparrow,\uparrow;\uparrow,\uparrow) = -(4\pi\alpha)\,\frac{4\,m_e m_\mu}{s}\cos\theta \quad \mathcal{M}(\downarrow,\uparrow;\uparrow,\uparrow) = -(4\pi\alpha)\,\frac{2\,m_\mu}{\sqrt{s}}\sin\theta \tag{5.174}$$

$$\mathcal{M}(\uparrow,\downarrow;\uparrow,\uparrow) = -(4\pi\alpha)\,\frac{2\,m_\mu}{\sqrt{s}}\sin\theta \qquad \mathcal{M}(\downarrow,\downarrow;\uparrow,\uparrow) = (4\pi\alpha)\frac{4\,m_e m_\mu}{s}\cos\theta \tag{5.175}$$

$$\mathcal{M}(\uparrow,\uparrow;\downarrow,\uparrow) = (4\pi\alpha)\,\frac{2\,m_e}{\sqrt{s}}\sin\theta \qquad \mathcal{M}(\downarrow,\uparrow;\downarrow,\uparrow) = -(4\pi\alpha)\,(1+\cos\theta) \tag{5.176}$$

$$\mathcal{M}(\uparrow,\downarrow;\downarrow,\uparrow) = (4\pi\alpha)\,(1-\cos\theta) \qquad \mathcal{M}(\downarrow,\downarrow;\downarrow,\uparrow) = -(4\pi\alpha)\,\frac{2\,m_e}{\sqrt{s}}\sin\theta, \tag{5.177}$$

$$\mathcal{M}(\uparrow,\uparrow;\uparrow,\downarrow) = (4\pi\alpha)\,\frac{2\,m_e}{\sqrt{s}}\sin\theta \qquad \mathcal{M}(\downarrow,\uparrow;\uparrow,\downarrow) = (4\pi\alpha)\,(1-\cos\theta) \tag{5.178}$$

$$\mathcal{M}(\uparrow,\downarrow;\uparrow,\downarrow) = -(4\pi\alpha)\,(1+\cos\theta) \qquad \mathcal{M}(\downarrow,\downarrow;\uparrow,\downarrow) = -(4\pi\alpha)\,\frac{2\,m_e}{\sqrt{s}}\sin\theta \tag{5.179}$$

$$\mathcal{M}(\uparrow,\uparrow;\downarrow,\downarrow) = (4\pi\alpha)\,\frac{4\,m_e m_\mu}{s}\cos\theta \qquad \mathcal{M}(\downarrow,\uparrow;\downarrow,\downarrow) = (4\pi\alpha)\,\frac{2\,m_\mu}{\sqrt{s}}\sin\theta \tag{5.180}$$

$$\mathcal{M}(\uparrow,\downarrow;\downarrow,\downarrow) = (4\pi\alpha)\,\frac{2\,m_\mu}{\sqrt{s}}\sin\theta \qquad \mathcal{M}(\downarrow,\downarrow;\downarrow,\downarrow) = -(4\pi\alpha)\frac{4m_e m_\mu}{s}\cos\theta \tag{5.181}$$

5.9 Show that the operator

$$P(h,s) = \frac{1 + h\,\gamma_5 \not{s}}{2}\ , \tag{5.182}$$

where the spin 4-vector is,

$$s^\mu = (\gamma\beta, \gamma\hat{\beta})\ , \tag{5.183}$$

is an helicity projector for a particle moving in the direction $\hat{\beta}$ with velocity $\vec{\beta}$. Show this explicitly for the helicity spinors defined in section 1.8.2.

5.10 Consider the process $e^- + e^+ \to \mu^- + \mu^+$ in QED. Using the trace technique and the helicity projector defined in Problem 5.9, show that, we can project the helicity amplitudes as

$$\mathcal{M}(h_1,h_2;h_3,h_4) = \frac{4\pi\alpha}{s}\overline{v}(p_2)P(h_2,s_2)\gamma^\mu P(h_1,s_1)u(p_1)\overline{u}(p_3)P(h_3,s_3)\gamma_\mu P(h_4,s_4)v(p_4) \tag{5.184}$$

Using this and `FeynCalc` for `Mathematica` show that one gets

$$|\mathcal{M}(\uparrow,\uparrow;\uparrow,\uparrow)|^2 = (4\pi\alpha)^2 \frac{16\, m_e^2 m_\mu^2}{s^2}\cos^2\theta \quad |\mathcal{M}(\downarrow,\uparrow;\uparrow,\uparrow)|^2 = (4\pi\alpha)^2 \frac{4\, m_\mu^2}{s}\sin^2\theta \tag{5.185}$$

$$|\mathcal{M}(\uparrow,\downarrow;\uparrow,\uparrow)|^2 = (4\pi\alpha)^2 \frac{4\, m_\mu^2}{s}\sin^2\theta \quad |\mathcal{M}(\downarrow,\downarrow;\uparrow,\uparrow)|^2 = (4\pi\alpha)^2 \frac{16\, m_e^2 m_\mu^2}{s}\cos^2\theta \tag{5.186}$$

$$|\mathcal{M}(\uparrow,\uparrow;\downarrow,\uparrow)|^2 = (4\pi\alpha)^2 \frac{4\, m_e^2}{s}\sin^2\theta \quad |\mathcal{M}(\downarrow,\uparrow;\downarrow,\uparrow)|^2 = (4\pi\alpha)^2 (1+\cos\theta)^2 \tag{5.187}$$

$$|\mathcal{M}(\uparrow,\downarrow;\downarrow,\uparrow)|^2 = (4\pi\alpha)^2 (1-\cos\theta)^2 \quad |\mathcal{M}(\downarrow,\downarrow;\downarrow,\uparrow)|^2 = (4\pi\alpha)^2 \frac{4\, m_e^2}{s}\sin^2\theta \tag{5.188}$$

$$|\mathcal{M}(\uparrow,\uparrow;\uparrow,\downarrow)|^2 = (4\pi\alpha)^2 \frac{4\, m_e^2}{s}\sin^2\theta \quad |\mathcal{M}(\downarrow,\uparrow;\uparrow,\downarrow)|^2 = (4\pi\alpha)^2 (1-\cos\theta)^2 \tag{5.189}$$

$$|\mathcal{M}(\uparrow,\downarrow;\uparrow,\downarrow)|^2 = (4\pi\alpha)^2 (1+\cos\theta)^2 \quad |\mathcal{M}(\downarrow,\downarrow;\uparrow,\downarrow)|^2 = (4\pi\alpha)^2 \frac{4\, m_e^2}{s}\sin^2\theta \tag{5.190}$$

$$|\mathcal{M}(\uparrow,\uparrow;\downarrow,\downarrow)|^2 = (4\pi\alpha)^2 \frac{16\, m_e^2 m_\mu^2}{s^2}\cos^2\theta \quad |\mathcal{M}(\downarrow,\uparrow;\downarrow,\downarrow)|^2 = (4\pi\alpha)^2 \frac{4\, m_\mu^2}{s}\sin^2\theta \tag{5.191}$$

$$|\mathcal{M}(\uparrow,\downarrow;\downarrow,\downarrow)|^2 = (4\pi\alpha)^2 \frac{4\, m_\mu^2}{s}\sin^2\theta \quad |\mathcal{M}(\downarrow,\downarrow;\downarrow,\downarrow)|^2 = (4\pi\alpha)^2 \frac{16\, m_e^2 m_\mu^2}{s^2}\cos^2\theta \tag{5.192}$$

Compare with the results of Problem 5.8.

5.11 Show that for polarized photons the Klein-Nishina formula becomes,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4\, m^2}\left(\frac{k'}{k}\right)^2 \left[\left(\frac{k'}{k}\right) + \left(\frac{k}{k'}\right) + 4(\varepsilon\cdot\varepsilon')^2 - 2\right] \tag{5.193}$$

where $\varepsilon, \varepsilon'$ are the polarization vectors for the two photons. As the result should be gauge invariant one can choose a gauge where,

$$\varepsilon = (0,\vec{\varepsilon}), \quad \varepsilon' = (0,\vec{\varepsilon}'), \quad p\cdot\varepsilon = p\cdot\varepsilon' = 0 \tag{5.194}$$

Show that the previous expression leads to Eq. (5.49) when we sum over the polarizations of the final photon and average for the polarizations of the initial photon (see Ref. [20]).

5.12 Consider Compton scattering.

a) Integrate the Klein-Nishina formula, Eq. (5.49), for the Compton scattering, to get the total cross section,

$$\sigma(x) = \frac{2\pi\alpha^2}{m_e^2}\frac{1}{x}\left[\left(1-\frac{4}{x}-\frac{8}{x^2}\right)\ln(1+x) + \frac{1}{2} + \frac{8}{x} - \frac{1}{2(1+x)^2}\right] \tag{5.195}$$

where $x = 2k/m_e$.

b) Show that in the limit $x \ll 1$, that is, $k \ll m_e$, one gets the Thomson classical expression, that is

$$\lim_{x\to 0}\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m_e^2}\left(1+\cos\theta^2\right), \quad \lim_{x\to 0}\sigma(x) = \frac{8\pi\alpha^2}{3m_e^2} \tag{5.196}$$

c) Redo the problem in a reference frame where the electron is not at rest. Use the Mandelstam variables to show that the differential cross section can be written as,

$$\frac{d\sigma}{dt} = \frac{2\pi\alpha^2}{(s-m_e^2)^2}\left[4\left(\frac{m_e^2}{s-m_e^2}+\frac{m_e^2}{u-m_e^2}\right)^2+4\left(\frac{m_e^2}{s-m_e^2}+\frac{m_e^2}{u-m_e^2}\right) - \left(\frac{s-m_e^2}{u-m_e^2}+\frac{u-m_e^2}{s-m_e^2}\right)\right] \tag{5.197}$$

where

$$\begin{aligned} s &= (p+k)^2 = (p'+k')^2 \\ t &= (p-p')^2 = (k-k')^2 \\ u &= (p-k')^2 = (p'-k)^2 \end{aligned} \tag{5.198}$$

d) Now go the limit where the initial electron is at rest and rediscover the Klein-Nishina formula.

5.13 Consider the process $e^-e^+ \to \gamma\gamma$ without neglecting the fermion masses.

a) Show that instead of Eq. (5.92) we get

$$\sum_{\text{spins}}\sum_{\lambda_1,\lambda_2}|\mathcal{M}|^2 = 8e^4\left[-4\left(\frac{m_e^2}{m_e^2-t}+\frac{m_e^2}{m_e^2-u}\right)^2+4\left(\frac{m_e^2}{m_e^2-t}+\frac{m_e^2}{m_e^2-u}\right) + \frac{m_e^2-t}{m_e^2-u}+\frac{m_e^2-u}{m_e^2-t}\right] \tag{5.199}$$

with

$$\begin{aligned} s &= (p_1+p_2)^2 = (k_1+k_2)^2 \\ t &= (p_1-k_1)^2 = (p_2-k_2)^2 \\ u &= (p_1-k_2)^2 = (p_2-k_1)^2 \end{aligned} \tag{5.200}$$

and therefore we get for the differential cross section,

$$\frac{d\sigma}{dt} = \frac{2\pi\alpha^2}{s(s-4m_e^2)}\left[-4\left(\frac{m_e^2}{m_e^2-t}+\frac{m_e^2}{m_e^2-u}\right)^2+4\left(\frac{m_e^2}{m_e^2-t}+\frac{m_e^2}{m_e^2-u}\right) + \frac{m_e^2-t}{m_e^2-u}+\frac{m_e^2-u}{m_e^2-t}\right] \tag{5.201}$$

b) Show that we get for the total cross section (do not forget to divide by two for identical particles in the final state),

$$\sigma_{\rm tot} = \frac{2\pi\alpha^2}{s^2\,(s-4m_e^2)}\left[\left(s^2+4sm_e^2-8m_e^4\right)\ln\left(\frac{\sqrt{s}+\sqrt{s-4m_e^2}}{\sqrt{s}-\sqrt{s-4m_e^2}}\right)\right.$$
$$\left.-(s+4m_e^2)\sqrt{s(s-4m_e^2)}\right] \tag{5.202}$$

This result was obtained for the first time by Dirac in 1930 [45].

c) Show that in the limit $\sqrt{s} \gg m_e$ one gets,

$$\sigma_{\rm tot} \simeq \frac{2\pi\alpha^2}{s}\left(\ln\frac{s}{m_e^2}-1\right) \tag{5.203}$$

d) What is the total cross section for the process $\gamma\gamma \to e^-e^+$?

5.14 Consider the elastic scattering $e^-e^- \to e^-e^-$ without neglecting the electron masses.

a) Show that the differential cross section can be written as

$$\frac{d\sigma}{dt} = \frac{\pi\alpha^2}{s(s-4m_e^2)}\left[\frac{f(s,u)}{t^2}+\frac{f(s,t)}{u^2}+\frac{f(s,s)}{tu}\right] \tag{5.204}$$

where the symmetric function f is defined in Eq. (5.109). The important part to retain here for future applications (see Problems 5.15 and 5.16) is that

$$\frac{1}{4}\sum_{\rm spins}|\mathcal{M}|^2 = e^4\left[\frac{f(s,u)}{t^2}+\frac{f(s,t)}{u^2}+\frac{f(s,s)}{tu}\right] \tag{5.205}$$

b) Show that in the limit $s \gg m_e^2$ the previous expression coincides with that of Prob. 5.3.

c) Show that in the limit $s \gg m_e^2$ the total cross section is,

$$\sigma_{\rm tot} = \frac{2\pi\alpha^2}{s}\left[\cos\theta_0 + 8\,\frac{\cos\theta_0}{\sin^2\theta_0}\right] \tag{5.206}$$

where we have integrated in the interval $\theta_0 < \theta < \pi-\theta_0$ to avoid the collinear divergence that appears for, $\theta = 0, \pi$, due the fact that the photon is massless.

d) Use CalcHEP (see Chapter 7) to evaluate this process and compare with the previous analytical result.

5.15 Consider the Bhabha scattering without neglecting the electron mass.

a) Show that we get for the differential cross section,

$$\frac{d\sigma}{dt} = \frac{\pi\alpha^2}{s(s-4m_e^2)}\left[\frac{f(t,u)}{s^2} + \frac{f(s,u)}{t^2} + \frac{f(u,u)}{st}\right] \tag{5.207}$$

where the symmetric function f was defined in Prob. 5.14.

b) Show that in the limit $s \gg m_e^2$ the previous expression reduces to that of Prob. 5.4.

c) Show that in the limit $s \gg m_e^2$ we get for the total cross section,

$$\sigma_{\text{tot}} = \frac{\pi\alpha^2}{24s}\left[111\cos\theta_0 + 6\cos(2\theta_0) + \cos(3\theta_0) + \frac{96}{\sin^2(\theta_0/2)} + 192\log\left(\sin^2(\theta_0/2)\right)\right] \tag{5.208}$$

d) Use CalcHEP (see Chapter 7) to evaluate this process and make a plot comparing with the previous result.

e) Study the dependance of the cross section on the angle θ_0. Reproduce the following figure.

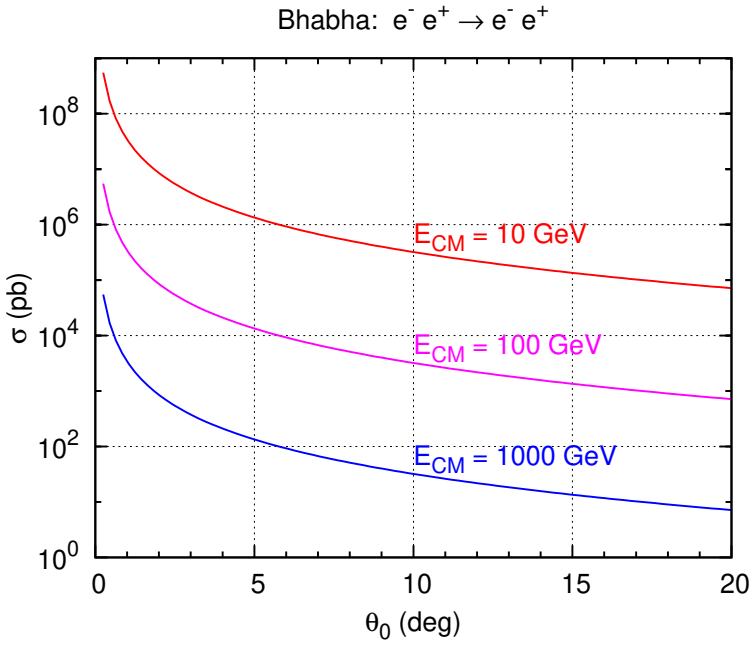


Figure 5.13: Variation of total cross section for Bhabha scattering with the angle θ_0 for several values of the CM energy.

f) Show that in the limit $s \gg m_e^2$ the Bhabha and Møller total cross sections are related in the CM frame by the relation,

$$d\sigma_{e^+e^-} = \cos^4 \frac{\theta}{2} d\sigma_{e^-e^-} \tag{5.209}$$

In this limit make a plot as a function of θ and explain why the Møller scattering has the symmetry $\theta \to \pi - \theta$ and the Bhabha scattering does not.

5.16 The results from Prob. 5.14 and Prob. 5.15 it is general for any process in QED that has two fermions in the initial state and two fermions in the final state. Verify this statement for the processes $e^-\mu^- \to e^-\mu^-$ and $e^-e^+ \to \mu^-\mu^+$ studied in section 5.7.

5.17 We are going in this problem evaluate the lifetime of the para-positronium, that is the positronium in the state $^1S_0$.

a) Explain, using the conservation laws for angular momenta and charge conjugation, that we should have (see Refs. [43] or [46])

$$^1S_0 \to \gamma\gamma, \quad ^3S_1 \to \gamma\gamma\gamma \tag{5.210}$$

b) Show that the cross section for an electron and positron in a state $^1S_0$ is

$$\sigma_{\text{singleto}} = 4\ \overline{\sigma} \tag{5.211}$$

where $\overline{\sigma}$ the non polarized cross section evaluated Eq. (5.202).

c) The speed of the particles in positronium is not relativistic. Therefore it is a good approximation to consider a reference frame where the positron is at rest and the electron has small velocity, $\beta \ll 1$. Use this approximation to evaluate,

$$\sigma_{\text{singleto}} = \frac{4\pi\alpha^2}{m^2} \frac{1}{\beta} \tag{5.212}$$

d) Show that the decay width is given by (why?),

$$\Gamma = \sigma_{\text{singleto}}\ \beta |\psi(0)|^2 \tag{5.213}$$

where

$$\psi(r) = \frac{1}{\sqrt{\pi a}} e^{-r/a}, \quad a = \frac{2}{m\alpha} \tag{5.214}$$

e) Evaluate the positronium lifetime.

5.18 Consider Coulomb scattering with polarized electrons. The incident beam has right-handed polarization, that is,

$$u_R(p_i) = \frac{1 + \gamma_5 \not{s}_i}{2}\, u(p_i) \tag{5.215}$$

where

$$s_i = (\gamma\beta, 0, 0, \gamma)$$

and the electron in the final state is measured with two polarizations,

$$u_R(p_f) = \frac{1 + \gamma_5 \not{s}_f}{2}\, u(p_f), \quad u_L(p_f) = \frac{1 - \gamma_5 \not{s}_f}{2}\, u(p_f)$$

where, with the usual kinematics, we have ($\gamma = E_i/m$),

$$s_f = (\gamma\beta, \gamma \sin\theta, 0, \gamma\cos\theta)$$

a) Show that the 4-vectors s_i and s_f, defined above, obey the relations, $s_i^2 = s_f^2 = -1$ and $s_i \cdot p_i = s_f \cdot p_f = 0$.

b) If we define the degree of polarization of the scattered electrons by,

$$P_R = \frac{N_R - N_L}{N_R + N_L} \tag{5.216}$$

where N_R (N_L) is the number of scattered electrons with right (left) polarization show that we have,

$$P_R = 1 - \left[\frac{2m^2 \sin^2(\theta/2)}{E^2\cos^2(\theta/2) + m^2 \sin^2(\theta/2)}\right] \tag{5.217}$$

c) Show that in relativistic limit there is no depolarization of the incident electrons.

Chapter 6

The Helicity Amplitudes

6.1 Introduction

When the number of diagrams, N, becomes large it is clear that the difficulty with the trace technique will increase as N^2. This makes a process with a large number of diagrams difficult to handle. There is however another technique known as the helicity amplitude technique, where the complexity of the problem only scales with N. The idea is that for each helicity combination each diagram is to be represented by a complex number and the total amplitude by the sum of N complex numbers. This is therefore a technique to be used numerically in all cases of interest. This method is very simple in the case of massless fermion and gauge bosons, but it can be extended for the case of massive fermions and massive gauge bosons. As we will see there a price to pay, the number of phase space integrations increases, but if we use a Monte Carlo integration method (like Vegas) the efficiency does not change much with the number of integrations.

6.2 Spinor products

We begin with a clarification on the term *helicity amplitude*. In the original articles of Gastmans, Wu and collaborators [47–50] are used heliciy spinors for mass less fermions with the notation

$$\frac{1 \pm \gamma_5}{2} u_\pm(p) = u_\pm(p), \quad \frac{1 \mp \gamma_5}{2} v_\pm(p) = v_\pm(p), \quad . \tag{6.1}$$

For helicity we use $\uparrow, \downarrow$ with (see Eqs. (1.259) and (1.260))

$$P_R u_\uparrow = u_\uparrow \quad ; \quad P_L u_\uparrow = 0 \quad ; \quad P_R u_\downarrow = 0 \quad ; \quad P_L u_\downarrow = u_\downarrow \tag{6.2}$$

$$P_R v_\uparrow = 0 \quad ; \quad P_L v_\uparrow = v_\uparrow \quad ; \quad P_R v_\downarrow = v_\downarrow \quad ; \quad P_L v_\downarrow = 0 \; . \tag{6.3}$$

Therefore the correspondence is

$$+ = \uparrow, \quad - = \downarrow \tag{6.4}$$

justifying the name of helicity spinors (and amplitudes). However, from the point of view of the numerical implementation we prefer to follow the work developed by Kleiss[1] and collaborators [51–53]. They use the notation $\pm$ for chirality and not for the helicity, that is,

$$\frac{1 \pm \gamma_5}{2} u_\pm(p) = u_\pm(p), \;\; \frac{1 \pm \gamma_5}{2} v_\pm(p) = v_\pm(p) \tag{6.5}$$

and this, even in the massless limit, differs from Eq. (6.1) for the antiparticles. we will follow Kleiss notation which means that the spinors are really eigenstates of chirality, not helicity. We will retain the notation $\uparrow\downarrow$ for helicity. For massless fermions the spinors are eigenstates of chirality and there is no distinction between spinors u and v for the definition of chirality. We will designate these spinors by $u_\pm(p)$ with $p^2 = 0$ and $\not{p} u_\pm(p) = 0$. Although they are chiral spinors we will abuse the language and will continue to speak, as usual, of helicity spinors and helicity amplitude technique[2]. It is convenient to define the right and left projectors [3],

$$\gamma_\pm = \frac{1 \pm \gamma_5}{2} \tag{6.6}$$

Then our spinors satisfy

$$\gamma_+ \not{p} = u_+(p)\overline{u}_+(p), \quad \gamma_- \not{p} = u_-(p)\overline{u}_-(p), \quad \not{p} = u_+(p)\overline{u}_+(p) + u_-(p)\overline{u}_-(p) \tag{6.7}$$

With these spinors we can form two independent *spinor products* [52],

$$\begin{aligned} s(p_1, p_2) &= \overline{u}_+(p_1) u_-(p_2) = -s(p_2, p_1) \\ t(p_1, p_2) &= \overline{u}_-(p_1) u_+(p_2) = s^*(p_2, p_1) \end{aligned} \tag{6.8}$$

with the normalization

$$|s(p_1, p_2)|^2 = 2 p_1 \cdot p_2 \tag{6.9}$$

To be able to do numerical calculations it is necessary to have a formula to evaluate the spinor products knowing the components of the two 4-vectors. This formula is [51],

$$s(p_1, p_2) = \left(p_1^2 + i p_1^3\right) \sqrt{\frac{p_2^0 - p_2^1}{p_1^0 - p_1^1}} - \left(p_2^2 + i p_2^3\right) \sqrt{\frac{p_1^0 - p_1^1}{p_2^0 - p_2^1}} \tag{6.10}$$

and using it one can trivially verify Eq. (6.9). A very important relation, known as identity of Chisholm, is ($\sigma = \pm$),

$$\overline{u}_\sigma(p_1) \gamma_\mu u_\sigma(p_2)\, \gamma^\mu = 2 u_\sigma(p_2) \overline{u}_\sigma(p_1) + 2 u_{-\sigma}(p_1) \overline{u}_{-\sigma}(p_2) \tag{6.11}$$

[1] Also a collaborator of the original papers.

[2] For particles $u_+ = u_\uparrow$ but for antiparticles $u_+ = v_\downarrow$.

[3] This notation is more practical in our calculations than the more usual R, L, therefore we will use it following Kleiss [51, 52] and not the original notation of Gastmans etal. [47–49] in Eq. (6.1).

which shows that the normalization of spinors is such that

$$\overline{u}_\sigma(p)\gamma_\mu u_\sigma(p) = 2p_\mu \ . \tag{6.12}$$

In Complement 6.1 you can find a derivation of Eqs. (6.10) an (6.11). Using Eq. (6.7) we can show the following useful relations:

$$\begin{aligned}
\overline{u}_+(p_1)\not{p}_2\not{p}_3\cdots\not{p}_{2n-1}u_-(p_{2n}) &= s(p_1,p_2)s^*(p_3,p_2)s(p_3,p_4)\cdots s(p_{2n-1},p_{2n})\\
\overline{u}_-(p_1)\not{p}_2\not{p}_3\cdots\not{p}_{2n-1}u_+(p_{2n}) &= s^*(p_2,p_1)s(p_2,p_3)s^*(p_4,p_3)\cdots s^*(p_{2n},p_{2n-1})\\
\overline{u}_+(p_1)\not{p}_2\not{p}_3\cdots\not{p}_{2n}u_+(p_{2n+1}) &= s(p_1,p_2)s^*(p_3,p_2)s(p_3,p_4)\cdots s^*(p_{2n+1},p_{2n})\\
\overline{u}_-(p_1)\not{p}_2\not{p}_3\cdots\not{p}_{2n}u_-(p_{2n+1}) &= s^*(p_2,p_1)s(p_2,p_3)s^*(p_4,p_3)\cdots s(p_{2n},p_{2n+1})
\end{aligned} \tag{6.13}$$

where p_i are momenta for massless particles, that is, $p_i^2 = 0$. When we have tow fermion lines connected by a contraction of two γ matrices we have to use Eq. (6.11). For example,

$$\overline{u}_+(p_1)\gamma_\mu u_+(p_2)\,\overline{u}_+(p_3)\gamma^\mu u_+(p_4) = 2s(p_3,p_1)s^*(p_4,p_2) \ . \tag{6.14}$$

with this relations we can transform all the amplitudes for massless fermions in products of spinor products. Before that we will extend the technique for other cases of interest we will give below the example of Bhabha scattering.

6.2.1 Bhabha scattering

As an example let us evaluate the differential cross section for Bhabha scattering in QED in the limit of massless fermions (which is a good approximation for $\sqrt{s} \gg m_e$, see Prob. 5.4). We have the two diagrams of Fig 5.7, corresponding to the amplitudes,

$$\mathcal{M}_1 = \frac{e^2}{s}\overline{v}(p_2)\gamma^\mu u(p_1)\overline{u}(p_3)\gamma_\mu v(p_4), \quad \mathcal{M}_2 = -\frac{e^2}{t}\overline{u}(p_3)\gamma^\mu u(p_1)\overline{v}(p_2)\gamma_\mu v(p_4) \tag{6.15}$$

with $s = (p_1+p_2)^2$, $t = (p_1-p_3)^2$. The result obtained with the trace technique was given in Eq. (5.85). Let us now use the helicity amplitudes technique to recover this result.

We start by noticing that from the 16 possible helicity amplitudes[4] $\mathcal{M}(\sigma_1\sigma_2;\sigma_3\sigma_4)$ only 6 are non-vanishing. These are $\mathcal{M}(++;++)$, $\mathcal{M}(--;--)$, $\mathcal{M}(+-;+-)$, $\mathcal{M}(-+;-+)$, $\mathcal{M}(++;--)$ e $\mathcal{M}(--;++)$. Using the compact notation,

$$s_{ij} = s(p_i,p_j) \tag{6.16}$$

[4]We again call the attention of the reader for the abuse of language here. When we compare with the helicity discussed earlier we should have, for instance, $\mathcal{M}(++;++) = \mathcal{M}(\uparrow,\downarrow;\uparrow,\downarrow)$ and so on. We will use always $\uparrow,\downarrow$ to refer to helicity states and $\pm$ (or R, L) when we refer to chirality.

and relations similar to Eq. (6.14), we get

$$\begin{aligned}
\mathcal{M}(++;++) &= \mathcal{M}_1(++;++) + \mathcal{M}_2(++;++) = 2e^2 \left[\frac{s_{32}s_{41}^*}{s} - \frac{s_{23}s_{41}^*}{t} \right] \\
\mathcal{M}(--;--) &= \mathcal{M}_1(--;--) + \mathcal{M}_2(--;--) = 2e^2 \left[\frac{s_{23}^*s_{14}}{s} - \frac{s_{32}^*s_{14}}{t} \right] \\
\mathcal{M}(+-;+-) &= \mathcal{M}_2(+-;+-) = -2e^2 \ \frac{s_{12}^*s_{34}}{t} \\
\mathcal{M}(-+;-+) &= \mathcal{M}_2(-+;-+) = -2e^2 \ \frac{s_{21}s_{43}^*}{t} \\
\mathcal{M}(++;--) &= \mathcal{M}_1(++;--) = 2e^2 \ \frac{s_{13}^*s_{24}}{s} \\
\mathcal{M}(--;++) &= \mathcal{M}_1(--;++) = 2e^2 \ \frac{s_{31}s_{42}^*}{s}
\end{aligned} \tag{6.17}$$

Therefor we get,

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{1}{4} \left[|\mathcal{M}(++;++)|^2 + |\mathcal{M}(--;--)|^2 + |\mathcal{M}(+-;+-)|^2 \right. \\
&\qquad \left. + |\mathcal{M}(-+;-+)|^2 + |\mathcal{M}(++;--)|^2 + |\mathcal{M}(--;++)|^2 \right] \\
&= e^4 \left[2\frac{|s_{23}|^2|s_{41}|^2}{t^2} + 2\frac{|s_{32}|^2|s_{41}|^2}{s^2} + 4\frac{|s_{23}|^2|s_{41}|^2}{st} \right. \\
&\qquad \left. + 2\frac{|s_{12}|^2|s_{34}|^2}{t^2} + 2\frac{|s_{13}|^2|s_{24}|^2}{s^2} \right]
\end{aligned} \tag{6.18}$$

and using

$$\begin{aligned}
|s_{23}|^2 &= -u = t + s = |s_{41}|^2 \\
|s_{12}|^2 &= s = |s_{34}|^2 \\
|s_{13}|^2 &= -t = |s_{24}|^2
\end{aligned} \tag{6.19}$$

we finally obtain Eq. (5.85). We should note that for this simple problem there is probably no gain in using this technique instead of that of the traces. However in more complex problems, like for instance, $e^-e^+ \to e^-e^+e^-e^+$, with 36 diagrams in lowest order, the gain is enormous. In these cases the amplitudes are calculated as complex numbers, using Eq. (6.10), and in the end we add the absolute value squared of those complex numbers, without the need to transform the spinor products in scalar products, as we have done here.

6.2.2 The scattering $e^-e^+ \to \mu^-\mu^+$

As another example we look at the process $e^-e^+ \to \mu^-\mu^+$. We consider the case of massless fermions. We start with the invariant amplitude, Eq. (5.51), that we write as

$$\mathcal{M} = \frac{e^2}{s}\overline{v}(p_2)\gamma^\mu u(p_1)\ \overline{u}(p_3)\gamma_\mu v(p_4) \tag{6.20}$$

Now each spinor can have two helicities (chiralities), so we can write

$$\mathcal{M}(\sigma_1,\sigma_2;\sigma_3,\sigma_4) = \frac{e^2}{s}\overline{u}_{\sigma_2}(p_2)\gamma^\mu u_{\sigma_1}(p_1)\ \overline{u}_{\sigma_3}(p_3)\gamma_\mu u_{\sigma_4}(p_4) \tag{6.21}$$

From the rules in section 6.2, one realizes the $\sigma_1 = \sigma_2$ and $\sigma_3 = \sigma_4$. Therefore there are only four non-vanishing helicity amplitudes. Using the relation in Eq. (6.11) we get

$$\begin{aligned}
\mathcal{M}(+,+;+,+) =& \frac{e^2}{s}\,2s_{32}s^*_{4,1}, \qquad \mathcal{M}(+,+;-,-) = \frac{e^2}{s}\,2s^*_{1,3}s_{2,4} \\
\mathcal{M}(-,-;+,+) =& \frac{e^2}{s}\,2s_{3,1}s^*_{4,2}, \qquad \mathcal{M}(-,-;-,-) = \frac{e^2}{s}\,2s^*_{2,3}s_{1,4}
\end{aligned} \tag{6.22}$$

where we defined

$$s_{ij} = s(p_i,p_j). \tag{6.23}$$

Therefore we obtain,

$$\begin{aligned}
\overline{|\mathcal{M}|^2} =& \frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{4s^2}\left[|\mathcal{M}(+,+;+,+)|^2 + |\mathcal{M}(+,+;-,-)|^2\right. \\
&\left. +|\mathcal{M}(-,-;+,+)|^2 + |\mathcal{M}(-,-;-,-)|^2\right] \\
=& \frac{e^4}{s^2}\left[2|s_{23}|^2|s_{14}|^2 + 2|s_{13}|^2|s_{24}|^2\right] \\
=& e^4\left[1+\cos^2\theta\right]
\end{aligned} \tag{6.24}$$

where we have used,

$$|s_{13}|^2 = |s_{24}|^2 = -t = -\frac{s}{2}(1-\cos\theta) \tag{6.25}$$

$$|s_{23}|^2 = |s_{14}|^2 = -u = -\frac{s}{2}(1+\cos\theta) \tag{6.26}$$

Inserting in the formula for the differential cross section we obtain

$$\begin{aligned}
\frac{d\sigma}{d\Omega} =& \frac{1}{64\pi^2 s}\,\overline{|\mathcal{M}|^2} \\
=& \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right)
\end{aligned} \tag{6.27}$$

in agreement with Eq. (5.56) in the limit $\beta \to 1$.

6.3 Polarizations for massless gauge fields

Up to now we know how to handle diagrams where the gauge fields are in internal lines of the digram. What happens if we have massless gauge bosons in external lines of the diagram? In that case we know the we have polarization vectors and we want these polarization vectors to obey the well known result,

$$\sum_{\lambda} \epsilon^{\mu}(k,\lambda)\epsilon^{*\mu}(k,\lambda) = -g^{\mu\nu} + \text{terms proportional to } k. \tag{6.28}$$

The terms proportional to k, can be neglected due to gauge invariance (see Complement 5.1). Following Kleiss [52, 53] this can be achieved with the identification

$$\epsilon^{\mu}(k,\lambda) = \frac{1}{(4k\cdot p)^{1/2}}\,\overline{u}_{\lambda}(k)\gamma^{\mu}u_{\lambda}(p) \tag{6.29}$$

where p is any 4-vector satisfying $p^2 = 0$ not proportional to k. As this polarization vector will always appear contracted with a matrix γ_μ we can use Eq. (6.11) to write ($N = \sqrt{4k\cdot p}$),

$$\not{\epsilon}(k,\lambda) = \frac{1}{N}\left[2u_{\lambda}(p)\overline{u}_{\lambda}(k) + 2u_{-\lambda}(k)\overline{u}_{-\lambda}(p)\right] \tag{6.30}$$

which shows that we can use massless spinors to describe the polarizations vectors of massless gauge bosons. It is useful also the write the expression for the complex conjugate. We get

$$\not{\epsilon}^{*}(k,\lambda) = \frac{1}{N}\left[2u_{\lambda}(k)\overline{u}_{\lambda}(p) + 2u_{-\lambda}(p)\overline{u}_{-\lambda}(k)\right] \tag{6.31}$$

Before we end this section we must show that Eq. (6.29) and Eq. (6.28) are indeed consistent. In fact we have,

$$\begin{aligned}
\sum_{\lambda} \epsilon^{\mu}(k,\lambda)\epsilon^{*\nu}(k,\lambda) =& \frac{1}{N^2}\Big[\overline{u}_{+}(k)\gamma^{\mu}u_{+}(p)\overline{u}_{+}(p)\gamma^{\nu}u_{+}(k) \\
&\qquad + \overline{u}_{-}(k)\gamma^{\mu}u_{-}(p)\overline{u}_{-}(p)\gamma^{\nu}u_{-}(k)\Big] \\
=& \frac{1}{N^2}\left(\mathrm{Tr}\left[\gamma^{\mu}\not{p}\gamma^{\nu}\gamma_{+}\not{k}\right] + \mathrm{Tr}\left[\gamma^{\mu}\not{p}\gamma^{\nu}\gamma_{-}\not{k}\right]\right) \\
=& -g^{\mu\nu} + \frac{p^{\mu}k^{\nu} + p^{\nu}k^{\mu}}{k\cdot p}
\end{aligned} \tag{6.32}$$

Choosing a different p will correspond to a gauge transformation, and will not affect the physics. We will give thee example of pair annihilation in the next section.

6.3.1 Pair annihilation

Let us now use the helicity amplitudes technique to evaluate pair annihilation into photons, $e^-e^+ \to \gamma\gamma$, that we have already seen in section 5.5. Looking at Eq. (5.87) we immediately realize that the electron and positron have to have the same chirality. For the photons it is more complicated as because of gauge invariance there is not a unique way of writing the polarization vectors. However we can use this to simplify the calculation, choosing the reference momenta in Eq. (6.29).

The only restriction is that in $\epsilon(k,\lambda)$ the vector p can not be proportional to k. We can even make different choices for different diagrams. For instance, if we choose for $\mathcal{M}_1$, p_1 for $\epsilon(k_1,\lambda)$ and p_2 for $\epsilon(k_2,\lambda)$ and for $\mathcal{M}_2$ the opposite choice, that is, p_2 for $\epsilon(k_1,\lambda)$ and p_1 for $\epsilon(k_2,\lambda)$, we get that the only non-vanishing amplitudes are,

$$\begin{aligned}
\mathcal{M}(++;+-) =& -4e^2\frac{s(p_1,k_1)s(p_2,k_2)s^*(k_1,p_2)s^*(p_1,k_1)}{t}\ \frac{1}{N(p_1,k_1)^2}\\
=& -2e^2\frac{s(p_2,k_2)s^*(k_1,p_2)}{t}\\
\mathcal{M}(++;-+) =& 4e^2\frac{s(p_1,k_2)s(p_2,k_1)s^*(k_2,p_2)s^*(p_1,k_2)}{u}\ \frac{1}{N(p_1,k_2)^2}\\
=& 2e^2\frac{s(p_2,k_1)s^*(k_2,p_2)}{u}\\
\mathcal{M}(--;-+) =& -4e^2\frac{s(p_1,k_1)s(p_2,k_1)s^*(k_2,p_2)s^*(p_1,k_1)}{t}\ \frac{1}{N(p_1,k_1)^2}\\
=& -2e^2\frac{s(p_2,k_1)s^*(k_2,p_2)}{t}\\
\mathcal{M}(--;+-) =& 4e^2\frac{s(p_1,k_2)s(p_2,k_2)s^*(k_1,p_2)s^*(p_1,k_2)}{u}\ \frac{1}{N(p_1,k_2)^2}\\
=& 2e^2\frac{s(p_2,k_2)s^*(k_1,p_2)}{u}
\end{aligned} \tag{6.33}$$

From here we get

$$\sum_{\sigma_1,\sigma_2,\lambda_1,\lambda_2}|\mathcal{M}(\sigma_1,\sigma_2;\lambda_1,\lambda_2)|^2 = 8e^4\frac{u}{t}+8e^4\frac{t}{u}=8e^4\frac{u^2+t^2}{ut} \tag{6.34}$$

in agreement with Eq. (5.92). If we had chosen another reference momentum we would get the same result, although this may be difficult to show. For instance, suppose we use the same convention for $\mathcal{M}_2$ e $\mathcal{M}_1$. In this case one can show that,

$$\begin{aligned}
\mathcal{M}(++;--) =& 0\\
\mathcal{M}(++;-+) =& 4e^2\ \frac{s(p_2,k_2)s(p_2,k_1)s^*(k_2,p_1)s^*(p_1,k_2)}{u}\ \frac{1}{N(p_1,k_1)^2}
\end{aligned}$$

$$
\begin{aligned}
\mathcal{M}(++;+-) =& 4e^2\ \frac{s(p_1,k_2)s(p_2,p_1)s^*(p_1,k_1)s^*(p_1,p_2)}{u}\ \frac{1}{N(p_1,k_1)^2} \\
&- 4e^2\ \frac{s(p_1,k_1)s(p_2,k_2)s^*(k_1,p_2)s^*(p_1,k_1)}{t}\ \frac{1}{N(p_1,k_1)^2} \\
\mathcal{M}(++;++) =& 4e^2\ \frac{s(p_1,p_2)s(p_2,p_1)s^*(p_1,k_1)s^*(p_1,k_2)}{u}\ \frac{1}{N(p_1,k_1)^2} \\
&+ 4e^2\ \frac{s(p_2,k_2)s(p_2,p_1)s^*(k_2,k_1)s^*(p_1,k_2)}{u}\ \frac{1}{N(p_1,k_1)^2} \qquad (6.35) \\
\mathcal{M}(--;--) =& \mathcal{M}(++;++)^* \\
\mathcal{M}(--;-+) =& \mathcal{M}(++;+-)^* \\
\mathcal{M}(--;+-) =& \mathcal{M}(++;-+)^* \\
\mathcal{M}(--;++) =& 0
\end{aligned}
$$

These expressions look different from those in Eq. (6.33). However choosing a kinematics and Eq. (6.10) we can show that we get the same final result. As an example we see that in Eq. (6.33) the amplitude $\mathcal{M}(++;++)$ vanishes, while in Eq. (6.35) it appears that it does not. However choosing the kinematics,

$$
\begin{aligned}
p_1 &= (\sqrt{s}/2, 0, 0, \sqrt{s}/2) \\
p_2 &= (\sqrt{s}/2, 0, 0, -\sqrt{s}/2) \\
k_1 &= (\sqrt{s}/2, 0, \sqrt{s}/2\sin\theta, \sqrt{s}/2\cos\theta) \qquad (6.36) \\
k_2 &= (\sqrt{s}/2, 0, -\sqrt{s}/2\sin\theta, -\sqrt{s}/2\cos\theta),
\end{aligned}
$$

we get from Eq. (6.10)

$$
\begin{aligned}
s(p_1,p_2) &= i\,2\frac{\sqrt{s}}{2} & s(p_1,k_1) &= i\,\frac{\sqrt{s}}{2} - i\,\frac{\sqrt{s}}{2}e^{-i\theta} \\
s(p_1,k_2) &= i\,\frac{\sqrt{s}}{2} + i\,\frac{\sqrt{s}}{2}e^{-i\theta} & s(p_2,k_1) &= -i\,\frac{\sqrt{s}}{2} - i\,\frac{\sqrt{s}}{2}e^{-i\theta} \\
s(p_2,k_2) &= -i\,\frac{\sqrt{s}}{2} + i\,\frac{\sqrt{s}}{2}e^{-i\theta} & s(k_2,k_1) &= -i\,2\,\frac{\sqrt{s}}{2}e^{-i\theta}.
\end{aligned}
$$

Using these explicit expressions we can show that

$$
\begin{aligned}
s(p_1,p_2)s^*(p_1,k_1) \;+\; & s(p_2,k_2)s^*(k_2,k_1) = \\
&= \left(\frac{\sqrt{s}}{2}\right)^2 \left[2i\left(-i + ie^{i\theta}\right) + \left(-i + ie^{-i\theta}\right)\left(2\,ie^{i\theta}\right)\right] \qquad (6.37) \\
&= \left(\frac{\sqrt{s}}{2}\right)^2 \left(2 - 2e^{i\theta} + 2e^{i\theta} - 2\right) = 0
\end{aligned}
$$

which is a necessary relation to verify that $\mathcal{M}(++;++)$ in Eq. (6.35) vanishes. In a similar way one can show that all the expressions in Eq. (6.35) are equivalent to those in Eq. (6.33). In summary, we can choose the reference in the most convenient way in order to simplify calculations, without affecting the final result.

6.4 Polarizations for massive gauge fields

In the previous section we have learned how to use the technique of helicity amplitudes for massless gauge fields, like the photon in QED. However for calculations in the *Standard Model* we need to learn how to use the helicity amplitude technique also for this case. As for the massless case, the key point is that we have to reproduce the well known result for the sum over polarizations, for massive gauge fields like the W and Z,

$$\sum_\lambda \epsilon^\mu(q,\lambda)\epsilon^{*\nu}(q,\lambda) = -g^{\mu\nu} + \frac{q^\mu q^\nu}{M_V^2} \tag{6.38}$$

where $q^2 = M_V^2$ and $V = W, Z$. This can be achieved if we make the identification [52],

$$\epsilon^\mu(q) \to a^\mu = \overline{u}_-(r_1)\gamma^\mu u_-(r_2) \tag{6.39}$$

where $r_{1,2}$ are two massless 4-vectors, such that

$$q = r_1 + r_2 \tag{6.40}$$

and also apply the correspondence

$$\sum_\lambda \epsilon^\mu(q,\lambda)\epsilon^{*\nu}(q,\lambda) \to \frac{3}{8\pi M_V^2}\int d\Omega\ a^\mu a^{*\nu} \tag{6.41}$$

where $d\Omega$ is the solid angle of one of the 4-vectors, say r_1, in the reference frame where the gauge boson is a rest. Let us show that this identification leads to the expected results. We have

$$\begin{aligned}
\int d\Omega\ a^\mu a^{*\nu} &= \int d\Omega\ \overline{u}_-(r_1)\gamma^\mu u_-(r_2)\overline{u}_-(r_2)\gamma^\nu u_-(r_1) \\
&= \int d\Omega\ \mathrm{Tr}\left[\gamma_- \not{r}_1 \gamma^\mu \gamma_- \not{r}_2 \gamma^\nu\right] \\
&= \int d\Omega\,(2r_1^\mu r_2^\nu + 2r_2^\mu r_1^\nu - 2r_1\cdot r_2 g^{\mu\nu})
\end{aligned} \tag{6.42}$$

The last integral can be easily done by noting that it can only depend on the metric and on the 4-vector q of the gauge boson (see Problem 6.3). Using $q^2 = 2r_1\cdot r_2 = M_V^2$ we get

$$\begin{aligned}
I^{\mu\nu} &= \int d\Omega\left(2r_1^\mu r_2^\nu + 2r_2^\mu r_1^\nu - M_V^2\, g^{\mu\nu}\right) \\
&= M_V^2 g^{\mu\nu} A + q^\mu q^\nu B
\end{aligned} \tag{6.43}$$

A and B can be obtained multiplying Eq. (6.43) by $g^{\mu\nu}$ and by $q^\mu q^\nu$ and using the fact that $2q\cdot r_1 = q^2 = M_V^2$. We get

$$4A + B = -8\pi$$

$$A + B = 0 \tag{6.44}$$

which gives $A = -B = -8\pi/3$. We then get

$$\int d\Omega \; a^\mu a^{*\nu} = \frac{8\pi M_V^2}{3}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{M_V^2}\right) \tag{6.45}$$

in agreement with Eqs. (6.38) and (6.41).

6.5 How to introduce massive fermions

To be able to solve any problem with the helicity amplitude technique we have to include massive fermions. This is of course a little bit *contra naturam* as the method was designed for massless fermions. However Kleiss and Stirling [52] showed that it is also possible to extend the method for this case.

The idea is similar to the case of massive gauge fields. If q is the 4-momentum of the massive fermion, $q^2 = m^2$, then we can choose two massless 4-momenta to construct q:

$$q^\mu = p_1^\mu + p_2^\mu, \quad p_1^2 = p_2^2 = 0 \tag{6.46}$$

The solution of the problem is [52],

$$\begin{aligned}
u(q,+) &= \frac{s(p_1,p_2)}{m}\; u_+(p_1) + u_-(p_2) \\
u(q,-) &= \frac{s^*(p_2,p_1)}{m}\; u_-(p_1) + u_+(p_2) \\
v(q,+) &= \frac{s(p_1,p_2)}{m}\; u_+(p_1) - u_-(p_2) \\
v(q,-) &= \frac{s^*(p_2,p_1)}{m}\; u_-(p_1) - u_+(p_2)
\end{aligned} \tag{6.47}$$

We can verify that the previous relations reproduce the results for the spin sums. In fact,

$$\begin{aligned}
\sum_\lambda u(q,\lambda)\overline{u}(q,\lambda) &= \left[\frac{s(p_1,p_2)}{m}\; u_+(p_1) + u_-(p_2)\right]\left[\frac{s^*(p_1,p_2)}{m}\; \overline{u}_+(p_1) + \overline{u}_-(p_2)\right] \\
&\quad + \left[\frac{s^*(p_2,p_1)}{m}\; u_-(p_1) + u_+(p_2)\right]\left[\frac{s(p_2,p_1)}{m}\; \overline{u}_-(p_1) + \overline{u}_+(p_2)\right] \\
&= u_+(p_1)\overline{u}_+(p_1) + u_-(p_2)\overline{u}_-(p_2) + u_-(p_1)\overline{u}_-(p_1) + u_+(p_2)\overline{u}_+(p_2) \\
&\quad + \frac{1}{m}\Big[s(p_1,p_2)u_+(p_1)\overline{u}_-(p_2) + s^*(p_1,p_2)u_-(p_2)\overline{u}_+(p_1) \\
&\qquad + s^*(p_2,p_1)u_-(p_1)\overline{u}_+(p_2) + s(p_2,p_1)u_+(p_2)\overline{u}_-(p_1)\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \not{p}_1 + \not{p}_2 + \frac{1}{m}\left(\not{p}_1\not{p}_2 + \not{p}_2\not{p}_1\right) \\
&= \not{q} + m
\end{aligned}
\tag{6.48}
$$

where we have used Eq. (6.8) and Eq. (6.9). In a similar way one can show

$$
\sum_{\lambda} v(q,\lambda)\overline{v}(q,\lambda) = \not{q} - m \ . \tag{6.49}
$$

Do not forget that in the end one has to do the phase space integrations of the momenta introduced, using the measure,

$$
\int \frac{d\Omega_{\vec{p}_i}}{4\pi} \tag{6.50}
$$

for each 4-momentum, with $p_i^2 = 0$. See specific examples in sections 6.5.1 and 8.2.3.

6.5.1 Compton scattering with helicity amplitudes

Although Compton scattering is more easily evaluated with the usual trace technique, as we have done in section 5.2, we are going here to use the more complicated, for this case, helicity amplitudes technique. As in the standard experimental setup the electron is at rest, we can not neglect its mass. Therefore this a case where we have to use the techniques of section 6.3 for photons and those of section 6.5 for the electrons.

Let us start by writing the amplitudes, Eq. (5.17), in the form (for convenience we designate the momenta of the electrons by p_1 and p_2 and those of the photons by k_1 e k_2),

$$
\begin{aligned}
\mathcal{M}_1(\sigma_1,\sigma_2;\lambda_1,\lambda_2) &= C_1\overline{u}(p_2,\sigma_2 s)\gamma_\nu\left(\not{p}_1 + \not{k}_1 + m\right)\gamma_\mu u(p_1,\sigma_1 s)\epsilon^{\mu}(k_1,\lambda_1)\epsilon^{\nu *}(k_2,\lambda_2) \\
\mathcal{M}_2(\sigma_1,\sigma_2;\lambda_1,\lambda_2) &= C_2\overline{u}(p_2,\sigma_2 s)\gamma_\mu\left(\not{p}_1 - \not{k}_2 + m\right)\gamma_\nu u(p_1,\sigma_1 s)\epsilon^{\mu}(k_1,\lambda_1)\epsilon^{\nu *}(k_2,\lambda_2)
\end{aligned}
\tag{6.51}
$$

where

$$
C_1 = -\frac{e^2}{(p_1+k_1)^2 - m^2}, \qquad C_2 = -\frac{e^2}{(p_1-k_2)^2 - m^2} \ . \tag{6.52}
$$

To continue we use for the spinors with mass the definition of Eq. (6.47) that we write as,

$$
\begin{aligned}
u(p_1,+s) &= \frac{s(r_1,r_2)}{m}\, u_+(r_1) + u_-(r_2) \\
u(p_1,-s) &= \frac{s^*(r_2,r_1)}{m}\, u_-(r_1) + u_+(r_2)
\end{aligned}
$$

$$\begin{aligned} u(p_2,+s) &= \frac{s(w_1,w_2)}{m}\, u_+(w_1) + u_-(w_2) \\ u(p_2,-s) &= \frac{s^*(w_2,w_1)}{m}\, u_-(w_1) + u_+(w_2) \end{aligned} \tag{6.53}$$

where we have defined,

$$p_1 = r_1 + r_2, \quad p_2 = w_1 + w_2, \quad r_i^2 = w_i^2 = 0 \tag{6.54}$$

Now we have to define the polarization vectors for the photons in terms of spinor products, using Eq. (6.29), that is,

$$\epsilon^\mu(k_i,\lambda) = \frac{1}{N_i}\,\overline{u}_\lambda(k_i)\gamma^\mu u_\lambda(r_1),\ N_i = \sqrt{(4k_i\cdot r_1)}, \quad i = 1,2 \tag{6.55}$$

where we have chosen the arbitrary momentum p in Eq. (6.29) as r_1. This allows us to use a momentum already defined in the problem, simplifying the calculation. We can verify that $k_i \cdot r_1 \neq 0$ and therefore this is a possible choice.

Inserting now the definitions of Eqs. (6.53), (6.54) and (6.55) in Eq. (6.51) we can get the 16 helicity amplitudes. We get then,

$$\begin{aligned} \mathcal{M}_1(+,+;+,-) =& \frac{C_1}{N_1N_2}\,[4s(k_2,k_1)s^*(r_1,k_1)s(k_1,r_2)s^*(r_1,w_2) \\ &+4s(k_1,r_2)s(k_2,r_2)s^*(r_1,r_2)s^*(r_1,w_2)] \end{aligned} \tag{6.56}$$

$$\begin{aligned} \mathcal{M}_2(+,+;+,-) =& \frac{C_2}{N_1N_2}\,[4s(k_1,r_2)s(k_2,r_2)s^*(r_1,r_2)s^*(r_1,w_2) \\ &-4s(k_1,k_2)s^*(r_1,k_2)s(k_2,r_2)s^*(r_1,w_2)] \end{aligned} \tag{6.57}$$

and in the same way for the other 15 amplitudes. We see that the problem becomes quite extensive and should handled by appropriate software. In section 7.10.3 one can find a program that allows to evaluate all the amplitudes and automatically obtain an *output* to `Fortran` file.

Using this program in another `Fortran` (see programs available in my site [37]) we get the result of Fig 6.1, where the line corresponds to the analytical form of Klein-Nishina, Eq. (5.49), and the dots the result of the numerical program. The agreement is excellent. See a more complete discussion in section 7.10.3.

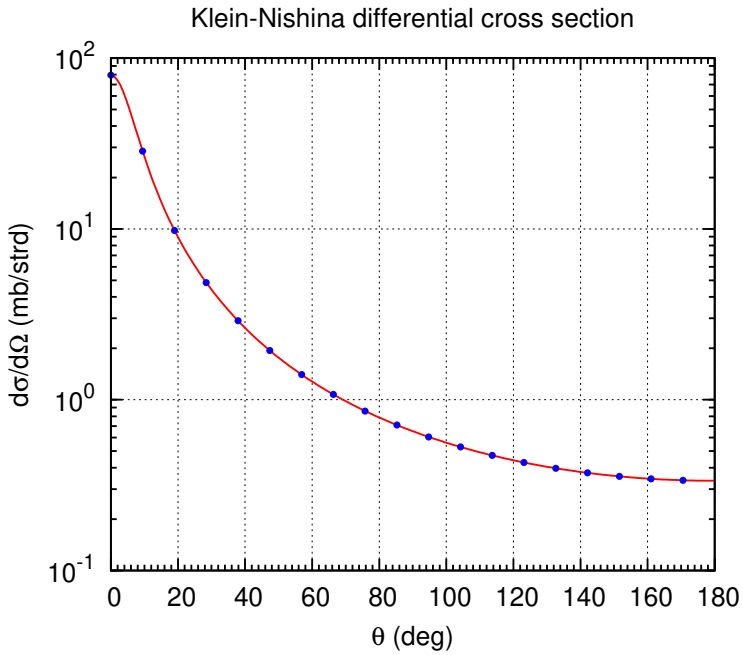


Figure 6.1: Comparison between the Klein-Nishina formula, Eq. (5.49) (red line) with the numerical result using the helicity amplitudes method (blue dots).

Complements

Complement 6.1 Proof of some spinor product relations

We have used, without proof, the definition of the basic spinor product in Eq. (6.10) and a very useful form of the Chisholm relation in Eq. (6.12). In this complement we are going to address these points.

Spinor product definition

We start by defining the properties that our massless ($p^2 = 0$) chiral spinors should obey, already given in Eq. (6.7), that we repeat here

$$\gamma_+ \not{p} = u_+(p)\overline{u}_+(p), \quad \gamma_- \not{p} = u_-(p)\overline{u}_-(p), \quad \not{p} = u_+(p)\overline{u}_+(p) + u_-(p)\overline{u}_-(p) \ . \tag{6.58}$$

Now we want to construct a basis for these spinors. We start by introducing two 4-vectors, k_0 and k_1, that should obey the conditions

$$k_0^2 = 0, \quad k_1^2 = -1, \quad k_0 \cdot k_1 = 0 \tag{6.59}$$

Now define two basic spinors of positive and negative chirality, $u_-(k_0)$ and $u_+(k_0)$ through the relations

$$u_-(k_0)\overline{u}_-(k_0) \equiv \gamma_- \not{k}_0 \tag{6.60}$$

$$u_+(k_0) \equiv \not{k}_1 u_-(k_0) \tag{6.61}$$

It is easy to show that these two basic spinors obey our defining rule in Eq. (6.58). In fact for $u_-(k_0)$ it is its definition. Let us show for $u_+(k_0)$. We have

$$\begin{aligned} u_+(k_0)\overline{u}_+(k_0) =& \not{k}_1 u_-(k_0)\overline{u}_-(k_0)\not{k}_1 \\ =& \not{k}_1\gamma_-\not{k}_0\not{k}_1 = \gamma_+ \not{k}_1\not{k}_0\not{k}_1 \\ =& -\gamma_+ \not{k}_0 k_1^2 = \gamma_+ \not{k}_0 \end{aligned} \tag{6.62}$$

where we have used Eq. (6.60). This shows that our basic spinors $u_\pm(k_0)$ obey the defining rules.

The next step is to use these basic spinors to define a general chiral spinor of momentum p. We define it by the relation ($p^2 = 0$),

$$u_\sigma(p) \equiv \not{p} u_{-\sigma}(k_0) \frac{1}{\sqrt{2p \cdot k_0}} \tag{6.63}$$

with the extra requirement that $p \cdot k_0 \neq 0$. Now we have to show that this definition is consistent with our rules in Eq. (6.58). We have

$$\begin{aligned} u_\sigma(p)\overline{u}_\sigma(p) =& \not{p}\, u_{-\sigma}(k_0)\overline{u}_{-\sigma}(k_0)\, \not{p}\, \frac{1}{2p \cdot k_0} \\ =& \not{p}\, \gamma_{-\sigma}\, \not{k}_0\, \not{p}\, \frac{1}{2p \cdot k_0} = \gamma_\sigma\, \not{p}\, \not{k}_0\, \not{p}\, \frac{1}{2p \cdot k_0} \end{aligned}$$

$$=\gamma_\sigma \not{p} \tag{6.64}$$

showing that it indeed obeys the defining rules. It is also clear that it obeys Dirac equation for massless fermions, $\not{p} u_\sigma(p) = 0$, because $\not{p}\not{p} = p^2 = 0$.

Now we have the tools to evaluate the spinor products. We have

$$\begin{aligned} s(p_1,p_2) =& \overline{u}_+(p_1)u_-(p_2) \\ =& \overline{u}_-(k_0)\not{p}_1\not{p}_2 u_+(k_0)\frac{1}{\sqrt{4(p_1\cdot k_0)(p_2\cdot k_0)}} \\ =& \overline{u}_-(k_0)\not{p}_1\not{p}_2\not{k}_1 u_-(k_0)\frac{1}{\sqrt{4(p_1\cdot k_0)(p_2\cdot k_0)}} \\ =& \mathrm{Tr}\left[\gamma_-\not{k}_0\not{p}_1\not{p}_2\not{k}_1\right]\frac{1}{\sqrt{4(p_1\cdot k_0)(p_2\cdot k_0)}} \\ =& 2\left[(p_1\cdot k_0)(p_2\cdot k_1) - (p_2\cdot k_0)(p_1\cdot k_1) + i\,\epsilon_{\mu\nu\alpha\beta}k_0^\mu k_1^\nu p_1^\alpha p_2^\beta\right]\frac{1}{\sqrt{4(p_1\cdot k_0)(p_2\cdot k_0)}} \end{aligned} \tag{6.65}$$

Now choose $k_0 = (1,1,0,0)$, $k_1 = (0,0,1,0)$ and after some algebra we get Eq. (6.10). Of course there is some degree of arbitrariness in the choice of reference vectors k_0, k_1. However they obey Eq. (6.59) and if the initial particles are in the z direction in principle the condition $p\cdot k_0 \neq 0$ is also verified. So the definition in Eq. (6.10) is a good choice. One can show that physical observables are not affected by this choice.

Chisholm relation

Now for the Chisholm relation. This name is an abuse of language because in strict sense that name applies to the relation,

$$\gamma^\mu\gamma^\nu\gamma^\rho = g^{\mu\nu}\gamma^\rho - g^{\mu\rho}\gamma^\nu + g^{\nu\rho}\gamma^\mu + i\varepsilon^{\mu\nu\rho\alpha}\gamma_\alpha\gamma_5\ . \tag{6.66}$$

The name comes from the fact that for the proof of Eq. (6.11) one needs a generalization of Eq. (6.66). Suppose that one has a string S of an odd number of Dirac matrices (slashed into 4-vectors to keep the indices simple). Then one can show that, when expressed in the basis of the Dirac matrices, it can always be written as

$$S = V_\mu\gamma^\mu + A_\mu\gamma^\mu\gamma_5 \tag{6.67}$$

for two 4-vectors V_μ and A_μ. The proof of this statement is left to Problem 6.2. Now let S^R be a string of the same Dirac matrices but written in reverse order. One can show (see again Problem 6.2) that we have

$$S^R = V_\mu\gamma^\mu + A_\mu\gamma_5\gamma^\mu \tag{6.68}$$

Adding and multiplying by 2 we have

$$2(S+S^R) = 4V_\mu\gamma^\mu + 2A_\mu\left(\gamma^\mu\gamma_5 + \gamma_5\gamma^\mu\right)$$

$$=4V_\mu\gamma^\mu = \mathrm{Tr}[S\gamma_\mu]\gamma^\mu \tag{6.69}$$

Now we choose for S the following expression

$$S = \not{p}_2\gamma_- \not{k}_0\not{p}_1 \frac{1}{\sqrt{4(p_1\cdot k_0)(p_2\cdot k_0)}} \tag{6.70}$$

Then we get

$$S = \not{p}_2 u_-(k_0)\overline{u}_-(k_0)\not{p}_1 \frac{1}{\sqrt{4(p_1\cdot k_0)(p_2\cdot k_0)}} = u_+(p_2)\overline{u}_+(p_1) \tag{6.71}$$

where we have used the definitions in Eqs. (6.60) and (6.63). In a similar way we have

$$S^R = u_-(p_1)\overline{u}_-(p_2) \tag{6.72}$$

and

$$2(S+S^R) = 2u_+(p_2)\overline{u}_+(p_1) + 2u_-(p_1)\overline{u}_-(p_2) \tag{6.73}$$

which is the right-hand side of Eq. (6.11). As for the left side we have, using Eq. (6.71),

$$\mathrm{Tr}[S\gamma_\mu]\gamma^\mu = \mathrm{Tr}[u_+(p_2)\overline{u}_+(p_1)\gamma_\mu]\gamma^\mu = \overline{u}_+(p_1)\gamma_\mu u_+(p_2)\ \gamma^\mu \tag{6.74}$$

which proves Eq. (6.11) for + chirality.

$$\overline{u}_+(p_1)\gamma_\mu u_+(p_2)\ \gamma^\mu = 2u_+(p_2)\overline{u}_+(p_1) + 2u_-(p_1)\overline{u}_-(p_2) \tag{6.75}$$

For the other case, − chirality, we use

$$\begin{aligned}\mathrm{Tr}[S\gamma_\mu]\gamma^\mu =& \mathrm{Tr}[S^R\gamma_\mu]\gamma^\mu \\ =& \mathrm{Tr}[u_-(p_1)\overline{u}_-(p_2)\gamma_\mu]\gamma^\mu \\ =& \overline{u}_-(p_2)\gamma_\mu u_-(p_1)\ \gamma^\mu\end{aligned} \tag{6.76}$$

Now if we relabel $p_1 \leftrightarrow p_2$ we get the desired result,

$$\begin{aligned}\overline{u}_-(p_1)\gamma_\mu u_-(p_2)\ \gamma^\mu &= 2u_+(p_1)\overline{u}_+(p_2) + 2u_-(p_2)\overline{u}_-(p_1) \\ &= 2u_-(p_2)\overline{u}_-(p_1) + 2u_+(p_1)\overline{u}_+(p_2)\end{aligned} \tag{6.77}$$

Complement 6.2 Angular dependance of the amplitudes

In section 5.3.2 we discussed the angular dependance of the amplitudes for the process $e^-e^+ \to \mu^-\mu^+$ using and explicit form for the helicity spinors. We can verify that using now the helicity amplitudes technique. From Eq. (5.72) we get,

$$\mathcal{M}(\uparrow\downarrow;\uparrow\downarrow) = -4\pi\alpha\,(1+\cos\theta) \tag{6.78}$$

and from Eq. (6.22)

$$\mathcal{M}(++;++) = \mathcal{M}(\uparrow\downarrow;\uparrow\downarrow) = 2e^2\frac{s_{32}s_{41}^*}{s} \tag{6.79}$$

Using the definition of Eq. (6.10) we get,

$$s_{32} = i\,\frac{\sqrt{s}}{2}\cos\theta\sqrt{\frac{1}{1-\sin\theta}} + i\,\frac{\sqrt{s}}{2}\sqrt{1-\sin\theta}$$

$$=i\,\frac{\sqrt{s}}{2}\,\frac{1}{\sqrt{1-\sin\theta}}\,(\cos\theta+1-\sin\theta) \tag{6.80}$$

$$\begin{aligned} s_{41} =& -i\,\frac{\sqrt{s}}{2}\cos\theta\sqrt{\frac{1}{1+\sin\theta}} - i\,\frac{\sqrt{s}}{2}\sqrt{1+\sin\theta} \\ =& -i\,\frac{\sqrt{s}}{2}\,\frac{1}{\sqrt{1+\sin\theta}}\,(\cos\theta+1+\sin\theta) \end{aligned} \tag{6.81}$$

and therefore

$$s_{32}s_{41}^{*} = -\frac{s}{2}(1+\cos\theta) \tag{6.82}$$

obtaining finally

$$\mathcal{M}(\uparrow\downarrow;\uparrow\downarrow) = -4\pi\alpha\,(1+\cos\theta) \tag{6.83}$$

in agreement with Eqs. (5.72) or (6.78).

With the trace technique it is not possible to get the amplitudes but only the square of the amplitudes. Nevertheless it is possible to separate the various amplitudes using appropriate projectors. The amplitude that we are discussing here corresponds to evaluate,

$$\begin{aligned} \overline{|\mathcal{M}|^2} =& \frac{e^4}{s^2}\mathrm{Tr}[P_R\not{p}_2\gamma^\mu P_R\not{p}_1\gamma^\nu]\mathrm{Tr}[P_R\not{p}_3\gamma_\mu P_R\not{p}_4\gamma_\nu] \\ =& \frac{4\pi\alpha}{s^2}\,16 p_1\cdot p_4\, p_2\cdot p_3 \end{aligned} \tag{6.84}$$

Using

$$p_1\cdot p_4 = p_2\cdot p_3 = \frac{s}{4}(1+\cos\theta) \tag{6.85}$$

we get

$$|\mathcal{M}_{RRRR}|^2 = |\mathcal{M}(++;++)|^2 = |\mathcal{M}(\uparrow\downarrow;\uparrow\downarrow)|^2 = (4\pi\alpha)^2(1+\cos\theta)^2\ . \tag{6.86}$$

in complete agreement.

Problems

6.1 Derive Eq. (6.14).

6.2 In the text we used the relations in Eqs. (6.67) and (6.68),

$$S = V_\mu \gamma^\mu + A_\mu \gamma^\mu \gamma_5, \quad S^R = V_\mu \gamma^\mu + A_\mu \gamma_5 \gamma^\mu \tag{6.87}$$

where S is any string with an odd number of gamma matrices and S^R is the string with the same Dirac matrices written in reversed order. Use the results of Problem 4.6 and Eq. (4.68) to prove these results.

6.3 In Eq. (6.42) the term in the trace proportional to γ_5 was discarded. This problem is intended to show that the term in fact does not contribute. After taking the trace this terms would be proportional to

$$\epsilon^{\mu\nu\alpha\beta} r_{1\alpha} r_{2\beta} \tag{6.88}$$

Let us see, in two different ways, that we have,

$$I^{\mu\nu} = \int d\Omega \epsilon^{\mu\nu\alpha\beta} r_{1\alpha} r_{2\beta} = 0 \tag{6.89}$$

a) Choose a reference frame where the gauge boson is at rest. Show that in that frame we have

$$r_1 = \frac{M_W}{2}(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{6.90}$$

$$r_2 = \frac{M_W}{2}(1, -\sin\theta\cos\phi, -\sin\theta\sin\phi, -\cos\theta) \tag{6.91}$$

Using the fact that $\alpha \neq \beta$ show explicitly that $I^{\mu\nu} = 0$.

b) The other way to argue is based in Lorentz arguments. Besides $g^{\mu\nu}$ and q^μ we also have at our disposal the tensor $\epsilon^{\mu\nu\alpha\beta}$. Explain why this contribution is zero. You do not have to do any calculation.

6.4 Derive Eq. (6.49).

6.5 Derive the results of Eq. (6.17).

6.6 Show that the s channel contribution to the amplitude $\mathcal{M}(+,+;+,+)$ in Eq. (6.17), that we denote by $\mathcal{M}_1(+,+;+,+)$ is equal to what you have obtained in Prob. 5.6 for the amplitude $\mathcal{M}_1(\uparrow,\downarrow;\uparrow,\downarrow)$. For this use the explicit form of the spinor product, Eq. (6.10).

6.7 In Complement 6.1 we derived an explicit expression for the spinor products, Eq. (6.10). This explicit expression is attached to a specific choice of the auxiliary dimensionless 4-vectors, k_0 and k_1. We can think that the choice of $k_0 = (1,1,0,0)$ is general if the reaction plane is taken as the yz plane, as this ensures that $p\cdot k_0 \neq 0$. However, even with this choice, we could take

$$k_1 = (0,0,\cos\alpha,\sin\alpha) \tag{6.92}$$

and still satisfy Eq. (6.59). On the other hand, following Ref. [54], we have defined helicity spinors for massless fermions in Eq. (1.257). Then the question arises how are these related and are the physical results independent of the choices? This problem aims at clarifying these points.

a) Consider an arbitrary massless positive energy fermion moving in the plane yz, that is, $p = (E,0,E\sin\theta,E\cos\theta)$. Show that the relation between the Kleiss and Thomson definitions is

$$u_-^{\text{Thomson}}(p) = e^{i\,\theta/2}e^{i\,\alpha}\; u_-^{\text{Kleiss}}(p) \tag{6.93}$$

$$u_+^{\text{Thomson}}(p) = e^{-i\,\theta/2}\; u_+^{\text{Kleiss}}(p) \tag{6.94}$$

b) Show that the absolute value of the spinor products does not depend on α and therefore physical variables like the scalar products are also independent of this choice. To show this derive the following relation,

$$s(p_1,p_2,\alpha) = e^{-i\,\alpha}s(p_1,p_2,\alpha=0)\ , \tag{6.95}$$

c) Consider two massless fermions of positive energy with momenta,

$$p_1 = (E_1,0,E_1\sin\theta_1,E1\cos\theta_1),\quad p_2 = (E_2,0,E_2\sin\theta_2,E1\cos\theta_2) \tag{6.96}$$

Use the definitions of Eq. (1.257) to evaluate explicitly

$$s^{\text{Thomson}}(p_1,p_2) \equiv \overline{u}_\uparrow(p_1)u_\downarrow(p_2) \tag{6.97}$$

Then use, for an arbitrary α, the definition of Eq. (6.95)

$$s^{\text{Kleiss}}(p_1,p_2) \equiv s(p_1,p_2,\alpha) \tag{6.98}$$

Show that

$$s^{\text{Thomson}}(p_1,p_2) = e^{i/2(\theta_1+\theta_2)}e^{i\alpha}s^{\text{Kleiss}}(p_1,p_2) \tag{6.99}$$

6.8 Consider again Bhabha scattering. For massless fermions we used the trace technique in section 5.4 and the helicity amplitude technique of Ref. [52] in section 6.2.1. The purpose of this problem is to show that one could get the same result using the massless helicity spinors of Ref. [54] as given, for instance, in Eq. (5.59) and Eq. (5.60). For instance, in this case define for positive energy spinors,

$$s_{ij}^{\text{Thomson}} = \overline{u}_{\uparrow}(p_i)u_{\downarrow}(p_j) \ , \tag{6.100}$$

and evaluate Eq. (6.17) with this definition. Show that you get the same result for the amplitude squared. Notice that there are some subtleties in going from a notation in terms of chiral spinors and one in terms of massless helicity spinors. For instance in this problem we have, in the usual notation, the spinors $u(p_1)$, $u(p_3)$ of positive energy and $v(p_2)$, $v(p_4)$ of negative energy. Then the correct definition of spinor products in the notation of Ref. [54] is,

$$s_{12} = \overline{u}_{\uparrow}(p_1)v_{\uparrow}(p_2), \quad s_{13} = \overline{u}_{\uparrow}(p_1)u_{\downarrow}(p_3), \quad s_{14} = \overline{u}_{\uparrow}(p_1)v_{\uparrow}(p_4) \tag{6.101}$$

$$s_{23} = \overline{v}_{\downarrow}(p_2)u_{\downarrow}(p_3), \quad s_{24} = \overline{v}_{\downarrow}(p_2)v_{\uparrow}(p_4), \quad s_{34} = \overline{u}_{\uparrow}(p_3)v_{\uparrow}(p_4) \ . \tag{6.102}$$

Then the advantage of the method of Kleiss [52] is that it is more convenient to implement from the computational point of view.

Chapter 7

Software for Quantum Field Theory

7.1 Introduction

In this chapter we will describe a large amount of software that we can use to help to do complicated calculations in Quantum Field Theory. This include the program `QGRAF` written in `Fortran` to find the Feynman diagrams for any model at any number of loops, packages for `Mathematica` like `FeynCal`, `FeynCalc`, `FeynRules` as well as other software like `CalcHEP` and `LoopTools`. We will also discuss `FeynMaster` that makes use of `QGRAF`, `FeynCalc` and `LoopTools` in a self contained way. We leave for more advanced texts the discussion `FeynArts` or `FormCalc`.

In the first part of this chapter we will describe the use of the software and then in the final sections of this chapter we will show how to apply it for the same examples in QED that we discussed in the first chapter. More complex problems will be dealt with in the following chapters.

7.2 QGRAF

7.2.1 Setup

QGRAF is an excellent program developed by Paulo Nogueira for the automatic generation of Feynman diagrams, including symmetry factors and fermionic signs. The program can be obtained from the webpage:

`http://cfif.ist.utl.pt/~paulo/qgraf.html`

The manual comes with the package. It generates symbolic expressions for the Feynman diagrams, which can then be imported into other programs. It does not present the Feynman diagrams graphically.

QGRAF is based on the method described in [55]. The program is easy to install in Linux, following the guidelines on the website. To compile you need a Fortran compiler. In Linux the easiest is to use `gfortran`. The command line is

```
> gfortran qgraf-3.x.y.f -o qgraf
```

where x and y are the last version numbers and qgraf is the executable file.

Running in Windows is more cumbersome. This is a program written in Fortran 77, so, if one decides to compile the latest version, one needs a Fortran compiler. Another option is to install a Virtual Machine in Windows running Linux. Then one uses the standard `gfortran` compiler. In the site CTQFT,

`https://porthos.tecnico.ulisboa.pt/CTQFT/`

one can find more information.

For the file management instructions, we assume a Linux (or Virtual Machine running Linux) installation. All other instructions are machine independent.

7.2.2 Simple tests

Available Models

In our installation, QGRAF is run from

```
> cd SOFTWARE/qgraf
```

The instruction

```
> ll Models
```

lists all Models available in the distribution, including QED (qed) and SM (gws-simple), which we will be using in this course. The qed file reads:

```
*  leptons
  [e,E,-]

*  photon
  [A,A,+]

*  fermion - gauge boson
  [E,e,A]
```

In this notation, the lowercase letter is the particle and the uppercase letter is the antiparticle. So `e` = electron and `E` = positron.

Since the photon, `A`, is its own antiparticle, one uses `A,A...`. In `[e,E,-]`, the minus sign means that the particle is a fermion, while in `[A,A,+]`, the plus sign means that the particle is a boson. Following a similar strategy, it is easy to build a new model file and study it.

Building the input file

The input file is named qgraf.dat. This name is **mandatory**. You can open the file in Linux with

```
> emacs qgraf.dat &
```

One example of the qgraf.dat file is:

```
output= list ;
style= Styles/sum.sty ;
model=  Models/qed;
in= e,e;
out=e,e;
loops= 0;
loop_momentum= ;
options=;
```

The first line describes the filename to which results will be sent to. In our case, "list".
The second line describes the style for the output (see the manual for details).
The third line describes the model to be considered. In our case, "qed" in the "Models" directory.
The fourth (fifth) line describes the incoming (outgoing) particles. In our case, it is the Möller scattering process $e^-e^- \to e^-e^-$.
The sixth line describes the number of loops. Zero loops is tree level, and, in that case, both "loop_momentum=" and "options=" should not be set to anything. In case one wants a calculation to two-loop (say), then one must set "loops= 2;". In such a case, setting "options=onepi;" guarantees that only one-particle-irreducible diagrams are considered. Further details can be found in the manual.

When unsure whether there will be many diagrams (ie, huge output file), one should not generate an output file. This is achieved by setting

```
output= '' ;
```

When running this from a terminal window (Linux or Windows), the number of diagrams will be shown on the terminal. For the above example we will get

```
------------------------------------------------------------
                          qgraf-3.4
------------------------------------------------------------
 output= list ;
 style= Styles/sum.sty ;
 model= Models/qed;
 in= e,e;
 out= e,e;
 loops= 0;
 loop_momentum= ;
 options= ;
------------------------------------------------------------
```

```
   v-degrees         #diagrams

      3^2      ....      2

      total =  2 diagrams
```

Running QGRAF

One can run the program in two ways: i) by executing

```
> ./qgraf
```

or, ii) by executing

```
> ./run
```

where `run` is an executable found on the CTQFT webpage. On the Windows distribution, just double-click on the `run` file. QGRAF will not write over the output file. So, one must first delete the output file in the working QGRAF directory; this is what the `run` instruction does. To keep an old result, just move the old output file into another directory or rename it.

Imagine one has set

```
output= '' ;
```

There will be no output file, but global information (for example, the number of diagrams) can be seen when running from a terminal window (Linux or Windows), as illustrated above.

For Windows, to get to the terminal window, do:

```
Winkey+R
cmd
enter
```

One can then use

```
cd ..
```

to go up one level in the directory,

```
dir
```

to find the subdirectories of that level, and

```
cd subdirectoryname
```

to go to the `subdirectoryname` subdirectory. Proceeding in this way, one can reach the directory where "qgraf.dat" and `run` are installed. As an example, using the qgraf.dat file

```
output=  ;
style= Styles/sum.sty ;
model=  Models/qed;
in= e,e;
out=e,e;
loops= 2;
loop_momentum= ;
options=onepi;
```

one can check that the two-loop $e^-e^- \to e^-e^-$ calculation has 52 one-particle-irreducible diagrams.

Understanding the output file

For the $e^-e^- \to e^-e^-$ process described above at tree level `loops=0`, the output file, therein named "list", reads:

```
+(1)*
 prop(A(1,-p1+q1),A(2,p1-q1))*
 vrtx(E(-2,-q1),e(-1,p1),A(1,-p1+q1))*
 vrtx(E(-4,-q2),e(-3,p2),A(2,p1-q1))

 -(1)*
 prop(A(1,-p1+q2),A(2,p1-q2))*
 vrtx(E(-4,-q2),e(-1,p1),A(1,-p1+q2))*
 vrtx(E(-2,-q1),e(-3,p2),A(2,p1-q2))
```

Notice the relative minus sign between the two diagrams, as expected from the anticommuting nature of the fermionic fields. The particles are identified by the letter included in the model. If no other choice is imposed (see manual), the momenta are identified as:

`p` = initial state momenta
`q` = final state momenta
`k` = loop momenta

where **all momenta in a vertex are incoming**.

Each time a particle is mentioned, it is identified as, for example, `E(-4,-q2)`. There is a number and a letter inside the parenthesis. Both have meaning. The letter indicates what is the notation for the momentum of the particle. The notation for the numbers respects the following rules:

negative number = external particle
    odd negative number = initial state particle
    even negative number = final state particle
positive number = internal particle

In our example, `e(-1,p1)` and `e(-3,p2)` are the two incoming electrons. Since all momenta are considered ingoing to a vertex, the outgoing electrons are represented

by `E(-2,-q1)` and `E(-4,-q2)`. To recapitulate, the minus sign means that these are exterior particles, the even numbers (and the momenta letters `q`) indicate that these are outgoing particles, and the positron `E` with momenta `-q` going into the vertex, corresponds to an electron, `e`, with momentum `+q` going away from the vertex, as befits a final state particle.

7.3 Symbolic Calculations with Mathematica

Nowadays `Mathematica` is an essential tool for any scientist. For QFT there are several useful packages. In this section we will introduce two of these packages, **FeynCalc** [38, 39, 56] and **FeynRules** [57, 58].

7.3.1 FeynCalc

This a very ambitious software [38, 39, 56] that allow us to perform Lorentz and Dirac algebra, but also to do calculations at *one-loop*.

Setup

This software works as a package of `Mathematica`. It can be downloaded from

`https://feyncalc.github.io/`

The installation is very simple, just follow the instructions on that page. After installation to use it, just open a new `Notebook` in a `Mathematica` session. The package is loaded as any other package,

```
In[1]:= << FeynCalc`
```

FeynCalc 9.3.0 (stable version). For help, use the documentation center, check out the wiki or write to the mailing list.

To save your and our time, please check our FAQ for answers to some common FeynCalc questions.

See also the supplied examples. If you use FeynCalc in your research, please cite

- V. Shtabovenko, R. Mertig and F. Orellana, P3H–20–002, TTP19–020, TUM–EFT 130/19, arXiv:2001.04407
- V. Shtabovenko, R. Mertig and F. Orellana, Comput. Phys. Commun., 207, 432–444, 2016, arXiv:1601.01167
- R. Mertig, M. Böhm, and A. Denner, Comput. Phys. Commun., 64, 345–359, 1991.

Lorentz Algebra

`FeynCalc` is very useful to do the Lorentz algebra. It can be done both in 4 dimensions as well as in D dimensions. This feature is only needed for one-loop calculations, and we postpone the discussion of the algebra in dimension D for chapter 9 when we discuss one-loop processes.

The names of the objects follow the Mathematica convention of having a long descriptive name, like MetricTensor or FourVector. However when we use this software in practice it is normally better to have a short name. This is also provided in FeynCalc. In the following table we give the name (full and short) for the various objects needed for Lorentz algebra. Internally there is still another represen-

Object	Long Name	Short Name
p^μ	FourVector[p,μ]	FV[p,μ]
$g^{\mu\nu}$	MetricTensor[μ,ν]	MT[μ,ν]
$(p\cdot q)=p_\mu q^\mu$	ScalarProduct[p,q]	SP[p,q]
$\epsilon^{\mu\nu\alpha\beta}$	LeviCivita[μ,ν,α,β]	LC[μ,ν,α,β]

Table 7.1: Objects for the Lorentz algebra

tation. One can toggle between these different forms using the FeynCalc functions FeynCalcInternal and FeynCalc functions FeynCalcExternal with short names FCI and FCE, respectively. These are very useful if one wants to make replacement lists. As in the front end one always sees the same representation the Mathematica command InputForm is very useful to tell us in which representation we are. We will exemplify below the various representations with the case of the FourVector.

```
In[2]:= FourVector[p, μ]
Out[2]= p^μ

In[3]:= FourVector[p, μ] // InputForm
Out[3]//InputForm=
Pair[LorentzIndex[μ], Momentum[p]]

In[4]:= MetricTensor[μ, ν] // InputForm
Out[4]//InputForm=
Pair[LorentzIndex[μ], LorentzIndex[ν]]

In[5]:= FourVector[p, μ] // InputForm
Out[5]//InputForm=
Pair[LorentzIndex[μ], Momentum[p]]

In[6]:= FourVector[p, μ] // FCE // InputForm
Out[6]//InputForm=
FV[p, μ]

In[7]:= FV[p, μ] // FCI // InputForm
Out[7]//InputForm=
Pair[LorentzIndex[μ], Momentum[p]]
```

After we do a calculation on has to be aware that the internal representation can have changed. Let us illustrate this with the use of the command Contract that performs the Lorentz contraction of indices.

```
In[9]:= ans = FV[p, μ] FV[p, μ] // Contract
Out[9]= p̄^2

In[10]:= ans // InputForm
Out[10]//InputForm=
Pair[Momentum[p], Momentum[p]]

In[11]:= ans // FCE // InputForm
Out[11]//InputForm=
SP[p, p]
```

This is particularly useful if one wants to make substitution rules, as in the following example

```
In[15]:= ans = FV[p, μ] FV[p, μ] // Contract // FCE
Out[15]= p̄^2

In[13]:= onshell = {SP[p, p] → m^2}
Out[13]= {p̄^2 → m^2}

In[16]:= ans /. onshell
Out[16]= m^2
```

The same result could be obtained with the function `Calc`. This function performs a variety of calculations (see the on-line documentation for this and other functions referred to here). We get

```
In[2]:= onshell = {SP[p, p] → m^2}
Out[2]= {p̄^2 → m^2}

In[5]:= (((FV[p, μ] FV[p, μ]) // Calc) // FCE) /. onshell
Out[5]= m^2
```

Notice that we can do everything in one line, but with the use of parenthesis. There is another function from `FeynCalc` that it is very useful, `ExpandScalarProduct`. Its usage is exemplified below,

```
In[17]:= onshell = {SP[p, p] → m^2, SP[q, q] → M^2};

In[10]:= FV[p - q, μ] FV[p + q, μ]
Out[10]= (p̄ - q̄)^μ (p̄ + q̄)^μ

In[14]:= ans = FV[p - q, μ] FV[p + q, μ] // ExpandScalarProduct
Out[14]= (p̄^μ - q̄^μ) (p̄^μ + q̄^μ)

In[16]:= (ans // Contract // FCE) /. onshell
Out[16]= m^2 - M^2
```

Note that `ExpandScalarProduct`, as well as `Contract` and `Calc` leave the result in

FeynCalc internal notation, so the need to apply FCE before applying the substitution rules. There is one more relevant piece of information when dealing with the LeviCivita tensor. In the short name in table 7.1 all the entries of the LeviCivita are Lorentz indices. However sometimes these indices are contract with four vectors. In those cases, FeynCalc has a convenient notation as shown below[1]

```
In[33]:= LC[μ, ν, α, β] // FCE // InputForm

Out[33]//InputForm=
LC[μ, ν, α, β]

In[27]:= LC[μ, ν, α, β] FV[r, β] // Contract // FCE // InputForm

Out[27]//InputForm=
LC[α, μ, ν][r]
```

or with more four vectors,

```
In[28]:= LC[μ, ν, α, β] FV[q, α] FV[r, β] // Contract // FCE // InputForm

Out[28]//InputForm=
LC[μ, ν][q, r]

In[29]:= LC[μ, ν, α, β] FV[p, ν] FV[q, α] FV[r, β] // Contract // FCE // InputForm

Out[29]//InputForm=
LC[μ][p, q, r]

In[30]:= LC[μ, ν, α, β] FV[k, μ] FV[p, ν] FV[q, α] FV[r, β] // Contract // FCE // InputForm

Out[30]//InputForm=
LC[][k, p, q, r]
```

Dirac Algebra

FeynCalc allows for very efficient computations with the Dirac algebra. The objects can be used both in 4 dimensions as well as in dimension D. This last feature is only useful for one-loop calculations and we postpone its explanation to chapter 9. The more important objects are given in the following table,

Object	Long Name	Short Name
$u(p,m), \overline{u}(p,m)$	Spinor[p,m]	Spinor[p,m]
$v(p,m), \overline{v}(p,m)$	Spinor[-p,m]	Spinor[-p,m]
γ^μ, γ_μ	DiracMatrix[μ]	GA[μ]
γ_5	DiracMatrix[5]	GA[5]
$\not{p}$	DiracSlash[p]	GS[p]

Table 7.2: Basic objects for the Dirac algebra

[1] Be aware that Mathematica orders the symbols in alphabetical order, and that we have LC[α, μ, ν][r]=LC[μ, ν, α][r].

A few clarifications are in order. As with the Lorentz algebra there is no distinction between up and down Lorentz indices. There is also no distinction between spinors u, v and $\overline{u}, \overline{v}$. Of course the objects are matrices and vectors in Dirac space and proper matrix multiplication (the .) in `Mathematica` has to be taken in account. For instance to write

$$\overline{v}(p_2, m)\gamma^\mu u(p_1, m)\ \overline{u}(p_3, m)\gamma_\mu v(p_4, m) \tag{7.1}$$

we enter the code,

```
In[10]:= Spinor[-p2, m].GA[μ].Spinor[p1, m] Spinor[p3, m].GA[μ].Spinor[-p4, m]
Out[10]= (φ(-p2, m)).γ^μ.(φ(p1, m)) (φ(p3, m)).γ^μ.(φ(-p4, m))
```

This is the easiest way to handle spinors, but there are other equivalent ways. In particular the more usual notation `SpinorU`, `SpinorUBar`, `SpinorV`, `SpinorVBar` can be used, but now with the true momenta. To see this one has to look at the internal form to realize that they mean the same,

```
In[14]:= Spinor[p, m] // FCI // InputForm
Out[14]//InputForm=
Spinor[Momentum[p], m, 1]

In[15]:= SpinorU[p, m] // FCI // InputForm
Out[15]//InputForm=
Spinor[Momentum[p], m, 1]

In[16]:= Spinor[-p, m] // FCI // InputForm
Out[16]//InputForm=
Spinor[-Momentum[p], m, 1]

In[17]:= SpinorV[p, m] // FCI // InputForm
Out[17]//InputForm=
Spinor[-Momentum[p], m, 1]

In[18]:= SpinorUBar[p, m] // FCI // InputForm
Out[18]//InputForm=
Spinor[Momentum[p], m, 1]
```

Therefore the code that corresponds to Eq. (7.1) could be entered in the more familiar way,

```
In[19]:= SpinorVBar[p2, m].GA[μ].SpinorU[p1, m] SpinorUBar[p3, m].GA[μ].SpinorV[p4, m]
Out[19]= v̄(p2, m).γ^μ.u(p1, m) ū(p3, m).γ^μ.v(p4, m)
```

Before we explain a number of useful operations on Dirac objects let us give two more definitions already in `FeynCalc`. They concern the left and right projectors. The definitions are given in table 7.3

Object	Long Name	Short Name
γ_5	DiracMatrix[5]	GA[5]
$P_R = \frac{1}{2}(1+\gamma_5)$	DiracMatrix[6]	GA[6]
$P_L = \frac{1}{2}(1-\gamma_5)$	DiracMatrix[7]	GA[7]

Table 7.3: Left and right projectors.

One can easily check that these obey the known rules, as in the following code, where we have used DiracSimplify, the first of a number of FeynCalc functions designed to handle the Dirac algebra in an efficient way.

```
In[22]:= GA[6].GA[7] // DiracSimplify
Out[22]= 0

In[23]:= GA[6].GA[6] // DiracSimplify
Out[23]= γ̄6

In[24]:= GA[6].(1 - GA[5]) // DiracSimplify
Out[24]= 0
```

DiracSimplify performs a number of simplifications for the Dirac algebra, as exemplified in the following code

```
In[31]:= ans = SpinorUBar[p2, m2].GA[mu].(GS[p2 - p1] + m2).GA[mu].SpinorU[p1, m1]
Out[31]= ū(p2, m2).γ̄^mu.(γ̄·(p̄2 − p̄1) + m2).γ̄^mu.u(p1, m1)

In[32]:= DiracSimplify[ans]
Out[32]= 2 m1 (φ(p̄2, m2)).(φ(p̄1, m1)) + 2 m2 (φ(p̄2, m2)).(φ(p̄1, m1))
```

DiracSimplify has many options (27 in total) that can be seen using the Mathematica usual way

```
Options[DiracSimplify]
```

The user should get familiar with these options. Here we just refer to two of them, that allow to write the results in terms of γ_5 or alternatively in terms of P_L and P_R, as shown below,

```
In[11]:= ans1 = SpinorUBar[p, m].GA[μ].(gV - gA GA[5]).SpinorU[p, m]
Out[11]= ū(p, m).γ̄^μ.(gV − gA γ̄^5).u(p, m)

In[14]:= DiracSimplify[ans1, DiracSubstitute5 → True]
Out[14]= −gA (φ(p̄, m)).γ̄^μ.γ̄^6.(φ(p̄, m)) + gA (φ(p̄, m)).γ̄^μ.γ̄^7.(φ(p̄, m)) + gV (φ(p̄, m)).γ̄^μ.(φ(p̄, m))
```

or

```
In[12]:= ans2 = SpinorUBar[p, m].GA[μ].GA[7].SpinorU[p, m]
Out[12]= ū(p, m).γ̄^μ.γ̄^7.u(p, m)

In[13]:= DiracSimplify[ans2, DiracSubstitute67 → True]
Out[13]= 1/2 (φ(p̄, m)).γ̄^μ.(φ(p̄, m)) - 1/2 (φ(p̄, m)).γ̄^μ.γ̄^5.(φ(p̄, m))
```

Four our calculations in this chapter, at tree level, there is on other very important function that evaluates the traces of Dirac γ matrices. Its usage is very simple as shown in the following code,

```
In[15]:= Tr[GA[α].GA[β].GA[μ].GA[ν]]
Out[15]= 4 (ḡ^αν ḡ^βμ - ḡ^αμ ḡ^βν + ḡ^αβ ḡ^μν)

In[16]:= Tr[GA[α].GA[β].GA[μ].GA[ν].GA[5]]
Out[16]= -4 i ε̄^αβμν
```

Before we close this section there are two extra functions that are very useful. They relate to the fermion spin sum. We just give the example of $e^-e^+ \to \mu^-\mu^+$ that we have studied before. We saw in Eq. (4.57) how to transform the fermion spin sums in traces. Now we can do this automatically. Let us consider again the amplitude of Eq. (5.51), that we recall here for convenience,

$$\mathcal{M} = \frac{e^2}{s}\,\overline{v}(p_2, s_e')\gamma^\mu u(p_1, s_e)\overline{u}(p_3, s_\mu')\gamma_\mu v(p_4, s_\mu) \tag{7.2}$$

We start by writing the amplitude in `FeynCalc`,

```
In[14]:= M = e^2/s SpinorVBar[p2, me].GA[μ].SpinorU[p1, me]
            SpinorUBar[p3, mm].GA[μ].SpinorV[p4, mm]
Out[14]= e^2 v̄(p2, me).γ̄^μ.u(p1, me) ū(p3, mm).γ̄^μ.v(p4, mm) / s
```

Then we use the function `ComplexConjugate` to write the adjoint of $\mathcal{M}$. This function is very useful because it understands the Dirac algebra but also takes the complex conjugate of the other quantities assuming everything real except if declared complex. We get

```
In[3]:= Mc = ComplexConjugate[M] /. {μ → ν}
Out[3]= e^2 (φ(p1̄, me)).γ̄^$AL($24).(φ(-p2̄, me)) (φ(-p4̄, mm)).γ̄^$AL($24).(φ(p3̄, mm)) / s
```

Finally we use the function `FermionSpinSum` to transform in traces and use `Calc` to evaluate the traces and do the contractions. We obtain

```
In[16]:= Msqavg = 1 / 4 FermionSpinSum[ M Mc] // Calc // FCE // Simplify
Out[16]= 8 e^4 (me^2 (p3 · p4) + mm^2 (p1 · p2) + (p1 · p4) (p2 · p3) + (p1 · p3) (p2 · p4) + 2 me^2 mm^2) / s^2
```

in agreement with Eq. (5.52). This section is just a brief introduction to `FeynCalc` using processes at tree level to describe the most important functions. For a full list one should consult the online reference guide,

`https://feyncalc.github.io/reference`

We will come back to this in chapter 9, while studying higher order loop processes.

7.3.2 FeynRules

`FeynRules` [57,58] is a `Mathematica` package that allows the calculation of Feynman rules in momentum space for any QFT physics model. The user needs to provide FeynRules with the minimal information required to describe the new model, contained in the so-called model-file. This information is then used to calculate the set of Feynman rules associated with the Lagrangian. It create files that that be handled by `FeynArts`, `MadGraph` or `FeyMaster`. It can be downloaded from

`https://feynrules.irmp.ucl.ac.be/`

Just follow the installation instructions on that web page. The program already comes with the model files for QED and the Standard Model. More models can be downloaded from the web page.

Although all the information can be given in just one model file, it is normally better to separate the definition of the particles from the parameters and from the Lagrangian. For instance, just consider the Lagrangian of QED

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + i\overline{\psi}\not{D}\psi - m_e\overline{\psi}\psi, \quad D_\mu \equiv \partial_\mu + ieQ_eA_\mu \tag{7.3}$$

Then the model file will look like

```
(********** This is the FeynRules model file for QED **********)


(***** Initial information *****)
M$ModelName = "QED";

FR$DSign = -1;

M$Information = {Authors -> "Duarte Fontes",
                 Institutions -> "Instituto Superior Tecnico",
                 Emails -> "duartefontes@tecnico.ulisboa.pt",
                 Date -> "16.07.2019"};
```

```
(***** Gauge group list *****)
M$GaugeGroups = {
  U1 == {
                Abelian -> True,
                GaugeBoson -> A,
                Charge -> Q,
                CouplingConstant -> ee}  }

Get["Parameters.fr"];
Get["ParticleClasses.fr"];
Get["Lagrangean.fr"];
```

In this model file we just define the gauge group and then load the files for the parameters, for the particle content and finally the Lagrangian. Notice also the assignment `FR$DSign = -1`. This is important to be in agreement with our conventions for the covariant derivative, as we will show below. The file for the parameters will very simple in QED,

```
(***** Parameter list *****)
M$Parameters = {
  mf == {TeX -> Subscript[m,f]},
  ee == {TeX -> e},
  xiA == {TeX -> Subscript[\[Xi],A]}}
```

Next we have to define the particle classes. `FeynRules` has a naming convention shown in Table 7.4. With these conventions we have, the for the file

Spin	0	1/2	1	2	Ghost
Symbol	S	F	V	T	U

Table 7.4: Names for the particle classes in `FeynRules`.

```
(***** Particle classes list *****)
M$ClassesDescription = {
  F[1] == {
                ClassName -> l,
                SelfConjugate -> False,
                QuantumNumbers -> {Q-> Q},
                Mass -> {me,0.000511},
                Width -> 0},
  V[1] == {
                ClassName -> A,
                Mass -> 0,
                SelfConjugate -> True}  }
```

Finally we write the Lagrangian using the conventions are built in in `FeynRules`

```
(***** QED Lagrangian *****)
```

```
LGauge := -1/4 FS[A, \[Mu], \[Nu]] FS[A, \[Mu], \[Nu]]

LFermions:= I lbar.Ga[\[Mu]].DC[l, \[Mu]] - ME[l] lbar.l

LGF := -1/2/xiA del[A[mu], mu] del[A[nu], nu]

L:= LGauge + LFermions + LGF
```

Now in the same directory where the model files are, one opens a `Mathematica` Notebook and loads `FeynRules` and the model. For this use the following code in the Notebook,

```
$FeynRulesPath = SetDirectory["~/work/Software/FeynRules"];
<< FeynRules`
SetDirectory[$FeynRulesPath <> "/Models/QED"];

LoadModel["QED.fr"]
```

Now we can look at the Lagrangian and check that the covariant derivative has our conventions [59], as it is shown below,

The Lagrangian and Covariant Derivative

The Lagrangian is

L

$$-\frac{\partial_{mu}[A_{mu}]\,\partial_{nu}[A_{nu}]}{2\,\xi_A} - \frac{1}{4}\,(-\partial_\nu[A_\mu] + \partial_\mu[A_\nu])^2 + i\,\bar{l}.\gamma^\mu.(i\,e\,l\,Q\,A_\mu + \partial_\mu[l]) - \bar{l}.l\,\eta_l$$

DC[l, μ]

$$i\,e\,l\,Q\,A_\mu + \partial_\mu[l]$$

Calculating Feynman rules

The screen output can be turned off using the option ScreenOutput → False

```
In[7]:= vertsQED = FeynmanRules[L, ScreenOutput → False];
Starting Feynman rule calculation.
Expanding the Lagrangian...
Collecting the different structures that enter the vertex.
1 possible non-zero vertices have been found -> starting the computation: 1 / 1.
1 vertex obtained.
```

The vertices can be accessed via

```
In[8]:= vertsQED
Out[8]= {{{{l̄, 1}, {l, 2}, {A, 3}}, -i e Q γ_{s1,s2}^μ3}}
```

Finally these rules can be exported in various formats to be handled by other software. The most important are given in Table 7.5. The reader should see the

`WriteCHOutput[ L1, L2, . . . , options ]`	CalcHEP
`WriteFeynArtsOutput[ L1, L2, . . . , options ]`	FeynArts/FormCalc
`WriteUFO[ L1, L2, . . . , options ]`	MadGraph5

Table 7.5: `Mathematica` commands for the Interfaces.

manual for a full explanation of these interfaces.

7.4 CalcHEP

7.4.1 Setup

CalcHEP is an excellent package for the calculation of Feynman diagrams and integration over multi-particle phase space developed by Alexander Pukhov, Alexander Belyaev, and Neil Christensen [60] The program and manual can be obtained from the webpage:

`https://theory.sinp.msu.ru/~pukhov/calchep.html`

It is restricted to tree level, presenting the Feynman diagrams graphically.
The program must be installed in Linux (or on a Virtual Machine in Windows running Linux), following the guidelines on the website. We know of no hack to install it in Windows directly. Further help can be found on the Computational Techniques in QFT (CTQFT) page at:

`https://porthos.tecnico.ulisboa.pt/CTQFT/`

For the file management instructions, we assume a Linux (or Virtual Machine running Linux) installation. Following the manual, the files in the CalcHEP root directory will be used only for reading and execution during the user session. Each user has to create his own directory to work with CalcHEP. This is done by executing the following command in the CalcHEP root directory

```
./mkWORKdir /home/yourusername/work/calchep
```

In our case, CalcHEP was compiled and installed in "/home/yourusername/Software/calchep" and it will be used in "/home/yourusername/work/calchep". Thenceforth "/home/yourusername/" will be assumed.

Installing some scripts

In our installation, CalcHEP is run in the directory `~/work/calchep`. So we move there

```
> cd ~/work/calchep
```

If CalcHEP has been run before, it assumes the previous state. If one wishes to start a new project, one must delete the working files, including all files and subdirs in the directory "results" and "tmp" and a possible "lock" file. Thus, we have developed a script "NewProject", placed in the "work/calchep" directory, which makes all the cleanup necessary before starting a new calculation. It just does the following

```
rm -fr results/* tmp/* lock
```

This and other useful scripts for the use of CalcHEP in our course can be obtained from

```
https://porthos.tecnico.ulisboa.pt/CTQFT/files/scripts_calchep.tar.gz
```

where one finds also the scripts "cleanup", "Ecm_cycle", and "p1Lab_cycle" and the file "cleanup.sed" needed to be placed in the same directory as "cleanup". We will explain the use of these scripts below. These will be installed in a directory that we created

```
> mkdir ~/work/calchep/MyResults
```

This directory is important to keep your scripts and results as the directory `~/work/calchep/results` is erased for each new process. The scripts "Ecm_cycle" and "p1Lab_cycle" must be altered according to your installation, pointing towards your CalcHEP root directory.[2] In the "MyResults" directory, you can edit the "Ecm_cycle" executable with

[2] If you are uncertain what your CalcHEP root directory is, go to your CalcHEP working directory, "work/calchep" in our case, where you can identify your CalcHEP root directory using the command
`> ls -l .`

```
> emacs Ecm_cycle &
```

You will find a piece

```
if(test -z "$CALCHEP") then
CALCHEP=/usr/local/lib/calchep
export CALCHEP
fi
```

You should trade "/usr/local/lib/calchep" by your CalcHEP root directory. In our case, the line

```
CALCHEP=/usr/local/lib/calchep
```

is traded for

```
CALCHEP=/home/yourusername/Software/calchep
```

As always with emacs, CTRL-X-C closes and saves the altered file by answering yes to the prompt. Do the same with the executable file "p1Lab_cycle". You are now ready to run CalcHEP in the QFT course.

7.4.2 Simple tests

Running a new project

The instruction

```
> ./NewProject
```

makes the cleanup necessary to start a new calculation. Then, to run CalcHEP, use

```
> ./calchep
```

which opens up the CalcHEP console. You can scroll up and down with up and down arrow keys; choose with the return key; and go back with the Esc key. We choose the model "SM" and then "Enter Process". You will see a table with the names used for particles and antiparticles. For example, it uses "e" and "E" for electro and positron, respectively. Similarly, it uses "m" and "M" for muon and anti-muon, respectively. We enter the process "e,E -> m,M" (return). The instruction "Exclude diagrams with" appears, and we just hit return. We then choose "View diagrams". We find two diagrams, one with a photon in the s-channel and another with a Z boson in the s-channel. At this stage we can choose any diagram with the arrow keys. Pressing the keyboard letter "o" ("O") will turn that diagram off (back on). On this first exercise, hit "o" on the Z diagram, thus keeping only the photon diagram which exists in QED.

Then "Esc" goes back, and choose "Square diagrams", followed by "Make n_calchep" (return). That closes the CalcHEP program and creates the executable "n_calchep" in the "work/calchep/results" directory. Move to that directory with

```
> cd results
```

The executable "n_calchep" has to be run one first time before performing either the `ECM_cycle` or `p1Lab_cycle`. Run it by simply typing

```
> ./n_calchep
```

A new window opens up, with title "calcHEP_3.7.4/num". Scroll down to "Monte Carlo simulation" and then scroll down to "*Start integration" (return). Then "Esc" three times and, upon the query "Quit session", answer "Y" (Yes). This has created the important file "session.dat" in the same directory "results".

One would now like to have a list of CM energies and cross-sections for this process. That is the purpose of the script `Ecm_cycle`. We assume that we have placed it in the directory "MyResults". Typing

```
> ../MyResults/Ecm_cycle
```

will show the description and usage for `Ecm_cycle`. The latter is

```
USAGE: Ecm_cycle StartValue Step NumberPoints m1 m2
```

This is to be applied to a two-to-two scattering, where m1 and m2 are the masses of the incoming particles. The CM energy and masses are in GeV. As the first example, take

```
> ../MyResults/Ecm_cycle 30 8 15 0 0 > ../MyResults/output.dat
```

This starts with a 30GeV CM energy, and does 15 points, increasing each time by 8GeV. The masses are taken to vanish. The result is written on a file "output.dat", in a directory which is up one level from the current "work/calchep/results" directory, and down one level into MyResults; thus, in the "work/calchep/MyResults" directory we had created before.

Move to that directory with

```
> cd ~/work/calchep/MyResults
```

or

```
> cd ../MyResults
```

Using

```
> more output.dat
```

you will see the contents of the file. The first two lines read

```
Ecm=30[GeV] sigma=1.1061E+02[pb]
Ecm=3.800E+01[GeV]  sigma=6.8942E+01[pb]
```

So, output.dat contains both numbers, the symbols indicating what those numbers refer to, and the respective units. CalcHEP always gives the results in picobarn. So, when comparing, for example, with analytical results, one must find those also in pb. Also when comparing one should make sure that the same values for the physical constants are chosen. The ones used by CalcHEP can be looked while running the `n_calchep` executable.

One would now like to import the two columns of numbers in output.dat (the first column referring to Ecm, and the second to the cross-section) into some graphics program, such as Excel, Mathematica, gnuplot, etc. For that, one must get rid of all indications other than the columns of numbers. This is achieved via the script "cleanup", as in

```
> ./cleanup output.dat
```

The first two lines of output.dat now read

```
30    1.1061E+02
3.800E+01    6.8942E+01
```

Do not worry about the alignment of the columns. To safe keep this file, move it, for example as

```
>  mv output.dat test-qed.dat
```

A second example

In order to study a second example, we follow the previous steps. To recap:

`> ./NewProject` *(cleans old CalcHEP files)*

`> ./calchep` *(to run CalcHEP, opening its console)*

Within CalcHEP:

"Choose the model"; "SM"
"Enter Process"; `e,E -> m,M` (return/return)
"View diagrams" *(shows the diagrams)*; "Esc"
"Square diagrams"
"Make n_calchep" (return) *(creates "n_calchep" in the "work/calchep/results" directory)*

```
> cd results
```

```
> ./n_calchep
```

In the new window `n_calchep` opens:

"Monte Carlo simulation"
"*Start integration" (return)
"Esc" (thrice); "Quit session-Yes" *(creates "session.dat"in the "work/calchep-/results" directory)*

```
> ../MyResults/Ecm_cycle 10 1 191 0 0 > ../MyResults/output.dat
> cd ../MyResults
> ./cleanup output.dat
```

where the last step keeps only numbers in the file. Finally

```
> mv output.dat test-SM.dat
```

safe keeps results in file "test-SM.dat".

Graphics with gnuplot

Gnuplot is a very powerful portable command-line driven program to produce graphics and you are well advised to learn it. Information may be found at

`http://www.gnuplot.info/`

We have a pre-created file for this job. To see the file open it with

```
> emacs plotXY.gp &
```

The relevant features are

```
set output "plot.eps"
```

which chooses the output file for the plot as "plot.eps";

```
set logscale y
```

which sets the log scale for the cross section; and the final four lines

```
set xrange [0:150] ;\
set yrange [0:10000];\
plot "test-SM.dat" u 1:2 with lines ls 1,\
"test-qed.dat" u 1:2 with lines ls 3
```

Note that there should be no space between the characters in `;\` and `,\` and no space after the `\`. This is run by

```
> gnuplot plotXY.gp
```

which creates the file *plot.eps*. To see the result use, for instance `gv` (ghostview)

```
> gv plot.eps
```

The result is shown in Fig. 7.1. At low energies, the Z propagator is heavily suppressed by its mass, and the SM results practically coincides with QED. When one approaches the Z mass, the result is completely dominated by the Z resonance. Finally, at very large energies, one can neglect the Z mass and its propagator equals the photon's. The difference between the amplitudes arises merely from the differing couplings.

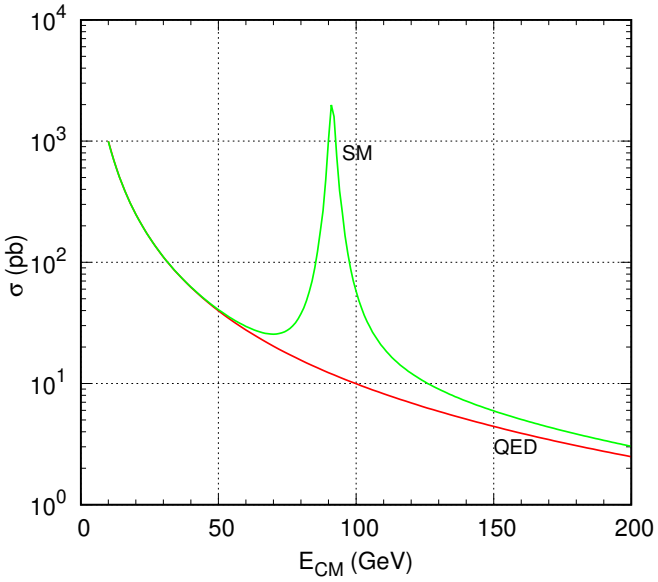


Figure 7.1: Cross section for $e^+e^- \to \mu^+\mu^-$ in QED (blue) and the SM (red) calculated with `CalcHEP`.

7.5 MadGraph

`MadGraph` [61] is a very powerful software framework that aims at providing all the elements necessary for SM and BSM phenomenology, such as the computations of cross sections, the generation of hard events and their matching with event generators, and the use of a variety of tools relevant to event manipulation and analysis. `MadGraph5_aMC@NLO` is the new version of both `MadGraph5` and `aMC@NLO` that unifies the LO and NLO lines of development of automated tools within the `MadGraph` family. As described in their wiki page `MadGraph5_aMC@NLO Wiki`

`https://cp3.irmp.ucl.ac.be/projects/madgraph/`

the processes can be simulated to LO accuracy, and to NLO accuracy in the case of QCD corrections to SM processes. Matrix elements at the tree- and one-loop-level can also be obtained. It can intake models in the `UFO` format that can be produced by other packages such as `FeynRules` and `SARAH`. The software can be downloaded from

`https://launchpad.net/mg5amcnlo`

After downloading follow the instructions on the web page for the installation and read the manual that can be obtained at the the site

`https://cp3.irmp.ucl.ac.be/projects/madgraph/wiki/ManualAndHelp`

In this introductory text we are only to give a very brief introduction on how to use it to evaluate simple processes. As an example we will calculate the tree level cross section for the process $e^-e^+ \to \mu^-\mu^+$, that we have evaluated with `CalcHEP` and shown in Fig. 7.1.

`MadGraph` provides an interactive python interface, which can be accessed by running the main executable. In this interactive interface the command `tutorial` will guide new users to the basic syntax, which we also cover here. We think the tutorial in `MG5` is quite complementary to these notes and talk through a bit more about the parameter and run cards, the extra steps (analysis, hadronisation, and detector simulation) which are not relevant for theoretical inclusive cross sections calculations. Here we will explain how to write and run `MG5` through a script. Suppose we call such a script `eEmM.txt`. Then we in the base directory of `MadGraph5_aMC@NLO` we run the script

```
$ ./bin/mg5_aAC eEmM.txt
```

A minimal working script would be

```
# File eEmM.txt for MadGraph input
generate e- e+ > mu- mu+
output
launch
```

This script illustrates the three main steps needed, `generate`, `output` and `launch`. `MG5` has a plenitude of default options which are a hindrance for this particular exercise, but are reasonable if you study either an experimental analysis or a phenomenological sensitivity test, that may require more features. For instance the browser is set to default because people are suppposed to run the Tutorial in the interactive session before doing anything else more complicated. But if you are making a scan, it is better to look only in the end. For this you have to uncomment the line

```
# automatic_html_opening = False
```

in `input/mg5_configuration.txt` by removing the `#`. Second, as you have not given a name for the output, `MadGraph` chooses one. In this case `PROC_sm_0`. If you run again you get `PROC_sm_1` and so on. Notice `_sm` meaning the Standard Model. As one has not given a model it defaults to `sm`. This script is then equivalent to

```
# File eEmM.txt for MadGraph input
import model sm
generate e- e+ > mu- mu+
output
launch
```

The next point is the `ECM`. `MadGraph` defaults to `ECM=1000 GeV` for lepton colliders. This can be changed with the following input file

```
# File eEmM.txt for MadGraph input
```

```
import model sm
generate e- e+ > mu- mu+
set lep  100.0
output
launch
```

for an `ECM=100 GeV`. There are several ways of getting the same result. The following commands are all equivalent

```
set lep  100.0
set ebeam 50.0
set ebeam1 50.0
set ebeam2 50.0
```

The next point is that `MadGraph` has some default cuts, that in this case will give a result that does not correspond to what we are looking for. Also we want just the cross section for a given `ECM` written in a file. The file with all these criteria would look like

```
# File eEmM.txt for MadGraph input
import model sm
generate e- e+ > mu- mu+
output eEmMdir
launch
analysis=OFF
set nevents 1000
set aEWM1 128.023
set MZ 91.29264
set gf 1.1951e-05
set drll 0.0 # No cut on distance between letons
set etal -1.0 # No eta cut for leptons
set ptl -1.0 # No Pt cut for leptons
set lep 100.0
set run_tag 100.0
launch eEmMdir -i
print_results --path=./eEmMdir/cross_section_eEmM.txt --format=short
```

Let us explain the meaning of the new commands. First we gave a name to directory where the information is stored, in this case `eEmMdir`. This stores the Fortran code that will be compiled and used to perform the MC generation and calculation of the cross section, `MG5` is only a python interface for a lot of different `C/C++/Fortran` routines that are otherwise different software packages. Next we make `analysis=OFF` to disable the `MadAnalysis` as we just want the cross section. Notice that this is optional, you do not need `MadAnalysis` for a full simulation. It exists to derive/compute histograms and to allow for an early-stage data exploration, which for p-p or other discovery machines can be quite interesting (as you have a range of `ECM` and therefore richer distributions of the final state observables). The following line `set aEWM1 128.023` it is only important because we want to compare with the result of `CalcHEP` that uses $\alpha^{-1} = 128.023$ while `MadGraph` uses $\alpha^{-1} = 132.507$. Beware that `MadGraph` recalculates the W mass for consistency. In

fact MadGraph allows for the so-called "internal" parameter which is a parameter given by an analytical expressions of other parameters. In the SM this is one of the few cases, but if you have a BSM model, many parameters are often related to another and this way you can generate events without having to numerically enforce those relations. The following table gives the MadGraph and CalcHEP defaults and the parameters you have to use in MadGraph for the SM to compare with CalcHEP. So to compare the results with those of CalcHEP you have to use the inputs for

Par	MadGraph	CalcHEP	Input MadGraph	Input CalcHEP	MadGraph from CalcHEP
α^{-1}	132.507	128.023	Yes	Yes	128.023
M_Z	91.188	91.29264	Yes	No	91.29264
G_F	1.1664×10^{-5}	1.1951×10^{-5}	Yes	No	1.1951×10^{-5}
M_W	80.419	80.385	No	Yes	Calculated
s_W	Calculated	0.474	No	Yes	Calculated

Table 7.6: Input values for MadGraph and CalcHEP.

MadGraph in the last column. This correspond to

```
set aEWM1 128.023
set MZ 91.29264
set gf 1.1951e-05
```

The next three lines remove the default cuts on leptons. These are mainly used for the experimental situation, but if we want to calculate the theorethical exact cross section these have to be removed, otherwise the phase space will not be completely covered. Finally we want to save the results in a file to be able to use it to make a plot. This is achieved by the last two lines. When we run this script we get

```
# run_name tag cross error Nb_event cross_after_matching
#  nb_event_after matching
run_01 100.0 56.8 0.09264843789832616 1000
```

which means that for ECM=100 GeV the cross section is 56.8 ± 0.09 pb. Now we want to repeat this for a range of values for ECM. There is a command scan, but it is not very useful here. The best way is to create a script file with all the values of ECM. As this is quite repetitive one can use a shell script to produce the MadGraph script. For instance the following code will give the desidered result

```
#!/bin/bash
export LC_ALL=C

myfile="eEmM_scan.txt"
rm -f $file

touch $myfile
echo "generate e- e+ > mu- mu+" >> $myfile
```

```
echo "output eEmMdir" >> $myfile

for i in `seq -f "%.1f" 5 2.5 100`
do
ECM=`expr "$i + $i" | bc`
echo "launch" >> $myfile
echo "analysis=OFF" >> $myfile
echo "set nevents 10000" >> $myfile
echo "set aEWM1 128.023">> $myfile
echo "set MZ 91.29264">> $myfile
echo "set gf 1.1951e-05">> $myfile
echo "set drll 0.0 # no minimal separation for leptons" >> $myfile
echo "set etal -1.0 # no cut in eta" >> $myfile
echo "set ptl -1 # no cut in Pt of leptons" >> $myfile
echo "set lep $ECM" >> $myfile
echo "set run_tag $ECM" >> $myfile
echo "done" >> $myfile
echo "" >> $myfile

done

echo "launch eEmMdir -i" >> $myfile
echo "print_results --path=./eEmMdir/cross_section_eEmM.txt
--format=short" >> $myfile
```

This will produce the file `eEmM_scan.txt` that we illustrate below

```
generate e- e+ > mu- mu+
output eEmMdir
launch
analysis=OFF
set nevents 10000
set aEWM1 128.023
set MZ 91.29264
set gf 1.1951e-05
set drll 0.0 # no minimal separation for leptons
set etal -1.0 # no cut in eta
set ptl -1 # no cut in Pt of leptons
set lep 10.0
set run_tag 10.0
done

launch
...
set lep 15.0
set run_tag 15.0
...
set lep 200.0
set run_tag 200.0
...
launch eEmMdir -i
print_results --path=./eEmMdir/cross_section_eEmM.txt --format=short
```

where the dots mean that we repeat the code. Running this code

```
./bin/mg5_aMC eEmM_scan.txt
```

we get a file `cross_section_eEmM.txt` that we illustrate here

```
# run_name tag cross error Nb_event cross_after_matching
# nb_event_after matching
run_01 10.0 996.389 3.95370521718552 1000
run_02 15.0 442.94 0.845193697148766 1000
...
run_39 200.0 3.0311 0.005190219676468425 1000
```

Using these values we can compare with the result of `CalcHEP`. This is shown in Fig. 7.2, showing a perfect agreement.

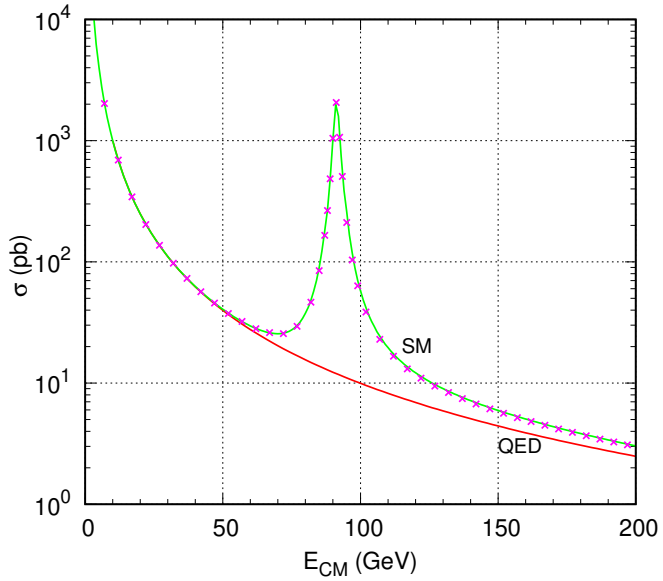


Figure 7.2: Cross section for $e^+e^- \to \mu^+\mu^-$ in QED (red) and the SM (green) calculated with `CalcHEP`. The result of `MadGraph` are the magenta crosses.

The reader should read the manual as well as a variety of online presentations to be able to understand the full power of `MadGraph`.

7.6 FeynMaster

`FeynMaster` [62] is a multi-tasking software for particle physics studies. By making use of already existing programs (`FeynRules`, `QGRAF`, `FeynCalc`), `FeynMaster` automatically generates Feynman rules, generates and draws Feynman diagrams, generates amplitudes, performs both loop and algebraic calculations, and fully renormalizes models. In parallel with this automatic character, `FeynMaster` allows the

user to manipulate the generated results in `Mathematica` notebooks in a flexible and consistent way. It uses the power of the `phyton` language to automatically create the necessary input files for the various programs used. It can be downloaded at

`https://porthos.tecnico.ulisboa.pt/FeynMaster/`

where one can also find the last version of the manual, as well as some video tutorials. The program has been shown to work in `Linux`, `Mac OS X` and `Windows`. The examples below will be given supposing a `Linux` environment. Here we assume that the user has already a structure `/home/user/Software`. Then you move to that directory and unpack the code using

```
cd /home/user/Software
tar xzf FM-Linux-2.x.y.tar.gz
```

This creates a directory tree

```
Software/
|-- FeynMaster
    |-- Control.m
    |-- Models/
    |-- Nucleus/
    |-- RUN-FeynMaster.sh
```

showing the main subdirectories of `FeynMaster`, `Models` and `Nucleus`. The files `RUN-FeynMaster.sh` and `Control.m` used to run and control the program will be explained below. The contents of the subdirectories are for `Models`

```
Models/
|-- C2HDM/
|-- QED/
|-- SM/
|-- SQED/
```

and for `Nucleus`,

```
Nucleus/
|-- Addendum.m
|-- Amperate.py
|-- ControlExtract.py
|-- Converter.py
|-- Definitions.m
|-- Drawing.py
|-- Finals.m
|-- FRExtract.py
|-- FunctionsFM.m
|-- General.py
|-- IntGauss.f
|-- JustConvert.py
```

```
|-- Preamble.m
|-- Printer.py
|-- Renapp.m
|-- SupReno.m
|-- WriteToPreTeX.m
```

We see that the distribution comes already with an implementation of QED, SQED (Scalar QED), the Standard Model (SM) and the complex two Higgs doublet model (C2HDM). All these models are prepared for renormalization. The subdirectory `Nucleus` has the necessary programs needed to run `FeynMaster` and the user does not need to change anything there. The directory `Models` has the model files for each model. These are really files for `FeynRules` with some features added. To use a model in `FeynMaster` one needs:

- The `FeynMaster` model file
- The file that controls the program, named `Control.m`, where you choose
  - The model.
  - The process.
  - Details of the process, for instance tree-level or loop process, if you want to exclude particles from the run, if you want to calculate the Feynman rules, see the diagrams, do renormalization and so on. We will show how this works with the real examples below.
- The script `RUN-FeynMaster.sh`, that you have to adapt, once and for all, to your software structure. We show here what you have to do, assuming our `FeynMaster` location:

```
# Directories:
#dirFR: /home/user/Software/FeynRules/
#dirQ: /home/user/Software/qgraf/
#dirFMout: /home/user/Sofware/FeynMaster/FM-Output/


# Do not edit:
#!/bin/bash
cd "$(dirname "$0")"/Nucleus/
python3 General.py $1
rm -f FRinter.sh
rm -f recorrente.sh
```

  You just indicate the path to the `FeynRules` and `QGRAF` software and the directory where you want `FeynMaster` to write the output.

  There is one more point here that needs some clarification. The models for `FeynMaster` are also the models for `FeynRules`. By default, the folder with the models for `FeynMaster` is inside the `FeynMaster` folder and is named

`Models`; this can be changed by defining a variable `dirFRmod` in the beginning of `RUN-FeynMaster.sh` and setting a path for it. In a similar way, the folder with the models for `QGRAF` (that is, the folder that shall contain the `QGRAF` models, which will be automatically generated by `FeynMaster`) is automatically generated and set inside the `QGRAF` folder with the default name `Models`; as before, this can be changed by defining a variable `dirQmod` in the beginning of `RUN-FeynMaster.sh` and setting a path for it.

Before we discuss how to add a model to `FeynMaster` let us see how to use it using the QED implementation already provided.

7.6.1 QED as an example

Adjust `Control.m`

We choose to start with a tree level process, Bhabha scattering, $e^- + e^+ \to e^- + e^+$. So we write in the `Control.m`

```
(* This is the Control.m file of FeynMaster *)

model: QED

inparticles: f,fbar
outparticles: f,fbar
loops: 0
options:

FRinterLogic: T
RenoLogic: F
Draw: T
Comp: T
FinLogic: T
DivLogic: F
SumLogic: T
MoCoLogic: F
LoSpinors: T
```

The `Control.m` has many options, that we will review briefly.

- `inparticles`, the input particles. In our model file we chose `f` for the electron and `fbar` for its antiparticle, the positron. If you do not know the names of the particles for a given model, just do, for QED

  ```
  ./RUN-FeynMaster.sh QED
  ```

  and you will get the output

  ```
  ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
  FeynMaster
  created by Duarte Fontes and Jorge C. Romao
  ```

```
version 2.0.2 (September 28, 2021)
~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
Particles of the FeynMaster model QED:
[f, A]
```

- `outparticles`, the output particles. The names are the names of the particles with a `bar` added. So in our case the positron will be `fbar`.
- `loops`, the number of loops.
- `options`, are options to be passed, for instance to `QGRAF`. For instance, at one loop one might want to set `options = onepi` to just calculate one particle irreducible diagrams. There are other options. To see all the options just write

```
    ./RUN-FeynMaster.sh help
```

and you get a list of options and optional commands.

```
    ~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
FeynMaster
created by Duarte Fontes and Jorge C. Romao
version 2.0.2 (September 28, 2021)
~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~

********* Variables for Control **********
Variable          Default          Example
- - - - - - - - - - - - - - - - - - - - - -
inparticles      (empty)           e,ebar
outparticles     (empty)           H,Z
loops            0                 1
parsel           (empty)           {avoid,Z,1,3},{keep,e,1,1}
factor           1                 16 Pi^2/I
options          (empty)           onepi

****************** Avaliable options ******************
Options      Description                 Converse option
- - - - - - - - - - - - - - - - - - - - - - - - - - - - -
ugauge       unitary gauge               (none)
noPVauto     PaVeAutoReduce -> False     (none)
onepi        1PI diagrams only           onepr
onshell      no external self-energies   offshell
nosigma      no self-energies at all     sigma
nosnail      no snails                   snail
notadpole    no tadpoles                 tadpole
simple       at most one propagator      notsimple
             between different vertices
```

Next we have other commands,

```
FRinterLogic: T
RenoLogic: F
Draw: T
Comp: T
FinLogic: T
DivLogic: F
SumLogic: T
MoCoLogic: F
LoSpinors: T
```

which mean the following:

- `FRinterLogic`. If you want to establish an interface with `FeynRules` to generate the FeynMan rules, choose `FRinterLogic` as `T` for `True`. You only have to do that once for your model.
- `RenoLogic`. If you want to do renormalization, choose `RenoLogic` as `T`.
- `Draw`. If you want to draw the Feynman diagrams, choose `Draw` as `T`.
- `Comp`. If you want to compute the final expressions, choose `Comp` as `T`.
- `FinLogic`. If `Comp = T` and you want to compute the final result of each diagram, choose `FinLogic` as `T`.
- `DivLogic`. If `Comp = T` and you want to compute the divergent part of each diagram, choose `DivLogic` as `T`. This applies only to one-loop processes.
- `SumLogic`. If `Comp = T` and you want to compute the sum of the expressions, choose `SumLogic` as `T`.
- `MoCoLogic`. If `Comp = T` and if you want to apply momentum conservation, choose `MCoLogic` as `T`.
- `LoSpinors`. If `Comp = T` and if you want to include Spinors in the external legs, choose `LoSpinors` as `T`.

In the file above we chose the options appropriate to a tree-level process, therefore we do not evaluate the Feynman rules for the counterterms and do not evaluate the divergent part.

The directory `FM-Output`

On the first run of `FeynMaster` it is created a directory `FM-Output` (this is the name we chose in the `RUN-FeynMaster.sh` script) where all the results will be stored. This is shown in the following diagram,

```
FeynMaster/
    |-- Control.m
    |-- FM-Ouput/
    |-- Models/
    |-- Nucleus/
    |-- RUN-FeynMaster.sh
```

The directory `FM-Output` has the following structure

```
FM-Output
|-- QED/
    |-- Counterterms
    |-- FeynmanRules
    |-- Processes
    ...
```

For each model there is a subdirectory with the model name and three further subdirectories. In `FeynmanRules` are stored the Feynman rules, in `Counterterms` the Feynman rules for the counterterms (if `RenoLogic = T`) and finaly in `Processes` the processes that we want to study. If we look at the contents of the `FeynmanRules` directory we get

```
FeynmanRules/
|-- Extras.m
|-- Feynman-Rules-Main.m
|-- FRtoTeX.m
|-- Matrixind.m
|-- Nclist.m
|-- ParamsValues.m
|-- Propagators.m
|-- Rulesv3Fermions.m
|-- TeXs-drawing/
```

showing the files with the Feynman rules and other information needed to run the program. Inside there is another subdirectory, `TeXs-drawing`, where we can find a PDF, as well as the corresponding `TeX` file, with the Feynman rules for the theory. Once we have used `FRinterLogic = T` once, all the information on the Feynman rules is generated and saved and therefore for any other process you do not have to generate them again, so you should put `FRinterLogic = F`. This is not very important for QED with just one vertex, but for the Standard Model or other larger models it saves a lot of time. Finally let us look at the subdirectory `Processes`,

```
Processes/
|--1-ffbarffbar
    |-- Amplitudes.m
    |-- Control.m
    |-- Helper.m
    |-- Lists/
    |-- Notebook.nb
    |-- TeXs-drawing/
```

```
    |-- TeXs-expressions/
```

The file `Amplitudes.m` has the amplitudes for the diagrams, `Helper.m` information needed for `FeynMaster`. In the subdrectories `TeXs-drawing` and `TeXs-expressions` you find `LaTeX` expressions for the diagrams and results as well as a PDF, as well as the corresponding `TeX` file, file with the diagrams (if `Draw: T`). In the directory `Lists/` are kept the results of the calculation (if `Comp:T`) that can used latter, without running again. We left to the end the most important file, the `Mathematica` file `Notebook.nb`. You can open this file, generated automatically by `FeynMaster`, and look at the amplitudes and results. This is a `FeynCalc` Notebook, so one can use all the power of `FeynCalc` to further work with the results.

Look at `Notebook.nb`

If we open and evaluate the `Notebook.nb` we get the following

```
dirHome =
  "PATH_To_FeynMaster/FM-Output/QED/Processes/1-ffbarffbar/";
SetDirectory[dirHome];
 << FeynCalc`
 dirNuc = "PATH_To_FeynMaster/Nucleus/";
 dirFey = "PATH_To_FeynMaster/FM-Output/QED/FeynmanRules/";
 dirCT = "PATH_To_FeynMaster/FM-Output/QED/Counterterms/";
  << Helper.m
```

This means that first we load the definitions of the relevant directories and also `FeynCalc`. Note that all the information on this file was automatically written by `FeynMaster`. Also loaded is the file `Helper.m` where are kept all the choices that were done in `Control.m`. If needed this file can be edited and the Notebook evaluated again.
Next the Feynman rules are loaded as well as some needed functions (see the manual for a more complete explanation).

```
 Get["Feynman-Rules-Main.m", Path -> {dirFey}]
 Get["FunctionsFM.m", Path -> {dirNuc}]
 Get["Definitions.m", Path -> {dirNuc}]
 compNwrite = False;
 Get["Finals.m", Path -> {dirNuc}]
 If[RenoLogic, Get["SupReno.m", Path -> {dirNuc}]]
```

Just note the `compNwrite = False` means that when evaluating the Notebook, the results are read from the directory `Lists/` and not calculated again.

After these preliminaires we can look at the results. The first thing we can access are the amplitudes. The are named `amp1`, $\cdots$, `ampn`. In our case we just have two amplitudes and we can make a list. This is shown below. Notice that this list is just the replacement of the Feynman rules so technically is what normally is called $i\mathcal{M}$.

```
In[ ]:= amp = {amp1, amp2}
```

$$\text{Out[ ]}= \left\{\frac{i\, e^2\, (\varphi(-\text{p2}, \text{mf})).\gamma^{\text{J2}}.(\varphi(\text{p1}, \text{mf}))\,(\varphi(\text{q1}, \text{mf})).\gamma^{\text{J2}}.(\varphi(-\text{q2}, \text{mf}))}{(-\text{p1}-\text{p2})^2},\right.$$
$$\left.-\frac{i\, e^2\, (\varphi(\text{q1}, \text{mf})).\gamma^{\text{J1}}.(\varphi(\text{p1}, \text{mf}))\,(\varphi(-\text{p2}, \text{mf})).\gamma^{\text{J1}}.(\varphi(-\text{q2}, \text{mf}))}{(\text{q1}-\text{p1})^2}\right\}$$

Next we can access a list with the result, called `res`. In this tree-level example, this list is just the list of the amplitudes $\mathcal{M}_i$. This can be seen below.

```
In[ ]:= res
```

$$\text{Out[ ]}= \left\{\frac{e^2\,(\varphi(-\overline{\text{p2}}, \text{mf})).\overline{\gamma}^{\text{J2}}.(\varphi(\overline{\text{p1}}, \text{mf}))\,(\varphi(\overline{\text{q1}}, \text{mf})).\overline{\gamma}^{\text{J2}}.(\varphi(-\overline{\text{q2}}, \text{mf}))}{(-\text{p1}-\text{p2})^2},\right.$$
$$\left.-\frac{e^2\,(\varphi(\overline{\text{q1}}, \text{mf})).\overline{\gamma}^{\text{J1}}.(\varphi(\overline{\text{p1}}, \text{mf}))\,(\varphi(-\overline{\text{p2}}, \text{mf})).\overline{\gamma}^{\text{J1}}.(\varphi(-\overline{\text{q2}}, \text{mf}))}{(\text{q1}-\text{p1})^2}\right\}$$

Finally we can get the total amplitude, the sum of the individual amplitudes. This is shown below. Notice the relative minus sign between the s-channel and the t-

```
In[ ]:= resT = Total[res] /. {J1 → μ, J2 → ν}
```

$$\text{Out[ ]}= \frac{e^2\,(\varphi(-\overline{\text{p2}}, \text{mf})).\overline{\gamma}^{\nu}.(\varphi(\overline{\text{p1}}, \text{mf}))\,(\varphi(\overline{\text{q1}}, \text{mf})).\overline{\gamma}^{\nu}.(\varphi(-\overline{\text{q2}}, \text{mf}))}{(-\text{p1}-\text{p2})^2} -$$
$$\frac{e^2\,(\varphi(\overline{\text{q1}}, \text{mf})).\overline{\gamma}^{\mu}.(\varphi(\overline{\text{p1}}, \text{mf}))\,(\varphi(-\overline{\text{p2}}, \text{mf})).\overline{\gamma}^{\mu}.(\varphi(-\overline{\text{q2}}, \text{mf}))}{(\text{q1}-\text{p1})^2}$$

From here one can use the power of `FeynCalc` and `FeynMaster` to evaluate the differential cross section. Se below in section 7.9.4 how this can be done.

7.6.2 Adding a new Model

Let us consider a model that besides QED has neutral scalar with interactions with the fermions,

$$\mathcal{L} = \mathcal{L}_{\text{QED}} + \frac{1}{2}\partial_\mu H \partial^\mu H - \frac{1}{2}m_H^2 H^2 + g_{\text{HfF}}\,\overline{f}\,f\,H - g_{\text{HHH}}H^3 - \frac{1}{4!}\lambda H^4 \qquad (7.4)$$

The first thing one has to do is to choose a unique name for the model. We call it `QEDAndScalar`. We have to start with the `FeynMaster` definition of the model. As this model is a simple extension of QED, it is best to start with the QED. The `FeynMaster` model that comes with the distribution uses the fact that QED is so simple that we can write all the information in one file `QED.fr`. In section 7.3.2, we

argued that it is better to separate the definitions of the particles, parameters and the Lagrangian in different files. So we will copy that structure to a subdirectory of `Models` and change the name of the file `QED.fr` to `QEDAndScalar.fr`.

```
(* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
This is the FeynMaster model file for QED and Scalar model.
Created by: Jorge Romao
Email: jorge.romao@tecnico.ulisboa.pt
Last Update: 05.11.2021
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *)

(***** Gauge group list *****)
M$GaugeGroups = {
  U1EM == {
                Abelian -> True,
                GaugeBoson -> A,
                Charge -> Q,
                CouplingConstant -> ee}};


Get["Parameters.fr"];
Get["ParticleClasses.fr"];
Get["Lagrangean.fr"];

(***** Extra information for FeynMaster *****)
M$FCeqs = {xiA->1, ee->e};
M$PrMassFL = False;
```

Next we add the scalar to the file `ParticleClasses.fr`,

```
(***** Particle classes list *****)
M$ClassesDescription = {
  F[1] == {
       ...},
  ... ,
  S[1] == {     ClassName -> H,
                Mass -> mH,
                SelfConjugate -> True}}
```

where the dots refer to the fields already defined in QED. We just add the scalar `H`. After this we need to add the new parameters to the file `Parameters.fr`,

```
(***** Parameter list *****)
M$Parameters = {
  mf == {TeXName -> "m_f"},
  ee == {TeXName -> "e"},
  xiA == {TeXName -> "\\xi_A"},
  mH == {TeXName -> "m_H"},
  gHfF == {TeXName -> "g_{HfF}"},
  gHHH == {TeXName -> "g_{HHH}"},
  lbd == {TeXName -> "\\lambda"}
  };
```

and finally the new Lagrangean in file `Lagrangean.fr`,

```
(* Lagrangean *)
 LGauge := -1/4 FS[A, \[Mu], \[Nu]] FS[A, \[Mu], \[Nu]]

LFermions := I psi1Lbar.sibar[mu].del[psi1L, mu] +\
I chi1Rbar.si[mu].del[chi1R, mu] \
- mf (psi1Lbar.chi1R + chi1Rbar.psi1L) \
+ ee psi1Lbar.sibar[mu].psi1L A[mu] + ee chi1Rbar.si[mu].chi1R A[mu]

LGF := -1/2/xiA del[A[mu], mu] del[A[nu], nu]

LHiggs := 1/2 del[H,nu] del[H,nu] - 1/2 mH^2 H^2 - 1/3! gHHH H^3\
          - 1/4! lbd H^4

LYukawa := gHfF fbar.f H
```

Notice that you should keep the names, `LGauge`, `LFermions`, `LGF`, `LHiggs` and `LYukawa` , for the different parts of the Lagrangian. Also the `LYukawa` was written with 4-spinors, while the `LFermions` with two spinors. This is correct if we do not want to renormalize the model. For renormalization we should use the same convention in both terms. The first time you choose a process, for instance $f + \overline{f} \to f + \overline{f}$, the equivalent of Bhabha scattering in this model, you have to choose `FRinterLogic = True` to generate the Feynman rules. If you leave the option `Draw = True`, you can see a PDF file with the Feynman rules. These are also kept in the directory of the Feynman rules and you can edit the `LaTeX` file to obtain publishing quality figures. This is shown in Fig. 7.3.

Now we can look at the process. One can go to the subdirectory `1-fFfF` in the `FeynMaster-Output` directory,

```
QEDAndScalar
|-- Processes
     |-- 1-fFfF
```

There you can open the `Mathematica` notebook `Notebook.nb` and look at the final result for the four diagrams, with photon and Higgs exchange.

```
In[ ]:= resT = Total[res] /. {J1 → μ, J2 → ν}
```

$$\text{Out[ ]=}\ \frac{e^2\,(\varphi(-\overline{p2},\text{mf}))\cdot\bar{\gamma}^{\nu}\cdot(\varphi(\overline{p1},\text{mf}))\,(\varphi(\overline{q1},\text{mf}))\cdot\bar{\gamma}^{\nu}\cdot(\varphi(-\overline{q2},\text{mf}))}{(-p1-p2)^2} -$$

$$\frac{e^2\,(\varphi(\overline{q1},\text{mf}))\cdot\bar{\gamma}^{\mu}\cdot(\varphi(\overline{p1},\text{mf}))\,(\varphi(-\overline{p2},\text{mf}))\cdot\bar{\gamma}^{\mu}\cdot(\varphi(-\overline{q2},\text{mf}))}{(q1-p1)^2}$$

.

One can also look at the subdirectory `TeXs-drawings` and obtain the diagrams for the process, We leave for chapter 9 the use of `FeynMaster` in higher order processes, specially in one-loop processes.

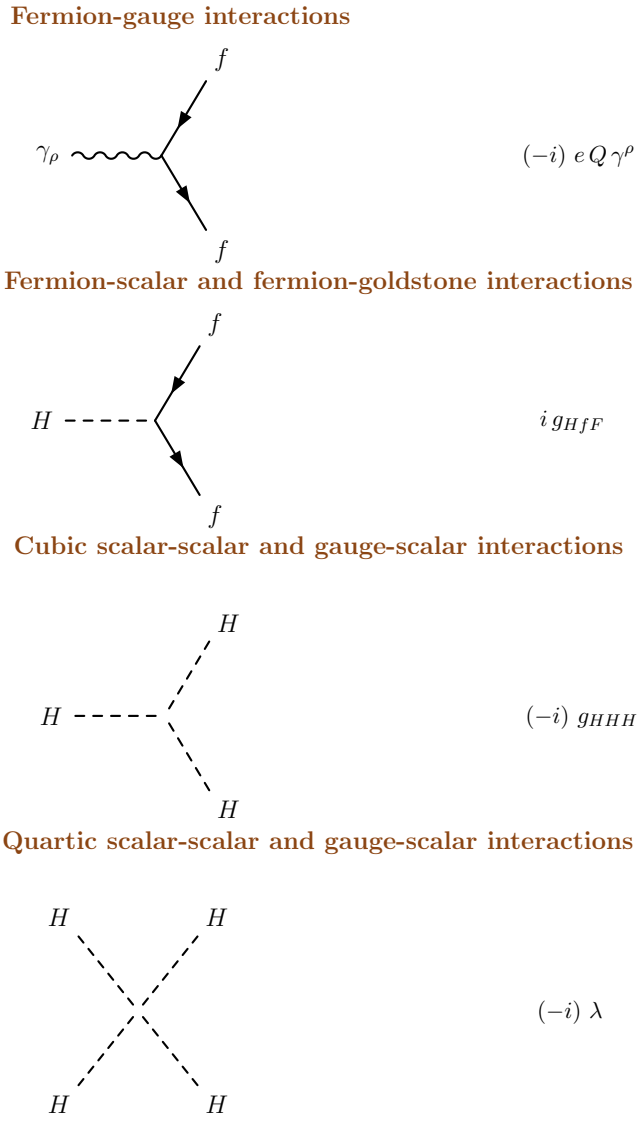


Figure 7.3: Feynman Rules for QED with an extra scalar.

7.7 Numerical Calculations

7.7.1 C/C++ and Fortran

In the previous sections, with the exception of MadGrap and SPheno, we covered mostly analytical tools that use Mathematica or FORM. This is quite good to study in detail many models of QFT. However if one wants to do phenomenological studies one normally has to scan over the parameter space of the model, and for each point calculate the mass matrices, the decay widths and other observables to compare with experimental results. This is normally much more efficient using programming languages that are very fast like C/C++ and Fortran. It is beyond the scope of this

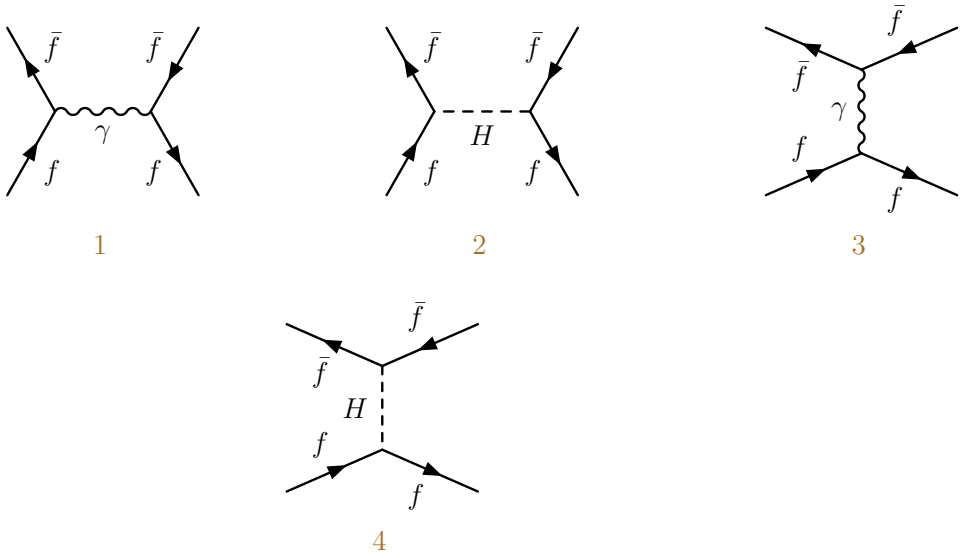


Figure 7.4: Diagrams for the process $f + \overline{f} \to f + \overline{f}$ in the `QEDAndScalar` model.

book to introduce these languages, but some of the numerical examples will make use of them.

7.7.2 CUBA library

Very rarely the integrals that are needed to obtain the cross sections can be done analytically. This specially true for three or more final state particles. Normally one uses numerical integrations. There are many ways of doing this, but one good choice is the library `CUBA` [63]. It can linked *linked* with programs in `C/C++` or in `Fortran`. One of the advantages is that, besides the more used `Vegas` method it has other methods that can be used to compare the precision of the results.

7.7.3 `LoopTools` and `Collier`

The packages `LoopTools` and `Collier` are designed to numerically evaluate the one-loop integrals. We will address them in chapter 9 when we will study the use of software to study one-loop processes.

7.8 Examples in QED: The scattering $e^-e^+ \to \mu^-\mu^+$

In these last three sections we are going to review the processes that we studied by the usual methods in chapters 5 and 6, but using now the software that we discussed before. Although these are simple processes, they will be enough to show how the software can be an enormous help.

7.8.1 QGRAF

We start by considering the process $e^-(p_1)+e^+(p_2) \to \mu^-(p_3)+\mu^+(p_4)$ in QED. We will first use `QGRAF` to find the diagram(s). In fact, as strictly speaking the μ is not part of QED we have to enlarge the model file. It will look like,

```
************** QGRAF QED enlarged Model File  **************
*  leptons

  [e,E,-]
  [m,M,-]

*  photon

  [A,A,+]

*  fermion - fermion - photon

  [E,e,A]
  [M,m,A]
***************** End of  QGRAF Model File  ****************
```

where `m`(`M`) corresponds to the $\mu^-(\mu^+)$, respectively. The next step is to write the input file `qgraf.dat` for `QGRAF`. We write

```
******************** QGRAF input file  *********************
output= list ;
style= Styles/sum.sty ;
model= Models/qed-enlarged;
in= e, E;
out=m, M ;
loops= 0;
loop_momentum= ;
options=  ;
******************       End of file     *******************
```

where `list` is the name of the output file. We run this code with the command (`>` is, in our case, the terminal prompt, it can vary with the system)

```
> ./qgraf
```

and get the following output in the file `list`.

```
****************** QGRAF ouptut  File  *****************
#
# file generated by qgraf-3.4
#
#  output= lista ;
#  style= Styles/sum.sty ;
#  model= Models/qed-enlarged;
#  in= e,E;
#  out= m,M;
#  loops= 0;
#  loop_momentum= ;
#  options= ;
#

 tsum := 0

 +(1)*
 prop(A(1,-p1-p2),A(2,p1+p2))*
 vrtx(E(-3,p2),e(-1,p1),A(1,-p1-p2))*
 vrtx(M(-2,-q1),m(-4,-q2),A(2,p1+p2))

 ;
# end
```

As expected we just have the diagram of Fig. 7.5. Using the Feynman rules we

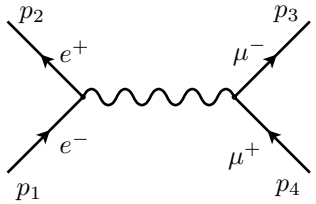


Figure 7.5: Scattering $e^-e^+ \to \mu^-\mu^+$ in QED.

get for the amplitude, (see also Eq. (5.51)),

$$\mathcal{M} = \frac{e^2}{s}\overline{v}(p_2)\gamma^\mu u(p_1)\ \overline{u}(p_3)\gamma_\mu v(p_4) \tag{7.5}$$

7.8.2 Using Traces

Now we want to calculate the spin averaged squared of the matrix element. As we saw in Eq. (5.52) we get,

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{4s^2}\text{Tr}\left[(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu\right]\text{Tr}\left[(\not{p}_3 + m_\mu)\gamma_\mu(\not{p}_4 - m_\mu)\gamma_\nu\right]$$

$$= \frac{e^4}{4s^2} \mathrm{Tr}\left[\not{p}_2 \gamma^\mu \not{p}_1 \gamma^\nu\right] \mathrm{Tr}\left[(\not{p}_3 + m_\mu)\gamma_\mu(\not{p}_4 - m_\mu)\gamma_\nu\right] \tag{7.6}$$

where we are going to neglect the electron mass as before. We have to evaluate the traces. We will use `FeynCalc` for that purpose. The code will look like,

```
In[1]:= << FeynCalc`;

In[2]:= L1 = GS[p2].GA[mu].GS[p1].GA[nu]

Out[2]= (γ·p2).γ^mu.(γ·p1).γ^nu

In[3]:= L2 = (GS[p3] + m).GA[mu].(GS[p4] - m).GA[nu]

Out[3]= (γ·p3 + m).γ^mu.(γ·p4 - m).γ^nu

In[4]:= MsqAux1 = e^4/(4 s^2) Contract[Tr[L1] Tr[L2]] // Simplify

Out[4]= 8 e^4 (m^2 (p1·p2) + (p1·p4)(p2·p3) + (p1·p3)(p2·p4)) / s^2

In[5]:= MsqAux2 = MsqAux1 // FCE

Out[5]= 8 e^4 (m^2 (p1·p2) + (p1·p4)(p2·p3) + (p1·p3)(p2·p4)) / s^2
```

After loading `FeynCalc` we define the two fermion lines, `L1` and `L2`. We could calculate each of the traces, but we can do everything in just one step, including the Lorentz contraction. There is just one point deserving explanation, what is the difference between `MsqAux1` and `MsqAux2`. In the screen they look exactly equal, but internally they are not. `FCE` means `FeynCalc External` and is more user friendly, as one can see in this piece of code,

```
In[6]:= MsqAux1 // InputForm

Out[6]//InputForm=
(8*e^4*(m^2*Pair[Momentum[p1], Momentum[p2]] +
   Pair[Momentum[p1], Momentum[p4]]*Pair[Momentum[p2],
     Momentum[p3]] + Pair[Momentum[p1], Momentum[p3]]*
    Pair[Momentum[p2], Momentum[p4]]))/s^2

In[7]:= MsqAux2 // InputForm

Out[7]//InputForm=
(8*e^4*(m^2*SP[p1, p2] + SP[p1, p4]*SP[p2, p3] +
   SP[p1, p3]*SP[p2, p4]))/s^2
```

Now we can use a substitution rule to obtain the final result. We use the kinematics of Eq. (5.54) to get

```
In[8]:= kin = {SP[p1, p2] → s / 2, SP[p1, p4] → s / 4 (1 + β Cos[θ]),
          SP[p2, p3] → s / 4 (1 + β Cos[θ]),
          SP[p1, p3] → s / 4 (1 - β Cos[θ]),
          SP[p2, p4] → s / 4 (1 - β Cos[θ])};

In[9]:= Msq = (MsqAux2 /. kin) // Simplify
```

$$\text{Out[9]=} \quad \frac{e^4 \left(4\, m^2 + \beta^2\, s \cos^2(\theta) + s\right)}{s}$$

in agreement with Eq. (5.56). The total cross section can easily be obtained, Eq. (5.57).

7.8.3 Using helicity amplitudes

To use the helicity amplitude technique we go back to the expression of the amplitude, Eq. (7.5). We will do this in the limit that we can also neglect the μ mass. We will exemplify the case of massive fermions in the Compton scattering below. We start by loading `FeynCalc` and making some definitions. These include the definition of helicity spinor and the Chisholm rule, Eq. (6.11),

```
<< FeynCalc`

(*Definitions*)
dp[s_] := (1 + s DiracMatrix[5]) / 2
U[p_, s_] := dp[s].Spinor[p, 0]
UBar[p_, s_] := SpinorUBar[p, 0].dp[-s]
delta[s1_, s2_] := If[s1 == s2, 1, 0]
```

```
UBarGammaU = Function[{p3, s3, p2, p1, s1, p4, s4},
    X1 = DiracSimplify[2 UBar[p3, s3].U[p1, s1]
        UBar[p2, s1].U[p4, s4]];
    X2 = DiracSimplify[2 UBar[p3, s3].U[p2, -s1]
        UBar[p1, -s1].U[p4, s4]];
    X1 + X2];
```

$$\overline{u}_\sigma(p_1)\gamma_\mu u_\sigma(p_2)\,\gamma^\mu = 2u_\sigma(p_2)\overline{u}_\sigma(p_1) + 2u_{-\sigma}(p_1)\overline{u}_{-\sigma}(p_2) \tag{7.7}$$

Next we need to define some simplifications and replacement rules,

```
In[9]:= (*Simplify*)
        vlist = {p1, p2, p3, p4};
        simp1 =
          Table[Spinor[vlist[[i]], 0].Spinor[vlist[[j]], 0] →
              sp[vlist[[i]], vlist[[j]]] + spc[vlist[[j]], vlist[[i]]],
            {i, 1, 4}, {j, 1, 4}] /. {sp[p_, p_] → 0, spc[q_, q_] → 0};
        simp2 =
          Table[Spinor[vlist[[i]], 0].DiracMatrix[5].
               Spinor[vlist[[j]], 0] →
              -sp[vlist[[i]], vlist[[j]]] + spc[vlist[[j]], vlist[[i]]],
            {i, 1, 4}, {j, 1, 4}] /. {sp[p_, p_] → 0, spc[q_, q_] → 0};
        simp = Flatten[{simp1, simp2}];
```

This will allow to evaluate the helicity amplitudes. Only four combinations are non-zero (for massless fermions)

```
In[13]:= (* Final Helicity Amplitudes *)

In[14]:= M[s1_, s2_, s3_, s4_] := Expand[res1[s1, s2, s3, s4] /. simp]

In[15]:= M[1, 1, 1, 1]

Out[15]= 2 e^2 spc(p4, p1) sp(p3, p2) / s

In[16]:= M[1, 1, -1, -1]

Out[16]= 2 e^2 spc(p1, p3) sp(p2, p4) / s

In[17]:= M[-1, -1, 1, 1]

Out[17]= 2 e^2 sp(p3, p1) spc(p4, p2) / s

In[18]:= M[-1, -1, -1, -1]

Out[18]= 2 e^2 sp(p1, p4) spc(p2, p3) / s
```

We get therefore

$$\begin{aligned}\overline{|\mathcal{M}|^2} =& \frac{e^4}{4s^2}\left[2|s_{14}|^2|s_{23}|^2 + 2|s_{13}|^2|s_{24}|^2\right] \\ =& e^4\left[1+\cos^2\theta\right] \end{aligned} \tag{7.8}$$

where we have used the definitions, for massless fermions,

$$|s_{14}|^2 = |s_{23}|^2 = 2(p_1\cdot p_4) = -u = -\frac{s}{2}(1+\cos\theta) \tag{7.9}$$

$$|s_{13}|^2 = |s_{24}|^2 = 2(p_1\cdot p_3) = -t = -\frac{s}{2}(1-\cos\theta) \tag{7.10}$$

We obtain again Eq. (5.56) in the limit $\beta \to 1$.

7.8.4 Using FeynMaster

We can use `FeynMaster` to evaluate directly the differential cross section. This process is not strictly QED, but instead of creating a new model, like we have done for `QGRAF`, we use the Standard Model and the possibilities of `FeynMaster` to exclude the Z and H exchange, as well as the Goldstones. The file `Control.m` is then

```
(* This is the Control.m file of FeynMaster *)

model: SM

inparticles: e,ebar
outparticles: m,mbar
loops: 0
parsel: {avoid,Z,1,1},{avoid,H,1,1}
options: ugauge

FRinterLogic: F
RenoLogic: F
Draw: T
Comp: T
FinLogic: F
DivLogic: F
SumLogic: F
MoCoLogic: F
LoSpinors: T
```

We note that the option `ugauge` chooses the unitary gauge and therefore eliminates the diagrams with the Goldstones (and uses the apropriate propagator for the gauge bosons), while the option `parsel: {avoid,Z,1,1},{avoid,H,1,1}` eliminates the diagrams with the exchange of the Z and H. See the manual for more details. In these way we are left with the diagram with the exchange of a photon as shown in Fig. 7.5.

Now you navigate to the directory `.../SM/Processes/1-eebarmmbar` and do the following calculations in the `Notebook.nb`.

```
In[ ]:= M = Total[res] // FeynAmpDenominatorExplicit;

In[ ]:= dsigdOmgAux = ((M // DiffXS) /. {e^4 → (4 Pi α)^2, S -> s, Theta -> theta, me -> 0}) // Simplify
```

$$\text{Out[ ]=}\ \frac{\alpha^2\ \sqrt{s\left(s-4\,\text{mm}^2\right)}\ \left(\left(s-4\,\text{mm}^2\right)\cos^2(\text{theta})+4\,\text{mm}^2+s\right)}{4\left(s^2\right)^{3/2}}$$

```
In[ ]:= dsigdOmg = Simplify[dsigdOmgAux /. mm^2 → s (1 - β^2)/(4), Assumptions -> {s > 0, β > 0}]
```

$$\text{Out[ ]=}\ \frac{\alpha^2\,\beta\left(-\beta^2+\beta^2\cos(2\,\text{theta})+4\right)}{8\,s}$$

```
(* Compare with Eq.5.56 *)

In[ ]:= Eq556 = α^2/(4 s) β (1 + Cos[theta]^2 + (1 - β^2) Sin[theta]^2)
```

$$\text{Out[ ]=}\ \frac{\alpha^2\,\beta\left(\left(1-\beta^2\right)\sin^2(\text{theta})+\cos^2(\text{theta})+1\right)}{4\,s}$$

```
In[ ]:= dsigdOmg - Eq556 // Simplify
Out[ ]= 0
```

We get the final result, in agreement with Eq. (5.56). We should emphasize the simplicity of the calculation. We only showed the relevant part of the Notebook. The function `DiffXS` calculates in the `CM` frame the differential cross section in terms of s and the scattering angle θ.

7.8.5 Using MadGraph5

This example was already studied in section 7.5, see the results in Fig. 7.2.

7.9 Examples in QED: Bhabha scattering

Now we turn to Bhabha scattering, $e^-(p_1) + e^+(p_2) \to e^-(p_3) + e^+(p_4)$ in QED.

7.9.1 QGRAF

We will use `QGRAF` to find the diagrams. The input file is

```
output= lista ;
style= Styles/sum.sty ;
model= Models/qed;
in= e,E;
out= e,E;
loops= 0;
loop_momentum= ;
```

```
 options= ;
```

and we get the output

```
# file generated by qgraf-3.4
#
#  output= lista ;
#  style= Styles/sum.sty ;
#  model= Models/qed;
#  in= e,E;
#  out= e,E;
#  loops= 0;
#  loop_momentum= ;
#  options= ;
#

 tsum := 0

 +(1)*
 prop(A(1,-p1-p2),A(2,p1+p2))*
 vrtx(E(-3,p2),e(-1,p1),A(1,-p1-p2))*
 vrtx(E(-2,-q1),e(-4,-q2),A(2,p1+p2))

 -(1)*
 prop(A(1,-p1+q1),A(2,p1-q1))*
 vrtx(E(-2,-q1),e(-1,p1),A(1,-p1+q1))*
 vrtx(E(-3,p2),e(-4,-q2),A(2,p1-q1))

 ;

# end
```

We have the two diagrams of Fig. 5.7, corresponding to the s-channel and t-channel, and these have a relative minus sign. By using `QGRAF` we get automatically the correct signs for the diagrams, without the need to go to Wick's theorem.

7.9.2 Using Traces

In section 5.4 we saw that the spin averaged square of the matrix element was given by

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{4}\left\{\frac{1}{t^2}\mathrm{Tr}\left[\not{p}_3\gamma^\mu\not{p}_1\gamma^\nu\right]\mathrm{Tr}\left[\not{p}_2\gamma_\mu\not{p}_4\gamma_\nu\right]\right.$$
$$\left.+\frac{1}{s^2}\mathrm{Tr}\left[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu\right]\mathrm{Tr}\left[\not{p}_3\gamma_\mu\not{p}_4\gamma_\nu\right]-\frac{2}{st}\mathrm{Tr}\left[\not{p}_3\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2\gamma_\mu\not{p}_4\gamma_\nu\right]\right\} \tag{7.11}$$

and we evaluated the traces *by hand*. Now we turn to `FeynCalc` to do this for us. We start with the definitions, then do the traces and the Lorentz contractions, and simplifications.

```
<< FeynCalc`

(* Definitions*)
L1 = GS[p3].GA[mu].GS[p1].GA[nu];
L2 = GS[p2].GA[mu].GS[p4].GA[nu];
L3 = GS[p2].GA[mu].GS[p1].GA[nu];
L4 = GS[p3].GA[mu].GS[p4].GA[nu];
L5 = GS[p3].GA[mu].GS[p1].GA[nu].GS[p2].GA[mu].GS[p4].GA[nu];
```

```
(*The Traces *)
Msqt = e^4/t^2 Contract[Tr[L1] Tr[L2]]
```

Out[7]=
$$\frac{16\, e^4 \left(2 \left(\overline{\text{p1}}\cdot\overline{\text{p4}}\right)\left(\overline{\text{p2}}\cdot\overline{\text{p3}}\right)+2 \left(\overline{\text{p1}}\cdot\overline{\text{p2}}\right)\left(\overline{\text{p3}}\cdot\overline{\text{p4}}\right)\right)}{t^2}$$

In[8]:=
```
Msqs = e^4/s^2 Contract[Tr[L3] Tr[L4]]
```

Out[8]=
$$\frac{16\, e^4 \left(2 \left(\overline{\text{p1}}\cdot\overline{\text{p4}}\right)\left(\overline{\text{p2}}\cdot\overline{\text{p3}}\right)+2 \left(\overline{\text{p1}}\cdot\overline{\text{p3}}\right)\left(\overline{\text{p2}}\cdot\overline{\text{p4}}\right)\right)}{s^2}$$

In[9]:=
```
Msqst = e^4/(s t) Contract[Tr[L5]]
```

Out[9]=
$$-\frac{32\, e^4 \left(\overline{\text{p1}}\cdot\overline{\text{p4}}\right)\left(\overline{\text{p2}}\cdot\overline{\text{p3}}\right)}{s\, t}$$

In[21]:=
```
MsqAux = (1/4 Msqt + 1/4 Msqs - 2/4 Msqst) // FCE
```

Out[21]=
$$\frac{4\, e^4 \left(2 \left(\overline{\text{p1}}\cdot\overline{\text{p4}}\right)\left(\overline{\text{p2}}\cdot\overline{\text{p3}}\right)+2 \left(\overline{\text{p1}}\cdot\overline{\text{p3}}\right)\left(\overline{\text{p2}}\cdot\overline{\text{p4}}\right)\right)}{s^2}+$$
$$\frac{16\, e^4 \left(\overline{\text{p1}}\cdot\overline{\text{p4}}\right)\left(\overline{\text{p2}}\cdot\overline{\text{p3}}\right)}{s\, t}+\frac{4\, e^4 \left(2 \left(\overline{\text{p1}}\cdot\overline{\text{p4}}\right)\left(\overline{\text{p2}}\cdot\overline{\text{p3}}\right)+2 \left(\overline{\text{p1}}\cdot\overline{\text{p2}}\right)\left(\overline{\text{p3}}\cdot\overline{\text{p4}}\right)\right)}{t^2}$$

```
(* Simplifications*)

kin = {SP[p1, p4] → -u/2, SP[p2, p3] → -u/2, SP[p1, p2] → s/2,
   SP[p3, p4] → s/2, SP[p1, p3] → -t/2, SP[p2, p4] → -t/2};

Msq = (MsqAux /. kin) /. u → -(s + t)
```

$$\frac{4\, e^4 \left(\frac{1}{2}(-s-t)^2+\frac{t^2}{2}\right)}{s^2}+\frac{4\, e^4 \left(\frac{s^2}{2}+\frac{1}{2}(-s-t)^2\right)}{t^2}+\frac{4\, e^4 (-s-t)^2}{s\, t}$$

We then get the result previously obtained in Eq. (5.85).

7.9.3 Using helicity amplitudes

Now we will illustrate the use of FeynCalc to evaluate the helicity amplitudes for Bhabha scattering. We will continue in the limit of massless fermions. The code is a simple generalization of what we got for $e^- + e^+ \to \mu^- + \mu^+$.

```
In[1]:= << FeynCalc`

In[2]:= (*Definitions*)
        dp[s_] := (1 + s DiracMatrix[5]) / 2
        U[p_, s_] := dp[s].Spinor[p, 0]
        UBar[p_, s_] := SpinorUBar[p, 0].dp[-s]
        delta[s1_, s2_] := If[s1 == s2, 1, 0]

In[6]:= UBarGammaU = Function[{p3, s3, p2, p1, s1, p4, s4},
           X1 = DiracSimplify[2 UBar[p3, s3].U[p1, s1]
              UBar[p2, s1].U[p4, s4]];
           X2 = DiracSimplify[2 UBar[p3, s3].U[p2, -s1]
              UBar[p1, -s1].U[p4, s4]];
           X1 + X2];
```

```
In[7]:= (* Amplitudes*)
        M1[s1_, s2_, s3_, s4_] :=
         e^2 / s delta[s1, s2] UBarGammaU[p3, s3, p2, p1, s1, p4, s4]

        res1[s1_, s2_, s3_, s4_] :=
         DiracSimplify[M1[s1, s2, s3, s4], DiracSubstitute67 → True]
        M2[s1_, s2_, s3_, s4_] :=
         delta[s1, s3] UBarGammaU[p2, s2, p3, p1, s1, p4, s4]
        res2[s1_, s2_, s3_, s4_] :=
         - e^2 / t DiracSimplify[M2[s1, s2, s3, s4],
           DiracSubstitute67 → True]
```

```
vlist = {p1, p2, p3, p4};
simp1 =
  Table[Spinor[vlist[[i]], 0].Spinor[vlist[[j]], 0] →
     sp[vlist[[i]], vlist[[j]]] + spc[vlist[[j]], vlist[[i]]],
    {i, 1, 4}, {j, 1, 4}] /. {sp[p_, p_] → 0, spc[q_, q_] → 0};
```

```
simp2 =
  Table[Spinor[vlist[[i]], 0].DiracMatrix[5].
      Spinor[vlist[[j]], 0] →
     -sp[vlist[[i]], vlist[[j]]] + spc[vlist[[j]], vlist[[i]]],
    {i, 1, 4}, {j, 1, 4}] /. {sp[p_, p_] → 0, spc[q_, q_] → 0};
simp = Flatten[{simp1, simp2}];
```

```
In[15]:= (* Final Helicity Amplitudes *)

In[16]:= M[s1_, s2_, s3_, s4_] :=
          Expand[(res1[s1, s2, s3, s4] + res2[s1, s2, s3, s4]) /. simp]

In[17]:= M[1, 1, 1, 1]

Out[17]= 2 e^2 spc(p4, p1) sp(p3, p2) / s  -  2 e^2 spc(p4, p1) sp(p2, p3) / t

In[22]:= M[-1, -1, -1, -1]

Out[22]= 2 e^2 sp(p1, p4) spc(p2, p3) / s  -  2 e^2 sp(p1, p4) spc(p3, p2) / t

In[24]:= M[1, -1, 1, -1]

Out[24]= -2 e^2 spc(p1, p2) sp(p3, p4) / t
```

```
In[25]:= M[-1, 1, -1, 1]

Out[25]= -2 e^2 sp(p2, p1) spc(p4, p3) / t

In[26]:= M[1, 1, -1, -1]

Out[26]= 2 e^2 spc(p1, p3) sp(p2, p4) / s

In[27]:= M[-1, -1, 1, 1]

Out[27]= 2 e^2 sp(p3, p1) spc(p4, p2) / s
```

We see that we obtain the result in agreement with Eq. (6.17).

7.9.4 Using FeynMaster

We can use `FeynMaster` to evaluate directly the differential cross section. This process is QED and the file `Control.m` is then

```
(* This is the Control.m file of FeynMaster *)
model: SM
inparticles: e,ebar
outparticles: e,ebar
loops: 0
parsel: {avoid,Z,1,3},{avoid,H,1,3}
options: ugauge
FRinterLogic: F
RenoLogic: F
Draw: T
Comp: T
FinLogic: F
DivLogic: F
SumLogic: F
MoCoLogic: F
LoSpinors: T
```

The Total Amplitude

```
In[ ]:= M = Total[res]
```

$$\text{Out[ ]}= \frac{e^2\,(\varphi(-\overline{p2},\text{me}))\cdot\bar{\gamma}^{j2}\cdot(\varphi(\overline{p1},\text{me}))\,(\varphi(\overline{q1},\text{me}))\cdot\bar{\gamma}^{j2}\cdot(\varphi(-\overline{q2},\text{me}))}{(-p1-p2)^2} - \frac{e^2\,(\varphi(\overline{q1},\text{me}))\cdot\bar{\gamma}^{j1}\cdot(\varphi(\overline{p1},\text{me}))\,(\varphi(-\overline{p2},\text{me}))\cdot\bar{\gamma}^{j1}\cdot(\varphi(-\overline{q2},\text{me}))}{(q1-p1)^2}$$

Differential xsection in the Limit me=0

```
In[ ]:= dsig = (M // DiffXS) /. me → 0 // Simplify
```

$$\text{Out[ ]}= \frac{e^4\,(\cos(2\,\text{Theta})+7)^2}{256\,\pi^2\,S\,(\cos(\text{Theta})-1)^2}$$

Result from Chapter 5

```
In[ ]:= dsigCh5 = α^2/2/s ((t^2+(s+t)^2)/s^2+(s^2+(s+t)^2)/t^2+ 2 (s+t)^2/s/t) // Simplify
```

$$\text{Out[ ]}= \frac{\alpha^2\,(s^2+s\,t+t^2)^2}{s^3\,t^2}$$

Compare with Chapter 5

```
In[ ]:= dsigCh5Aux = (dsigCh5 /. {α → e^2/(4 Pi), s → S, t → -S/2 (1-Cos[Theta])}) // Simplify
```

$$\text{Out[ ]}= \frac{e^4\,(\cos^2(\text{Theta})+3)^2}{64\,\pi^2\,S\,(\cos(\text{Theta})-1)^2}$$

```
In[ ]:= dsig - dsigCh5Aux // Simplify
```

$$\text{Out[ ]}= 0$$

We get the final result, in agreement with Eq. (5.86). We should emphasize the simplicity of the calculation. We only showed the relevant part of the Notebook.

7.10 Examples in QED: Compton scattering

7.10.1 QGRAF

In our last example we start again by using `QGRAF`. The relevant part of the input file is

```
model= Models/qed;
in= e,A;
out= e,A;
loops= 0;
```

giving the output

```
# file generated by qgraf-3.4
#
#  output= lista ;
#  style= Styles/sum.sty ;
#  model= Models/qed;
#  in= e,A;
#  out= e,A;
#  loops= 0;
#  loop_momentum= ;
#  options= ;
#
 tsum := 0

 +(1)*
 prop(e(1,p1+p2),E(2,-p1-p2))*
 vrtx(E(2,-p1-p2),e(-1,p1),A(-3,p2))*
 vrtx(E(-2,-q1),e(1,p1+p2),A(-4,-q2))

 +(1)*
 prop(e(1,p1-q2),E(2,-p1+q2))*
 vrtx(E(2,-p1+q2),e(-1,p1),A(-4,-q2))*
 vrtx(E(-2,-q1),e(1,p1-q2),A(-3,p2))
 ;
# end
```

We recover the two diagrams of Fig. 5.2, with a relative plus sign, because of Bose symmetry.

7.10.2 Using Traces

We saw in section that the sum over spins lead to the calculation of four traces, in Eq. (4.76), Eq. (5.28) and Eq. (5.29). We recall them here for convenience,

$$\sum_{s,s'} |\mathcal{M}_1|^2 = \mathrm{Tr}\left[(\not{p}' + m)\Gamma_1(\not{p} + m)\overline{\Gamma}_1\right], \quad \sum_{s,s'} |\mathcal{M}_2|^2 = \mathrm{Tr}\left[(\not{p}' + m)\Gamma_2(\not{p} + m)\overline{\Gamma}_2\right]$$

$$\sum_{s,s'} (\mathcal{M}_1\mathcal{M}_2^\dagger + \mathcal{M}_1^\dagger\mathcal{M}_2) = \mathrm{Tr}\left[(\not{p}' + m)\Gamma_1(\not{p} + m)\overline{\Gamma}_2\right] + \mathrm{Tr}\left[(\not{p}' + m)\Gamma_2(\not{p} + m)\overline{\Gamma}_1\right]$$

With our experience it is very easy to write a program to evaluate them with `FeynCalc`. We start by making some definitions.

```
<< FeynCalc`

(* Definitions*)
prop = Function[{p, m}, (GS[p] + m)];
Line1 := prop[pp, m].GA[mu].prop[p + k, m].GA[nu].prop[p, m].
   GA[nu].prop[p + k, m].GA[mu];
Line2 := prop[pp, m].GA[mu].prop[p - kp, m].GA[nu].prop[p, m].
   GA[nu].prop[p - kp, m].GA[mu];
Line12 := prop[pp, m].GA[nu].prop[p + k, m].GA[mu].prop[p, m].
   GA[nu].prop[p - kp, m].GA[mu];
Line21 := prop[pp, m].GA[mu].prop[p - kp, m].GA[nu].prop[p, m].
   GA[mu].prop[p + k, m].GA[nu];
```

```
(*Traces*)
ans1 = Simplify[Contract[Tr[Line1]] / (2 SP[p, k]) ^2];
ans2 = Simplify[Contract[Tr[Line2]] / (2 SP[p, kp]) ^2];
ans12 = Simplify[Contract[Tr[Line12]] / (-4 SP[p, k] SP[p, kp])];
ans21 = Simplify[Contract[Tr[Line21]] / (-4 SP[p, k] SP[p, kp])];

ans = 1 / 4 * (ans1 + ans2 + ans12 + ans21) // FCE
```

$$\frac{1}{4}\left(\frac{1}{(\overline{k}\cdot\overline{p})(\overline{\mathrm{kp}}\cdot\overline{p})} 8\left((\overline{k}\cdot\overline{\mathrm{kp}})\left(m^2 - 2\,(\overline{p}\cdot\overline{\mathrm{pp}})\right) - \right.\right.$$
$$2\,(\overline{k}\cdot\overline{p})\left(m^2 - \overline{p}\cdot\overline{\mathrm{pp}}\right) - m^2\,(\overline{k}\cdot\overline{\mathrm{pp}}) + 2\,m^2\,(\overline{\mathrm{kp}}\cdot\overline{p}) + m^2\,(\overline{\mathrm{kp}}\cdot\overline{\mathrm{pp}}) -$$
$$2\,(\overline{\mathrm{kp}}\cdot\overline{p})\,(\overline{p}\cdot\overline{\mathrm{pp}}) - 3\,m^2\,(\overline{p}\cdot\overline{\mathrm{pp}}) - 3\,m^2\,\overline{p}^2 + 2\,\overline{p}^2\,(\overline{p}\cdot\overline{\mathrm{pp}}) + 2\,m^4\big) +$$
$$\frac{1}{(\overline{k}\cdot\overline{p})^2} 4\left(2\,(\overline{k}\cdot\overline{p})\,(\overline{k}\cdot\overline{\mathrm{pp}} + 2\,m^2) + \overline{k}^2\left(4\,m^2 - \overline{p}\cdot\overline{\mathrm{pp}}\right) - 4\,m^2\,(\overline{k}\cdot\overline{\mathrm{pp}}) + \right.$$
$$2\,\overline{p}^2\,(\overline{k}\cdot\overline{\mathrm{pp}}) - 3\,m^2\,(\overline{p}\cdot\overline{\mathrm{pp}}) + \overline{p}^2\,(\overline{p}\cdot\overline{\mathrm{pp}}) + 4\,m^4\big) +$$
$$\frac{1}{(\overline{\mathrm{kp}}\cdot\overline{p})^2} 4\left((\overline{\mathrm{kp}}\cdot\overline{p})\left(2\,(\overline{\mathrm{kp}}\cdot\overline{\mathrm{pp}}) - 4\,m^2\right) + \overline{\mathrm{kp}}^2\left(4\,m^2 - \overline{p}\cdot\overline{\mathrm{pp}}\right) + 4\,m^2\,(\overline{\mathrm{kp}}\cdot\overline{\mathrm{pp}}) - \right.$$
$$\left.\left.2\,\overline{p}^2\,(\overline{\mathrm{kp}}\cdot\overline{\mathrm{pp}}) - 3\,m^2\,(\overline{p}\cdot\overline{\mathrm{pp}}) + \overline{p}^2\,(\overline{p}\cdot\overline{\mathrm{pp}}) + 4\,m^4\big)\right)$$

```
(*Kinematics*)
onshell = {SP[p, p] → m^2, SP[pp, pp] → m^2, SP[k, k] → 0,
   SP[kp, kp] → 0};

res = ans /. onshell;

kin = {SP[p, k] → m Ek, SP[p, kp] → m Ekp, SP[k, kp] → m (Ek - Ekp),
   SP[p, pp] → m^2 + m (Ek - Ekp), SP[pp, k] → m Ekp, SP[pp, kp] → m Ek};

aux = Expand[res / 2 /. kin]
```

$$\frac{m^2}{\text{Ek}^2}-\frac{2\,m^2}{\text{Ek}\,\text{Ekp}}+\frac{\text{Ekp}}{\text{Ek}}+\frac{\text{Ek}}{\text{Ekp}}+\frac{2\,m}{\text{Ek}}+\frac{m^2}{\text{Ekp}^2}-\frac{2\,m}{\text{Ekp}}$$

We then get the final result, the Klein-Nishina formula in agreement in Eq. (5.49)

```
(* Final Formula*)

aux1 = aux - Ek / Ekp - Ekp / Ek;

aux2 = Simplify[aux1 /. Ekp → Ek / (1 + Ek / m * (1 - Cos[teta]))];

final = 2 e^4 (Ek / Ekp + Ekp / Ek + aux2)
```

$$2\,e^4\left(\frac{\text{Ekp}}{\text{Ek}}+\frac{\text{Ek}}{\text{Ekp}}-\sin^2(\text{teta})\right)$$

7.10.3 Using helicity amplitudes

We have already discusses the helicity amplitudes for Compton scattering in section 6.5.1. This is a difficult problem because we have to use helicity spinors for the polarization sums for photons and have to consider fermions with mass like in section 6.5 for the electrons. The amplitudes were given in Eq. (6.51), and we recall them here for convenience,

$$\begin{aligned}\mathcal{M}_1(\sigma_1,\sigma_2;\lambda_1,\lambda_2) =& C_1\overline{u}(p_2,\sigma_2 s)\gamma_\nu\left(\not{p}_1+\not{k}_1+m\right)\gamma_\mu u(p_1,\sigma_1 s)\epsilon^\mu(k_1,\lambda_1)\epsilon^{\nu *}(k_2,\lambda_2)\\ \mathcal{M}_2(\sigma_1,\sigma_2;\lambda_1,\lambda_2) =& C_2\overline{u}(p_2,\sigma_2 s)\gamma_\mu\left(\not{p}_1-\not{k}_2+m\right)\gamma_\nu u(p_1,\sigma_1 s)\epsilon^\mu(k_1,\lambda_1)\epsilon^{\nu *}(k_2,\lambda_2)\end{aligned} \tag{7.12}$$

where C_i were given in Eq. (6.52). As before we use for massive spinors the definition of Eq. (6.47) that we recall here

$$\begin{aligned}u(p_1,+s) &= \frac{s(r_1,r_2)}{m}\,u_+(r_1)+u_-(r_2)\\ u(p_1,-s) &= \frac{s^*(r_2,r_1)}{m}\,u_-(r_1)+u_+(r_2)\end{aligned}$$

$$
\begin{aligned}
u(p_2,+s) &= \frac{s(w_1,w_2)}{m}\, u_+(w_1) + u_-(w_2) \\
u(p_2,-s) &= \frac{s^*(w_2,w_1)}{m}\, u_-(w_1) + u_+(w_2)
\end{aligned}
\tag{7.13}
$$

with

$$
p_1 = r_1 + r_2, \quad p_2 = w_1 + w_2, \quad r_i^2 = w_i^2 = 0 \tag{7.14}
$$

For the polarization vectors we use Eq. (6.29), that is,

$$
\epsilon^\mu(k_i,\lambda) = \frac{1}{N_i}\,\overline{u}_\lambda(k_i)\gamma^\mu u_\lambda(r_1),\ N_i = \sqrt{(4k_i\cdot r_1)}, \quad i = 1,2 \tag{7.15}
$$

where we have chosen the arbitrary momentum p in Eq. (6.29) as r_1, a momentum already in the problem (with $k_i\cdot r_1 \neq 0$). Now we will show a program for `Feyncalc` to evaluate these helicity amplitudes. We first define the massive spinors

```
In[1]:= << FeynCalc`

        (*Definitions*)
        dp[s_] := (1 + s GA[5]) / 2;

In[3]:= U[p_, s_] := dp[s].SpinorU[p, 0];
        UBar[p_, s_] := SpinorUBar[p, 0].dp[-s];
        Um[p1_, p2_, m_, s_] :=
          (If[s == 1, MySP[p1, p2], MySPc[p2, p1]] / m U[p1, s] + U[p2, -s]);
        Vm[p1_, p2_, m_, s_] :=
          (If[s == 1, MySP[p1, p2], MySPc[p2, p1]] / m U[p1, s] - U[p2, -s]);
        UmBar[p1_, p2_, m_, s_] :=
          (If[s == 1, MySPc[p1, p2], MySP[p2, p1]] / m UBar[p1, s] +
            UBar[p2, -s]);
        VmBar[p1_, p2_, m_, s_] :=
          (If[s == 1, MySPc[p1, p2], MySP[p2, p1]] / m UBar[p1, s] -
            UBar[p2, -s]);
```

Then we define the polarization vectors

```
In[9]:= (* More Definitions*)
        PolSConjU = Function[{k, p, l, i},
           If[i == 1, 2 U[k, l], If[i == 2, 2 U[p, -l]]]];
        PolSConjUBar = Function[{k, p, l, i},
           If[i == 1, UBar[p, l], If[i == 2, UBar[k, -l]]]];
        PolSU = Function[{k, p, l, i},
           If[i == 1, 2 U[p, l], If[i == 2, 2 U[k, -l]]]];
        PolSUBar = Function[{k, p, l, i},
           If[i == 1, UBar[k, l], If[i == 2, UBar[p, -l]]]];
        MydsU = Function[{p, i}, If[i == 1, U[p, 1], If[i == 2, U[p, -1]]]];
        MydsUBar = Function[{p, i},
           If[i == 1, UBar[p, 1], If[i == 2, UBar[p, -1]]]];
```

and some simplifications

```
        (*Simplifications*)
        vlist = {w1, w2, k1, k2, r1, r2};

In[16]:= simp1 =
          Table[Spinor[vlist[[i]], 0].Spinor[vlist[[j]], 0] →
             MySP[vlist[[i]], vlist[[j]]] + MySPc[vlist[[j]], vlist[[i]]],
            {i, 1, 6}, {j, 1, 6}] /. {MySP[p_, p_] → 0, MySPc[q_, q_] → 0};

In[17]:= simp2 =
          Table[Spinor[vlist[[i]], 0].DiracMatrix[5].
              Spinor[vlist[[j]], 0] →
             -MySP[vlist[[i]], vlist[[j]]] + MySPc[vlist[[j]], vlist[[i]]],
            {i, 1, 6}, {j, 1, 6}] /. {MySP[p_, p_] → 0, MySPc[q_, q_] → 0};

In[18]:= simp = Flatten[{simp1, simp2}];
```

Now the amplitudes

Amplitudes M1 and M2

```
MDiag1[s1_, s2_, l1_, l2_] :=
  Sum[
    (UmBar[w1, w2, m, s2].PolSConjU[k2, r1, l2, i1]
       PolSConjUBar[k2, r1, l2, i1].MydsU[r1, i2]
       MydsUBar[r1, i2] .PolSU[k1, r1, l1, i3]
       PolSUBar[k1, r1, l1, i3].Um[r1, r2, m, s1] +
      UmBar[w1, w2, m, s2].PolSConjU[k2, r1, l2, i1]
       PolSConjUBar[k2, r1, l2, i1].MydsU[r2, i2]
       MydsUBar[r2, i2] .PolSU[k1, r1, l1, i3]
       PolSUBar[k1, r1, l1, i3].Um[r1, r2, m, s1] +
      UmBar[w1, w2, m, s2].PolSConjU[k2, r1, l2, i1]
       PolSConjUBar[k2, r1, l2, i1].MydsU[k1, i2]
       MydsUBar[k1, i2] .PolSU[k1, r1, l1, i3]
       PolSUBar[k1, r1, l1, i3].Um[r1, r2, m, s1]), {i1, 1, 2},
    {i2, 1, 2}, {i3, 1, 2}] + m (
     Sum[ UmBar[w1, w2, m, s2].PolSConjU[k2, r1, l2, i1]
       PolSConjUBar[k2, r1, l2, i1].PolSU[k1, r1, l1, i2]
       PolSUBar[k1, r1, l1, i2].Um[r1, r2, m, s1], {i1, 1, 2},
      {i2, 1, 1}]);
```

```
MDiag2[s1_, s2_, l1_, l2_] :=
  Sum[
    (UmBar[w1, w2, m, s2].PolSU[k1, r1, l1, i1]
       PolSUBar[k1, r1, l1, i1].MydsU[r1, i2]
       MydsUBar[r1, i2] .PolSConjU[k2, r1, l2, i3]
       PolSConjUBar[k2, r1, l2, i3].Um[r1, r2, m, s1] +
      UmBar[w1, w2, m, s2].PolSU[k1, r1, l1, i1]
       PolSUBar[k1, r1, l1, i1].MydsU[r2, i2]
       MydsUBar[r2, i2] .PolSConjU[k2, r1, l2, i3]
       PolSConjUBar[k2, r1, l2, i3].Um[r1, r2, m, s1] -
      UmBar[w1, w2, m, s2].PolSU[k1, r1, l1, i1]
       PolSUBar[k1, r1, l1, i1].MydsU[k2, i2]
       MydsUBar[k2, i2] .PolSConjU[k2, r1, l2, i3]
       PolSConjUBar[k2, r1, l2, i3].Um[r1, r2, m, s1]),
    {i1, 1, 2}, {i2, 1, 2}, {i3, 1, 2}] + m (
     Sum[ UmBar[w1, w2, m, s2].PolSU[k1, r1, l1, i1]
       PolSUBar[k1, r1, l1, i1].PolSConjU[k2, r1, l2, i2]
       PolSConjUBar[k2, r1, l2, i2].Um[r1, r2, m, s1],
      {i1, 1, 2}, {i2, 1, 2}]);
```

Introduce the simplifications

Simplify the Amplitudes

In[21]:=
```
res1[s1_, s2_, l1_, l2_] :=
  DiracSimplify[MDiag1[s1, s2, l1, l2], DiracSubstitute67 → True];
res2[s1_, s2_, l1_, l2_] :=
  DiracSimplify[MDiag2[s1, s2, l1, l2], DiracSubstitute67 → True];

M1aux[s1_, s2_, l1_, l2_] := Expand[res1[s1, s2, l1, l2] /. simp];
M2aux[s1_, s2_, l1_, l2_] := Expand[res2[s1, s2, l1, l2] /. simp];
M1[s1_, s2_, l1_, l2_] :=
  Expand[M1aux[s1, s2, l1, l2] /.
    {MySP[p_, q_] → sp[p, q], MySPc[p_, q_] → spc[p, q]}];
M2[s1_, s2_, l1_, l2_] :=
  Expand[M2aux[s1, s2, l1, l2] /.
    {MySP[p_, q_] → sp[p, q], MySPc[p_, q_] → spc[p, q]}];
```

Now we test a few amplitudes, in agreement with Eq. (6.56)

Test a few amplitudes

```
In[27]:= M1[1, 1, 1, -1]
```

$$\text{Out[27]= } 4\,\text{sp(k2, k1)}\,\text{spc(r1, k1)}\,\text{sp(k1, r2)}\,\text{spc(r1, w2)} + 4\,\text{sp(k1, r2)}\,\text{sp(k2, r2)}\,\text{spc(r1, r2)}\,\text{spc(r1, w2)}$$

```
In[28]:= M2[1, 1, 1, -1]
```

$$\text{Out[28]= } 4\,\text{sp(k1, r2)}\,\text{sp(k2, r2)}\,\text{spc(r1, r2)}\,\text{spc(r1, w2)} - 4\,\text{sp(k1, k2)}\,\text{spc(r1, k2)}\,\text{sp(k2, r2)}\,\text{spc(r1, w2)}$$

But of course this problem has to be done numerically, so we output the amplitudes to a `Fortran` file.

Output to Fortran

```
If[True,
 M = {C1 M1[1, 1, 1, 1] + C2 M2[1, 1, 1, 1],
   C1 M1[1, 1, 1, -1] + C2 M2[1, 1, 1, -1],
   C1 M1[1, 1, -1, 1] + C2 M2[1, 1, -1, 1],
   C1 M1[1, 1, -1, -1] + C2 M2[1, 1, -1, -1],
   C1 M1[1, -1, 1, 1] + C2 M2[1, -1, 1, 1],
   C1 M1[1, -1, 1, -1] + C2 M2[1, -1, 1, -1],
   C1 M1[1, -1, -1, 1] + C2 M2[1, -1, -1, 1],
   C1 M1[1, -1, -1, -1] + C2 M2[1, -1, -1, -1],
   C1 M1[-1, 1, 1, 1] + C2 M2[-1, 1, 1, 1],
   C1 M1[-1, 1, 1, -1] + C2 M2[-1, 1, 1, -1],
   C1 M1[-1, 1, -1, 1] + C2 M2[-1, 1, -1, 1],
   C1 M1[-1, 1, -1, -1] + C2 M2[-1, 1, -1, -1],
   C1 M1[-1, -1, 1, 1] + C2 M2[-1, -1, 1, 1],
   C1 M1[-1, -1, 1, -1] + C2 M2[-1, -1, 1, -1],
   C1 M1[-1, -1, -1, 1] + C2 M2[-1, -1, -1, 1],
   C1 M1[-1, -1, -1, -1] + C2 M2[-1, -1, -1, -1]};
 stmp = OpenWrite["ComptonAmplitudes.f", FormatType → FortranForm,
   PageWidth → 60];
 Do[Write[stmp, "HelAmp(", i, ")=", M[[i]]], {i, 1, 16}];
 Close[stmp];]
```

Using the output inside a `Fortran` program (see programs in the CTQFT site [37]) we obtain the result in Fig 7.6, where the line corresponds to the Klein-Nishina analytical result, Eq. (5.49), and the points to the result of the numerical program. The agreement is complete.

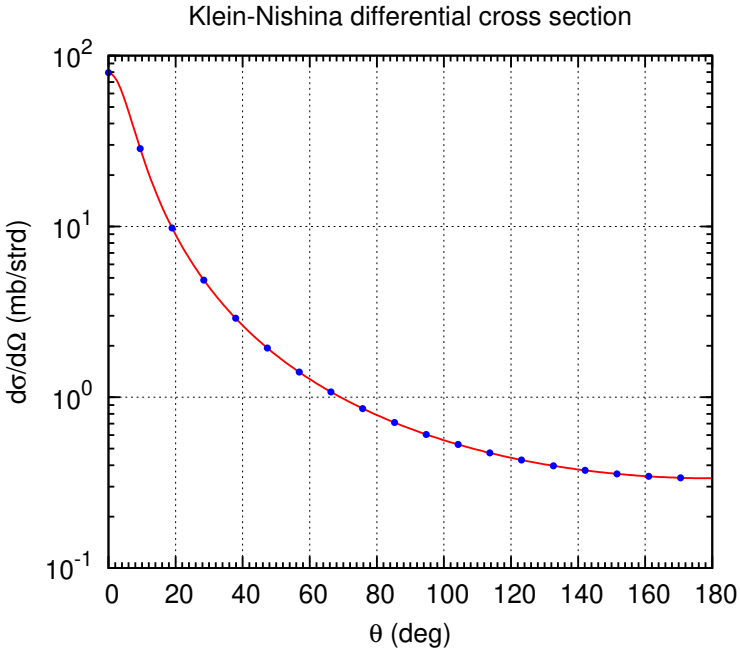


Figure 7.6: Comparison of the Klein-Nishina formula, Eq. (1.116) (red line) with the numerical result using helicity amplitudes (blue points).

In the numerical program there only one detail that it worth to discuss here. In the method used it was necessary two pairs of 4-vectors of the null type, that is,

$$p_1 = r_1 + r_2, \quad p_2 = w_1 + w_2, \quad r_i^2 = w_i^2 = 0 \tag{7.16}$$

This leads to the introduction of two additional integrations (the variables that define r_1, the angles θ_1 and φ_1, also define the 4-vector r_2 and similarly for w_2) in the form,

$$\sum_{\text{Pol}} |\mathcal{M}|^2 = \int \frac{d\Omega_{\vec{r}_1}}{4\pi} \frac{d\Omega_{\vec{r}_2}}{4\pi} \frac{d\Omega_{\vec{w}_1}}{4\pi} \frac{d\Omega_{\vec{w}_2}}{4\pi} \sum_{\sigma_1,\sigma_2,\lambda_1,\lambda_2} |M(\sigma_1,\sigma_2,\lambda_1,\lambda_2)|^2 \tag{7.17}$$

The question now is how to define, in the lab frame where the photon momenta are defined, these new four 4-vectors. Let us consider first the case of r_i. As $p_1 = r_1 + r_2$ corresponds to a particle at rest in the lab frame, we define the new 4-vectors in this same frame. We have then,

$$r_1 \;=\; \frac{m_e}{2}\,(1, \cos\varphi_1 \sin\theta_1, \sin\varphi_1 \sin\theta_1, \cos\theta_1)$$

$$r_2 = \frac{m_e}{2}(1, -\cos\varphi_1 \sin\theta_1, -\sin\varphi_1 \sin\theta_1, -\cos\theta_1) \tag{7.18}$$

that satisfy all the necessary conditions. The case of w_i, such that $p_2 = w_1 + w_2$, is more difficult as the scattered electron is not at rest in the lab frame. It is useful to recall the kinematics for Compton scattering shown in Fig. 7.7.

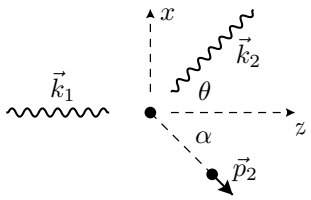


Figure 7.7: Kinematics for Compton scattering in the lab frame S.

We denote by S the lab frame, by S'' the frame where the scattered electron is at rest and by S' an auxiliary frame that corresponds to a rotation around the the axis y by an angle α in the lab frame, in such a way that z' coincides with the direction of the scattered electron. With these definitions we have

$$y = y' = y'', \quad z' = z'' \tag{7.19}$$

Then the relation between the coordinates in these three frames is (we use a compact notation where x represents a 4-vector),

$$x' = \mathbf{Boost_z}(\beta) \cdot x'', \quad x = \mathbf{Rot_y}(-\alpha) \cdot x' \tag{7.20}$$

where

$$\mathbf{Boost_z}(\beta) = \begin{bmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix} \tag{7.21}$$

and

$$\mathbf{Rot_y}(-\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & 0 & -\sin\alpha \\ 0 & 0 & 1 & 0 \\ 0 & \sin\alpha & 0 & \cos\alpha \end{bmatrix} \tag{7.22}$$

where α is the angle indicated in the figure and $\gamma = p_2^0/m_e$, $\beta = 1/\sqrt{1 - 1/\gamma^2}$.

We can now, finally, write the desired relation between the coordinates of the frame where the scattered electron is at rest, and those in the lab frame. We get

$$x = \mathbf{Rot_y}(-\alpha) \cdot \mathbf{Boost_z}(\beta) \cdot x'' \tag{7.23}$$

As in the frame S'' where the scattered electron is at rest we can write the 4-vectors w_i''

$$w_1'' = \frac{m_e}{2}(1, \cos\varphi_2 \sin\theta_2, \sin\varphi_2 \sin\theta_2, \cos\theta_2)$$

$$w_2'' = \frac{m_e}{2}(1, -\cos\varphi_2 \sin\theta_2, -\sin\varphi_2 \sin\theta_2, -\cos\theta_2) \tag{7.24}$$

the problem is then solved. In the site [37] there is an example of a `Fortran` program that implements this algorithm to produce the data for Fig. 7.6.

7.10.4 Using FeynMaster

Again this is QED, so the input file for `FeynMaster` looks like

```
(* This is the Control.m file of FeynMaster *)

model: SM

inparticles: e,A
outparticles: e,A
loops: 0
options: ugauge

FRinterLogic: F
RenoLogic: F
Draw: T
Comp: T
FinLogic: F
DivLogic: F
SumLogic: F
MoCoLogic: F
LoSpinors: T
```

Now you navigate to the directory `.../SM/Processes/1-eAeA` and do the following calculations in the `Notebook.nb`.

```
In[ ]:= M = Total[res];

In[ ]:= Aux = (M // DiffXS) /. {me → m, e^4 → (4 Pi α)^2} // Simplify
Out[ ]= (α^2 ((5 m^2 S^2 - 3 m^4 S + 3 m^6 + 3 S^3) cos^2(Theta) - (m^2 S^2 + 3 m^4 S + 3 m^6 - 7 S^3) cos(Theta) -
          m^2 S^2 - (m^2 - S)^3 cos^3(Theta) + 3 m^4 S + m^6 + 5 S^3)) / (4 S^2 ((S - m^2) cos(Theta) + m^2 + S)^2)

In[ ]:= dsigdOmeg = Function[{s, theta}, Aux /. {S → s, Theta → theta}];

In[ ]:= dsigdOmeg[s, theta]
Out[ ]= (α^2 ((5 m^2 s^2 - 3 m^4 s + 3 m^6 + 3 s^3) cos^2(theta) - (m^2 s^2 + 3 m^4 s + 3 m^6 - 7 s^3) cos(theta) -
          m^2 s^2 - (m^2 - s)^3 cos^3(theta) + 3 m^4 s + m^6 + 5 s^3)) / (4 s^2 ((s - m^2) cos(theta) + m^2 + s)^2)
.
```

Now we want to compare this with known results. As we are in the `CM` frame this will not be the Klein-Nishina formula but the result of Eq. (5.197). We have

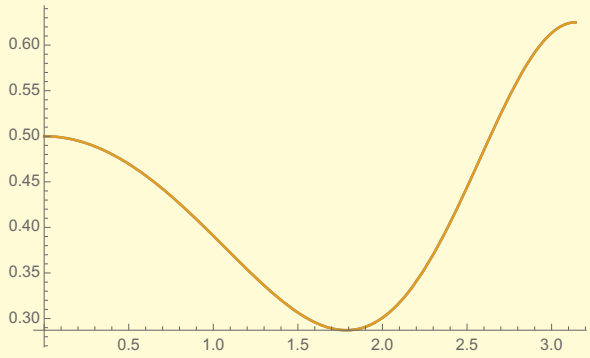
```
dsigdt = Function[{s, t}, u = -s - t + 2 m^2;
    2 Pi α^2/(s - m^2)^2
     (4 (m^2/(s - m^2) + m^2/(u - m^2))^2 + 4 (m^2/(s - m^2) + m^2/(u - m^2)) -
        ((s - m^2)/(u - m^2) + (u - m^2)/(s - m^2)))];

dsigdt2 = Function[{s, theta}, dsigdt[s, -(s - m^2)^2/2/s (1 - Cos[theta])]];

dsigdOmg3 = Function[{s, theta}, 1/(2 Pi) (s - m^2)^2/2/s dsigdt2[s, theta]];

dsigdOmg3[s, theta] // Simplify
```

$$\left(\alpha^2\left(m^6+3\,m^4\,s-m^2\,s^2-\left(m^2-s\right)^3\cos^3(\text{theta})+\left(3\,m^6-3\,m^4\,s+5\,m^2\,s^2+3\,s^3\right)\cos^2(\text{theta})-\left(3\,m^6+3\,m^4\,s+m^2\,s^2-7\,s^3\right)\cos(\text{theta})+5\,s^3\right)\right)\Big/\left(4\,s^2\left(\left(s-m^2\right)\cos(\text{theta})+m^2+s\right)^2\right)$$

We now plot the two expressions to verify that they coincide. We have used arbitrary constants.

```
In[26]:= Plot[{(dsigdOmg[2, theta] /. {m → 1, α → 1}), (dsigdOmg3[2, theta] /. {m → 1, α → 1})},
          {theta, 0, Pi}]
```

Out[26]=



Now we can integrate both expressions to get the total cross section and compare with the classical limit.

Total Cross Section

My Result

```
In[29]:= xs5 = 2 Pi Integrate[dsigdOmgAux Sin[theta], {theta, 0, Pi},
             Assumptions → {m > 0, s > 0, s > m^2}]
```

Out[29]=

$$\frac{\pi\,\alpha^2\left(2\,s^2\left(-3\,m^4-6\,m^2\,s+s^2\right)\log\left(\frac{m^2}{s}\right)+\left(m^2-s\right)\left(m^6-m^4\,s+15\,m^2\,s^2+s^3\right)\right)}{s^2\left(m^2-s\right)^3}$$

```
In[30]:= xs6 = xs5 /. s → m^2 (1 + x) // Simplify
```

$$\text{Out[30]}= -\frac{\pi\,\alpha^2\left(2\,(x+1)^2\left(x^2-4\,x-8\right)\log\left(\frac{1}{x+1}\right)-x\left(x^3+18\,x^2+32\,x+16\right)\right)}{m^2\,x^3\,(x+1)^2}$$

```
In[31]:= Limit[xs6, x → 0]
```

$$\text{Out[31]}= \frac{8\,\pi\,\alpha^2}{3\,m^2}$$

.

From ITC, Problem 5.14

```
In[32]:= xs3 = Integrate[dsigdt[s, t], {t, -(s - m^2)^2/s, 0},
           Assumptions → {m > 0, s > 0, s > m^2}]
```

$$\text{Out[32]}= \frac{\pi\,\alpha^2\left(2\,s^2\left(-3\,m^4-6\,m^2\,s+s^2\right)\log\left(\frac{m^2}{s}\right)+\left(m^2-s\right)\left(m^6-m^4\,s+15\,m^2\,s^2+s^3\right)\right)}{s^2\left(m^2-s\right)^3}$$

.

```
In[33]:= xs4 = xs3 /. s → m^2 (1 + x) // Simplify
```

$$\text{Out[33]}= -\frac{\pi\,\alpha^2\left(2\,(x+1)^2\left(x^2-4\,x-8\right)\log\left(\frac{1}{x+1}\right)-x\left(x^3+18\,x^2+32\,x+16\right)\right)}{m^2\,x^3\,(x+1)^2}$$

```
In[34]:= Limit[xs4, x → 0]
```

$$\text{Out[34]}= \frac{8\,\pi\,\alpha^2}{3\,m^2}$$

.

We get the final result, in agreement with other methods. Again we should emphasize the simplicity of the calculation.

Chapter 8

Simple Examples in the Standard Model

8.1 Introduction

We saw in Chapter 5 some examples of the application of the Feynman rules for QED. In fact there is nothing particular about QED, besides its simplicity. We will show in this chapter that the same type of reasoning can be applied to any theory if the Lagrangian is known and therefore also known are its Feynman rules.

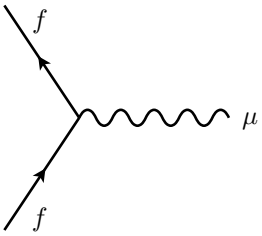
As an example we are going to consider the *Standard Model of the Electroweak Interactions* in the following called simply *Standard Model* (SM). This is a theory that was developed in the 60's of last century proposed by Glashow [64], Weinberg [65] and Salam [66] that unifies weak and electromagnetic interactions. It can be extended for the strong interactions although we will not cover that here. Nowadays the SM has been tested thoroughly at LEP and LHC colliders and agrees with experiment to better than 0.1%. It includes QED as a consistent subset of the theory. By QED we mean, in this context, something more general than what we studied so far, which is the study of the interactions of photons with *all* charged particles in Nature. For this the relevant part of the Lagrangian is then,

$$\mathcal{L}_{int} = -eQ_f \ \overline{\psi}_f \gamma^\mu \psi_f A_\mu \tag{8.1}$$

where Q_f is the charge of the fermion[1] in units of the charge of the e^+. It is easy to see that this leads to the Feynman rule shown in Fig. 8.1, which is an obvious generalization of the rule for QED.

As we saw, the photon is the carrier of the electromagnetic interaction. When we unify weak and electromagnetic interactions we are lead to introduce other spin 1 fields as carriers of that interaction. We are not going to study the SM (see for instance the excellent book of Thomson [54], or my own text [67]) but just give the main features. The carriers are the Z^0 boson with zero electric charge and the $W^\pm$

[1] With this definition we also include the quarks that ha fractional charges.



$$-ie\, Q_f \gamma^\mu$$

Figure 8.1: Interaction of the photon with charged fermions. Note that $e = |e|$.

bosons with electric charge $\pm$. One of the main differences between electromagnetic and weak interactions are that these are of short range, which implies that contrary to photon, the $W^\pm$ and Z^0 do have mass. The more recent values given in PDG [68] are

$$M_W \simeq 80.4 \text{ GeV} \quad ; \quad M_Z \simeq 91.2 \text{ GeV} \tag{8.2}$$

From the technical point of view, the SM of the electroweak interactions is a *non-abelian gauge theory* corresponding to local gauge group $SU(2) \times U(1)$. It is not our goal here to study this theory in detail (see Ref. [54] or Ref. [67]). However some aspects are quite simple and give us the possibility of applying techniques similar to what we have learned in QED. In a general way the complete Lagrangian for the SM can be written as,

$$\begin{aligned}\mathcal{L}_{MP} =& \mathcal{L}^{\text{kinetic}}_{\text{gauge fields}}(A^\mu, W^{\pm\mu}, Z_0^\mu) + \sum_f \mathcal{L}^{\text{Kinetic}}_{\text{fermions}}(\psi_f) \\ &+ \mathcal{L}^{CC}(W^{\pm\mu}, \psi_f) + \mathcal{L}^{CN}(Z_0^\mu, A^\mu, \psi_f) \\ &+ \mathcal{L}_{\text{remaining}}\end{aligned} \tag{8.3}$$

where $\mathcal{L}^{CC}$ and $\mathcal{L}^{CN}$ are, respectively, the Lagrangians for charged and neutral currents that we will describe below, and where $\mathcal{L}_{\text{remaining}}$ includes all other complications that we not going to address here[2]. For a better understanding of what follows we should describe the fermionic field content of the SM. A first group includes the charged leptons, that is, the electron (e^-), the muon (μ^-) and the tau (τ^-) and their antiparticles (e^+, μ^+, τ^+). After that we have the neutral leptons, the neutrinos. They come in three flavours, ν_e, ν_μ e ν_τ. Experimentally it was found that the neutrinos that participate in the electroweak interactions, through the the charged current interaction, are left-handed, that is,

$$\begin{aligned}P_R\ \nu_i =& \frac{1+\gamma_5}{2}\nu_i = 0 \\ P_L\ \nu_i =& \frac{1-\gamma_5}{2}\nu_i = \nu_i\end{aligned} \tag{8.4}$$

[2]Higgs fields, ghosts, Yukawa interactions,...

In the construction of the SM the left-handed and right-handed components of the fermions[3] are treated in a different way. The left-handed components are in the fundamental representation (*doublet*) of $SU(2)$ while the right-handed components are singlets of $SU(2)$. More precisely the doublets are

$$\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L \quad ; \quad \begin{pmatrix}\nu_\mu\\ \mu^-\end{pmatrix}_L \quad ; \quad \begin{pmatrix}\nu_\tau\\ \tau^-\end{pmatrix}_L \tag{8.5}$$

and e^-_R, μ^-_R and τ^-_R are singlets of $SU(2)$. In the initial proposal of the SM, neutrino were only left-handed and had no mass. Nowadays we know that neutrinos have a small mass (< 1 eV) and oscillate among the different flavours. As the mass is so very small, for the purposes of our study here (at high energy colliders) it is a very good approximation to assume the neutrinos to be massless. The structure in families or generations of doublets is similar for the quarks, the constituents of the hadrons, the particles that also interact through the strong force. This repetition in three families is accounted for in the SM, but its origin is not yet explained. We have,

$$\begin{pmatrix}u\\ d\end{pmatrix}_L \quad ; \quad \begin{pmatrix}c\\ s\end{pmatrix}_L \quad ; \quad \begin{pmatrix}t\\ b\end{pmatrix}_L \quad ; \quad u_R, d_R, c_R, s_R, t_R, b_R \tag{8.6}$$

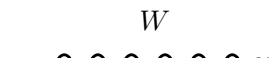
with the difference that all of them have a right-handed component, a singlet under $SU(2)$.

After this quick digression in the content of the theory, we are now going to describe, without proof, and using whenever possible the analogy with QED, some of the terms in Eq.(8.3).

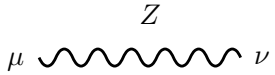
• *Lagrangian for the gauge fields*

The Lagrangian for the gauge fields is quite complicated and we are not going to write it here. For our goals it is enough to give, without proof, that they have the following propagators[4],

$$-i\frac{g_{\mu\nu}}{q^2+i\epsilon} \tag{8.7}$$



$$-i\frac{g_{\mu\nu}-\frac{q_\mu q_\nu}{M_W^2}}{q^2-M_W^2++iM_W\Gamma_W} \tag{8.8}$$



$$-i\frac{g_{\mu\nu}-\frac{q_\mu q_\nu}{M_Z^2}}{q^2-M_Z^2+iM_Z\Gamma_Z} \tag{8.9}$$

[3]Any fermion ψ can be written as the sum of its components ψ_L and ψ_R such that $\psi_L{=}P_L\psi$; $\psi_R{=}P_R\psi$.

[4]Technically we have to choose a gauge to write the propagators. The expressions here are correct in the so-called unitary gauge.

These expressions for the propagators are sufficient to calculate processes at tree-level where that gauge fields are not external particles. For the processes with gauge fields in the external lines, it is necessary to take in account that the Z^0 and $W^\pm$ have now three polarizations, has it happens with any massive spin 1 particle. In external lines they are described, in a similar way as the photon, by polarization vectors $\varepsilon^\mu(q,\lambda)$ but now we have for the polarization sums,

$$\sum_\lambda \varepsilon_\alpha(q,\lambda)\varepsilon^*_\beta(q,\lambda) = -g_{\alpha\beta} + \frac{q_\alpha q_\beta}{M_V^2} \qquad ; \qquad V = W, Z \tag{8.10}$$

The gauge bosons $W^\pm$ and Z^0 are not stable, they can decay. Therefore in some processes to avoid a divergence from the pole of the propagator it is necessary to include the decay width. This is done adding an imaginary part proportional to the width Γ_Z or Γ_W to the denominator of the propagator.

μ $\overset{V}{\sim\!\sim\!\sim}$ ν $\qquad -i\dfrac{g_{\mu\nu} - \frac{q_\mu q_\nu}{M_V^2}}{q^2 - M_V^2 + iM_V\Gamma_V} \qquad ; \qquad V = W, Z$ (8.11)

The experimentally measured values for these widths are [68],

$$\Gamma_Z \simeq 2.5 \text{ GeV} \qquad ; \qquad \Gamma_W \simeq 2.0 \text{ GeV} \tag{8.12}$$

These results, presented without proof, will be enough for our goals here.

• *Kinetic Lagrangian for fermions*

The kinetic Lagrangian for the fermions it is simply the repetition, for each fermion species, of Dirac Lagrangian for the electron. We can therefore write,

$$\mathcal{L}^{\text{cinético}}_{\text{fermiões}}(\psi_f) = \sum_f \left(i\overline{\psi}_f\gamma^\mu\partial_\mu\psi_f - m_f\overline{\psi}_f\psi_f\right) \tag{8.13}$$

to which corresponds the following propagator, for each fermion,

$\overset{p}{\longrightarrow}$ $\qquad i\dfrac{\not{p} + m_f}{p^2 - m_f^2 + i\epsilon}$ (8.14)

• *Lagrangian for the charged current*

The charged current is the interaction of the $W^\pm$ with the charged and neutral fermions belonging to one doublet of $SU(2)$. As we said before the most important point is that only left-handed fermions are involved. If we call,

$$\psi_L = \begin{pmatrix}\psi_u \\ \psi_d\end{pmatrix}_L \tag{8.15}$$

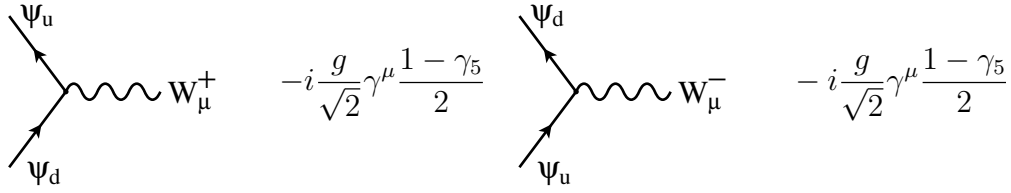


Figure 8.2: Feynman rules for the interaction of the $W^\pm$ with the doublets of the SM.

where $\psi_u = \nu_e, \nu_\mu, \ldots, u, c, \ldots$ and $\psi_d = e^-, \mu^-, \ldots, d, s, \ldots$ then we can write for each doublet [5],

$$\mathcal{L}^{CC} = -\frac{g}{\sqrt{2}}\ \overline{\psi}_u \gamma^\mu \frac{1-\gamma_5}{2}\psi_d\ W^+_\mu - \frac{g}{\sqrt{2}}\ \overline{\psi}_d \gamma^\mu \frac{1-\gamma_5}{2}\psi_u\ W^-_\mu \tag{8.16}$$

to which correspond the Feynman rules given in Fig. 8.2, where the coupling constant g is related to the electric charge through,

$$e = g \sin\theta_W \qquad ; \qquad \sin^2\theta_W \simeq 0.23 \tag{8.17}$$

and θ_W is the weak mixing angle.

- *Lagrangian for the neutral current*

Let consider finally the interactions of the fermions with the Z^0 boson and the photon. The form of the Lagrangian is

$$\begin{aligned}\mathcal{L}^{CN} &= -\sum_f eQ_f \overline{\psi}_f \gamma^\mu \psi_f A_\mu \\ &\quad -\frac{g}{\cos\theta_W}\ \sum_f \overline{\psi}_f \gamma^\mu \left(g_V^f - g_A^f \gamma_5\right)\psi_f\ Z^0_\mu\end{aligned} \tag{8.18}$$

where

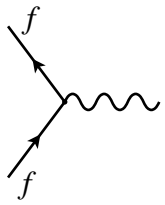
$$g_V^f = \frac{1}{2}\ T_3^f - Q_f\ \sin^2\theta_W \quad ; \quad g_A^f = \frac{1}{2}\ T_3^f \tag{8.19}$$

The values of T_3^f and Q_f for the known fermions are given in Table 8.1 The Feynman rule for the interaction of the Z^0 with fermions is then given in Fig. 8.3, while the photon interacts with all charged fermions through the rule given before in Fig. 8.1.

[5] For simplicity we are here making the approximation, $V_{\rm CKM} = \mathbb{1}$, where CKM is the Cabibbo [69], Kobayashi and Maskawa [70] matrix for the quark mixing.

Fermion	e^-, μ^-, τ^-	ν_e, ν_μ,ν_τ	u	d	c	s	b	t
T_3^f	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
Q_f	-1	0	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$

Table 8.1: Values of T_3^f and Q_f for the SM fermions.



$$-i\frac{g}{\cos\theta_W}\,\gamma^\mu(g_V^f - g_A^f\gamma_5)$$

Figure 8.3: Interaction of the Z^0 with fermions.

8.2 Decay width of Z^0 into fermions

After this introduction and knowing the propagators and relevant vertices for the SM, we are ready to do a first example. We will use first the trace technique an then the helicity amplitudes, first for massless fermions and then for fermions with mass.

8.2.1 $Z^0 \to f\overline{f}$ using traces

Lets us the consider the process,

$$Z^0 \to f\ \overline{f} \tag{8.20}$$

where f is any fermion in Table 8.1 with the exclusion of the quark t as this particle has a mass [68] $m_t \simeq 172.76$ GeV and therefore $m_t > M_Z$ which means that the Z^0 boson can not decay in $t\bar{t}$. The Feynman diagram is given in Fig.8.4, to which corresponds the amplitude

$$i\,\mathcal{M} = -\frac{ig}{\cos\theta_W}\,\epsilon^\mu(k,\lambda)\,\overline{u}(q_1)\gamma_\mu\left(g_V^f - g_A^f\gamma_5\right)v(q_2) \tag{8.21}$$

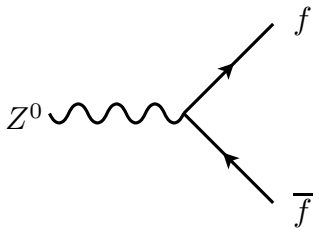


Figure 8.4: Decay of the Z.

The decay width is then

$$\Gamma = \int \frac{1}{2M_Z}\ \overline{|\mathcal{M}|^2}(2\pi)^4\delta^4(k-q_1-q_2)\prod_{i=1}^{2}\ \frac{d^3q_i}{(2\pi)^3 2E_i} \tag{8.22}$$

In he rest frame of the Z^0 we get,

$$\frac{d\Gamma}{d\Omega} = \frac{1}{64\pi^2}\ \frac{1}{M_Z}\ \overline{|\mathcal{M}|^2}\ \sqrt{1-\frac{4m_f^2}{M_Z^2}} \tag{8.23}$$

so we only need to evaluate the average value of the amplitude squared.

$$\begin{aligned}\overline{|\mathcal{M}|^2} =& \frac{1}{3}\sum_{\text{spins}}|\mathcal{M}|^2 \\ =& \frac{1}{3}\left(\frac{g}{\cos\theta_W}\right)^2\sum_{\lambda}\epsilon^{\mu}(k,\lambda)\epsilon^{*\nu}(k,\lambda) \\ & \text{Tr}\left[(\not{q}_1+m_f)\gamma_\mu\left(g_V^f-g_A^f\gamma_5\right)(\not{q}_2-m_f)\gamma_\nu\left(g_V^f-g_A^f\gamma_5\right)\right]\end{aligned} \tag{8.24}$$

Using now

$$\sum_{\lambda}\epsilon^{\mu}(k,\lambda)\epsilon^{*\nu}(k,\lambda) = -g^{\mu\nu}+\frac{k^\mu k^\nu}{M_Z^2} \tag{8.25}$$

and evaluating the traces of the γ matrices, (see Prob. 8.2)

$$\begin{aligned}&\text{Tr}\left[(\not{q}_1+m_f)\gamma_\mu\left(g_V^f-g_A^f\gamma_5\right)(\not{q}_2-m_f)\gamma_\nu\left(g_V^f-g_A^f\gamma_5\right)\right] \\ =\ & 4\left[\left(g_V^{f\,2}+g_A^{f\,2}\right)(q_{1\mu}q_{2\nu}+q_{1\nu}q_{2\mu}-g_{\mu\nu}\ q_1\cdot q_2)-g_{\mu\nu}\ m_f^2\left(g_V^{f\,2}-g_A^{f\,2}\right)\right. \\ & \left.-2i\epsilon^{\alpha\beta}{}_{\mu\nu}q_{1\alpha}q_{2\beta}\ g_V^f g_A^f\right]\end{aligned} \tag{8.26}$$

we finally get[6],

$$\overline{|\mathcal{M}|^2} = \frac{4}{3}\ \left(\frac{g}{\cos\theta_W}\right)^2\ M_Z^2\left[g_V^{f\,2}+g_A^{f\,2}+2\left(\frac{m_f}{M_Z}\right)^2\left(g_V^{f\,2}-2g_A^{f\,2}\right)\right] \tag{8.27}$$

[6]The last term in Eq.(8.26) does not contribute because it has an antisymmetric tensor in ν and μ that is contracted with a symmetric tensor in the same indices, as results from Eqs.(8.25) and (8.24).

which gives for the decay width (the integration in Ω gives 4π)

$$\Gamma = \frac{M_Z}{12\pi}\left(\frac{g}{\cos\theta_W}\right)^2\sqrt{1-\frac{4m_f^2}{M_Z^2}}\left[g_V^{f\,2}+g_A^{f\,2}+2\left(\frac{m_f}{M_Z}\right)^2\left(g_V^{f\,2}-2g_A^{f\,2}\right)\right] \tag{8.28}$$

This result is normally given in terms of the well measured Fermi constant defined through,

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} = \left(\frac{g}{\cos\theta_W}\right)^2\frac{1}{8M_Z^2} \tag{8.29}$$

where we have used the SM relation between the Z^0 and $W^\pm$ masses,

$$M_W = M_Z\cos\theta_W \tag{8.30}$$

From this we get,

$$\Gamma = \frac{2G_F M_Z^3}{3\sqrt{2}\pi}\sqrt{1-\frac{4m_f^2}{M_Z^2}}\left[g_V^{f\,2}+g_A^{f\,2}+2\left(\frac{m_f}{M_Z}\right)^2\left(g_V^{f\,2}-2g_A^{f\,2}\right)\right] \tag{8.31}$$

For the majority of the fermions, with the possible exception of the b quark, the ratio $\left(\frac{m_f}{M_Z}\right)^2$ is negligible. Even for the b quarks we have,

$$\left(\frac{m_f}{M_Z}\right)^2 \simeq 3\times 10^{-3} \tag{8.32}$$

It is therefore a good approximation to write,

$$\Gamma(Z\to f\overline{f}) = \frac{2G_F M_Z^3}{3\sqrt{2}\pi}\left(g_V^{f\,2}+g_A^{f\,2}\right) \tag{8.33}$$

which gives, for instance, for the electrons [7],

$$\Gamma(Z\to e^+e^-) \simeq 83.4\ \text{MeV} \tag{8.34}$$

which we can compare with the PDG [68] value,

$$\begin{aligned}\Gamma(Z\to e^+e^-) &= \Gamma_Z\times \text{Br}(Z\to e^+e^-)\\ &= (2.4952\pm 0.0023)\times 10^3\times(3.363\pm 0.004)\times 10^{-2}\ \text{MeV}\\ &= (83.914\pm 0.127)\ \text{MeV}\end{aligned} \tag{8.35}$$

[7] Note that this calculation is in lowest order in perturbation theory.

8.2.2 $Z^0 \to f\overline{f}$ with helicity amplitudes: massless fermions

We are now going to use this example to show how one can use the helicity amplitudes technique to handle massive gauge fields like the Z. We consider, for now, massless fermions, so in the end we should recover Eq. (8.33) for the width. Starting from Eq. (8.21) and making the substitutions

$$\epsilon^\mu(k,\lambda) \to a^\mu = \overline{u}_-(r_1)\gamma^\mu u_-(r_2) \tag{8.36}$$

with $k = r_1 + r_2$ and $r_i^2 = 0$ (see section 6.4) and

$$g_V^f - g_A^f\gamma_5 = g_+\gamma_+ + g_-\gamma_-, \quad \text{onde } g_+ = g_V^f - g_A^f \text{ e } g_- = g_V^f + g_A^f\ , \tag{8.37}$$

we get

$$\begin{aligned}\mathcal{M}(\sigma_1,\sigma_2) &= -\frac{g}{\cos\theta_W}\overline{u}_-(r_1)\gamma^\mu u_-(r_2)\overline{u}_{\sigma_1}(q_1)\gamma_\mu\left[g_+\gamma_+ + g_-\gamma_-\right]u_{\sigma_2}(q_2) \qquad (8.38)\\ &= -\frac{g}{\cos\theta_W}\overline{u}_-(r_1)\gamma^\mu u_-(r_2)\left[g_+\overline{u}_{\sigma_1}(q_1)\gamma_\mu\gamma_+ u_{\sigma_2}(q_2) + g_-\overline{u}_{\sigma_1}(q_1)\gamma_\mu\gamma_- u_{\sigma_2}(q_2)\right]\end{aligned}$$

Therefore the only non-vanishing amplitudes are,

$$\begin{aligned}\mathcal{M}(+,+) &= -\frac{2gg_+}{\cos\theta_W}\ s(q_1,r_2)s^*(q_2,r_1)\\ \mathcal{M}(-,-) &= -\frac{2gg_-}{\cos\theta_W}\ s^*(r_1,q_1)s(r_2,q_2)\end{aligned} \tag{8.39}$$

and therefore

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{1}{3}\left[|\mathcal{M}(+,+)|^2 + |\mathcal{M}(-,-)|^2\right]\\ &= \frac{1}{3}\frac{4g^2}{\cos\theta_W}\left[4g_+^2(q_1\cdot r_2)(q_2\cdot r_1) + 4g_-^2(q_1\cdot r_1)(q_2\cdot r_2)\right]\end{aligned} \tag{8.40}$$

The width will then be

$$\frac{d\Gamma}{d\Omega} = \frac{1}{64\pi^2 M_Z}\frac{3}{8\pi M_Z^2}\int d\Omega^* \frac{16g^2}{3\cos^2\theta_W}\left[g_+^2(q_1\cdot r_2)(q_2\cdot r_1) + g_-^2(q_1\cdot r_1)(q_2\cdot r_2)\right] \tag{8.41}$$

where Ω^* is the solid angle of r_1 in the frame where the Z is at rest (see section 6.4). To do this integration we can either use the kinematics of the CM frame and do the calculation explicitly, or use the properties of Lorentz invariance to show that,

$$\int d\Omega^*\ r_1^\alpha r_2^\beta = \frac{\pi}{3}\left(M_Z^2\, g^{\alpha\beta} + 2k^\alpha k^\beta\right) \tag{8.42}$$

Substituting now in Eq. (8.41) we get easily

$$\Gamma = \frac{M_Z}{12\pi}\left(\frac{g}{\cos\theta_W}\right)^2\left(g_V^{f\,2} + g_A^{f\,2}\right) \tag{8.43}$$

in agreement with Eq. (8.28) in the limit $m_f = 0$.

Although the example is very simple we can use `FeynCalc` to get the amplitudes. A simple program that does that is given in the section Software, Code 8.1.

8.2.3 $Z^0 \to f\overline{f}$ with helicity amplitudes: massive fermions

Although this is a very simple problem best handled by the trace technique, we are going to use it to illustrate the use of the helicity amplitudes technique for massive fermions explained in section 6.5.

The amplitude is then

$$\begin{aligned}\mathcal{M}(\sigma_1,\sigma_2) &= -\frac{g}{\cos\theta_W}\overline{u}_-(r_1)\gamma^\mu u_-(r_2)\overline{u}(q_1,\sigma_1 s)\gamma_\mu\left[g_+\gamma_+ + g_-\gamma_-\right]v(q_2,\sigma_2 s)\\ &= -\frac{g}{\cos\theta_W}\overline{u}_-(r_1)\gamma^\mu u_-(r_2)\left[g_+\overline{u}(q_1,\sigma_1 s)\gamma_\mu\gamma_+ v(q_2,\sigma_2 s)\right.\\ &\qquad \left. + g_-\overline{u}(q_1,\sigma_1 s)\gamma_\mu\gamma_- v(q_2,\sigma_2 s)\right] \qquad (8.44)\end{aligned}$$

where $u(q_1,\sigma_1 s)$ and $v(q_2,\sigma_2 s)$ are given in Eq. (6.47). A simple program for `FeynCalc` is given in the section Software, Code 8.2. With that program we get the following results,

$$\begin{aligned}\mathcal{M}(+,+) &= -\frac{2\,g}{\cos\theta_W}\left[\frac{g_+}{m^2}\; s(w_1,r_2)s^*(t_1,r_1)s(t_1,t_2)s^*(w_1,w_2)\right.\\ &\qquad \left. - g_-\; s^*(r_1,w_2)s(r_2,t_2)\right]\\ \mathcal{M}(+,-) &= -\frac{2\,g}{\cos\theta_W}\left[-\frac{g_+}{m}\; s(w_1,r_2)s^*(t_2,r_1)s^*(w_1,w_2)\right.\\ &\qquad \left. + \frac{g_-}{m}\; s^*(r_1,w_2)s(r_2,t_1)s^*(t_2,t_1)\right] \qquad (8.45)\\ \mathcal{M}(-,+) &= -\frac{2\,g}{\cos\theta_W}\left[-\frac{g_-}{m}\; s(r_2,t_2)s(w_2,w_1)s^*(r_1,w_1)\right.\\ &\qquad \left. + \frac{g_+}{m}\; s^*(t_1,r_1)s(w_2,r_2)s(t_1,t_2)\right]\\ \mathcal{M}(-,-) &= -\frac{2\,g}{\cos\theta_W}\left[\frac{g_-}{m^2}\; s(r_2,t_1)s^*(t_2,t_1)s(w_2,w_1)s^*(r_1,w_1)\right.\\ &\qquad \left. - g_+\; s^*(t_2,r_1)s(w_2,r_2)\right]\end{aligned}$$

where $q_1 = w_1 + w_2$, $q_2 = t_1 + t_2$ with $w_i^2 = 0, t_i^2 = 0$ e $2\,(w_1\cdot w_2) = 2\,(t_1\cdot t_2) = m^2$. The expression for the width will then be,

$$\begin{aligned}\frac{d\Gamma}{d\Omega} &= \frac{1}{64\pi^2 M_Z}\,\frac{3}{8\pi M_Z^2}\int d\Omega_{\vec{r}_1}\frac{1}{4\pi}\int d\Omega_{\vec{w}_1}\frac{1}{4\pi}\int d\Omega_{\vec{t}_1}\frac{1}{3}\left[|\mathcal{M}(+,+)|^2 + |\mathcal{M}(+,-)|^2\right.\\ &\qquad \left. + |\mathcal{M}(-,+)|^2 + |\mathcal{M}(-,-)|^2\right] \qquad (8.46)\end{aligned}$$

The angles of the vectors $\vec{w}_1$ ($\vec{t}_1$) are easily defined in the referential in which the fermions with momentum q_1 (q_2), respectively, are at rest. To do the problem

numerically it is necessary to make a Lorentz transformation to the referential where the Z is at rest and where the angles of the the vectors $\vec{r}_i$ are defined. In Fig. 8.5 it is shown a comparison between the analytical exact result of Eq. (8.31) with a numerical calculation using the Monte Carlo integration method Vegas [37]. The agreement is excellent.

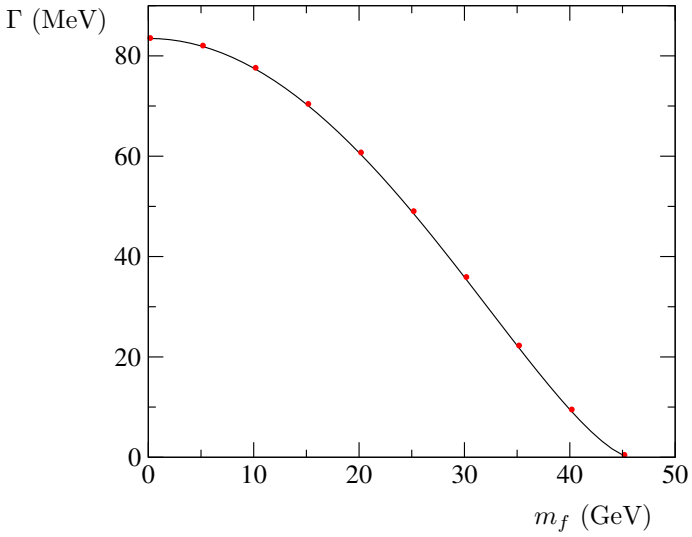


Figure 8.5: Width $Z \to f\overline{f}$ as a function of the fermion mass. The solid line represents the exact calculation of Eq. (8.31) and the dots the numerical calculation using the helicity amplitudes for massive fermions.

8.3 $e^-e^+ \to \mu^-\mu^+$ scattering in the SM

Let us now return to the process $e^-e^+ \to \mu^-\mu^+$ already studied in QED in Chapter 5. From what we have seen in the introduction to the SM, it is clear that the result obtained in QED is an approximation, as in lowest order there are the two diagrams of Fig. 8.6 and in section 5.3 we only considered the first diagram where a photon is exchanged. We are going to evaluate this process for the two diagrams to discover if and where QED gives a good approximation. In what follows we are going to neglect the electron mass but not[8] the mass of the μ^-. We also have to include the width Γ_Z in the Z^0 propagator so that we can consider the region $\sqrt{s} \simeq M_Z$. In these conditions the amplitude is,

$$i\,\mathcal{M} = (ie)^2 \frac{-ig^{\mu\nu}}{s}\ \overline{v}(p_2)\gamma_\mu u(p_1)\ \overline{u}(q_1)\gamma_\nu v(q_2) \tag{8.47}$$

[8] In the calculation we will consider the process $e^-e^+ \to f\overline{f}$ valid for any fermion f, except for the electron.

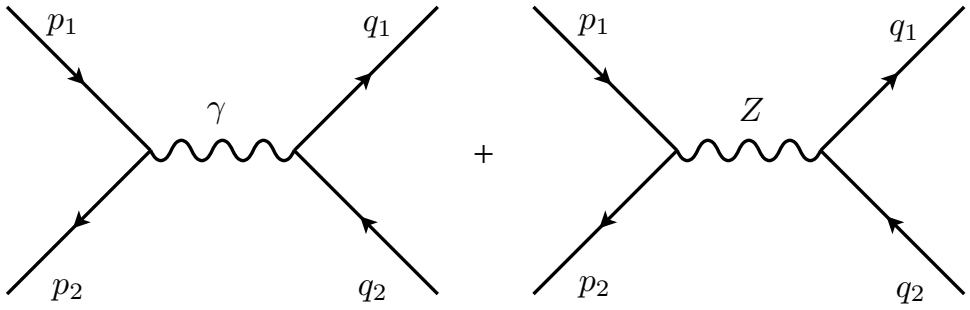


Figure 8.6: Diagrams for $e^-e^+ \to \mu^-\mu^+$ in the SM.

$$+\left(\frac{-ig}{\cos\theta_W}\right)^2 \overline{v}(p_2)\gamma^\mu \left(g_V^e - g_A^e\gamma_5\right) u(p_1) \frac{-i\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M_Z^2}\right)}{s - M_Z^2 + iM_Z\Gamma_Z} \overline{u}(q_1)\gamma^\nu \left(g_V^f - g_A^f\gamma_5\right) v(q_2)$$

Before we proceed we note a simplification due to the fact that we are considering $m_e = 0$. In these conditions the term in $k_\mu k_\nu / M_Z^2$ in the propagator of the Z^0 does not contribute. In fact this term is proportional to

$$\begin{aligned}
\overline{v}(p_2)\gamma^\mu \left(g_V^e - g_A^e\gamma_5\right) u(p_1) k_\mu &= \overline{v}(p_2)(\not{p}_1 + \not{p}_2) \left(g_V^e - g_A^e\gamma_5\right) u(p_1) \\
&= \overline{v}(p_2) \left(g_V^e + g_A^e\gamma_5\right) \not{p}_1 u(p_1) \\
&\quad + \overline{v}(p_2)\not{p}_2 \left(g_V^e - g_A^e\gamma_5\right) u(p_1) \\
&= 0
\end{aligned} \tag{8.48}$$

where we have used the Dirac equation in the limit $m_e = 0$, that is, $\not{p}_1 u(p_1) = 0$ and $\overline{v}(p_2)\not{p}_2 = 0$. We can therefore, without loss of generality, omit that term from the beginning. We then get for the amplitude,

$$\begin{aligned}
\mathcal{M} = &\frac{e^2}{s}\Big[\overline{v}(p_2)\gamma^\mu u(p_1)\ \overline{u}(q_1)\gamma_\mu v(q_2) \\
&+ F(s)\overline{v}(p_2)\gamma^\alpha(g_V^e - g_A^e\gamma_5)u(p_1)\ \overline{u}(q_1)\gamma_\alpha(g_V^f - g_A^f\gamma_5)v(q_2)\Big]
\end{aligned} \tag{8.49}$$

where

$$F(s) = \frac{1}{\sin^2\theta_W \cos^2\theta_W}\ \frac{s}{s - M_Z^2 + iM_Z\Gamma_Z} \tag{8.50}$$

Then we have

$$\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{1}{4}\sum_{\text{spins}} |M|^2 \\
&= \frac{e^4}{4s^2}\left\{\mathrm{Tr}\left[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu\right]\mathrm{Tr}\left[(\not{q}_1 + m_f)\gamma_\mu(\not{q}_2 - m_f)\gamma_\nu\right]\right. \\
&\quad \left.+|F(s)|^2\mathrm{Tr}\left[\not{p}_2\gamma^\alpha(g_V^e - g_A^e\gamma_5)\not{p}_1\gamma^\beta(g_V^e - g_A^e\gamma_5)\right]\right.
\end{aligned}$$

$$
\begin{aligned}
&\mathrm{Tr}\big[(\not{q}_1 + m_f)\gamma_\alpha(g_V^f - g_A^f\gamma_5)(\not{q}_2 - m_f)\gamma_\beta(g_V^f - g_A^f\gamma_5)\big] \\
&+2Re\Big[F(s)\mathrm{Tr}\big[\not{p}_2\gamma^\alpha(g_V^e - g_A^e\gamma_5)\not{p}_1\gamma^\mu\big] \\
&\qquad \mathrm{Tr}\big[(\not{q}_1 + m_f)\gamma_\alpha(g_V^f - g_A^f\gamma_5)(\not{q}_2 - m_f)\gamma_\mu\big]\Big]\Big\} \\
\equiv\ & \frac{e^4}{4s^2}\Big\{A + |F(s)|^2 B + 2Re\big[F(s)C\big]\Big\} \qquad (8.51)
\end{aligned}
$$

where the last line defines the quantities A, B e C. Using the theorems on traces of the γ matrices we get, (see Prob. 8.7),

$$
\begin{aligned}
A =& 32(p_1\cdot q_1\ p_2\cdot q_2 + p_1\cdot q_2\ p_2\cdot q_1 + m_f^2\ p_1\cdot p_2) \\
=& 4s^2\ \big[1 + \cos^2\theta + (1-\beta^2)\sin^2\theta\big] \qquad (8.52)
\end{aligned}
$$

$$
\begin{aligned}
B =& 32\Big\{g_V^{f\,2}(g_V^{e\,2} + g_A^{e\,2})(p_1\cdot q_1\ p_2\cdot q_2 + p_1\cdot q_2\ p_2\cdot q_1 + m_f^2\ p_1\cdot p_2) \\
&+ g_A^{f\,2}(g_V^{e\,2} + g_A^{e\,2})(p_1\cdot q_1\ p_2\cdot q_2 + p_1\cdot q_2\ p_2\cdot q_1 - m_f^2\ p_1\cdot p_2) \\
&- 4g_A^e g_V^e g_A^f g_V^f(p_1\cdot q_1\ p_2\cdot q_2 - p_1\cdot q_2\ p_2\cdot q_1)\Big\} \\
=& 4s^2\Big\{g_V^{f\,2}(g_V^{e\,2} + g_A^{e\,2})\big[1 + \cos^2\theta + (1-\beta^2)\sin^2\theta\big] \\
&+ g_A^{f\,2}(g_V^{e\,2} + g_A^{e\,2})\beta^2(1+\cos^2\theta) + 8g_A^e g_V^e g_A^f g_V^f\beta\cos\theta\Big\} \qquad (8.53)
\end{aligned}
$$

and

$$
\begin{aligned}
C =& 32\Big[g_V^e g_V^f(p_1\cdot q_1\ p_2\cdot q_2 + p_1\cdot q_2\ p_2\cdot q_1 + m_f^2\ p_1\cdot p_2) \\
&+ g_A^e g_A^f(-p_1\cdot q_1\ p_2\cdot q_2 + p_1\cdot q_2\ p_2\cdot q_1)\Big] \\
=& 4s^2\Big\{g_V^e g_V^f\big[1 + \cos^2\theta + (1-\beta^2)\sin^2\theta\big] \\
&+ 2g_A^e g_A^f\beta\cos\theta\Big\} \qquad (8.54)
\end{aligned}
$$

where we have used the same kinematics as in section 5.3[9]. Using now the expressions in Eqs. (8.51) and (8.54) into the differential cross section given in Eq. (5.56) we

[9]This means that $\sqrt{s}$ is total CM energy, θ is the scattering angle of the μ^- with respect to the direction of the e^- and $\beta = \sqrt{1 - 4m_\mu^2/s}$ is the velocity of fermion f in the CM frame.

get,

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{\alpha^2}{4s}\,\beta\Big\{Q_f^2\big[1+\cos^2\theta+(1-\beta^2)\sin^2\theta\big] \\ &\quad -2Q_f\chi_1(s)\Big[g_V^e g_V^f\big[1+\cos^2\theta+(1-\beta^2)\sin^2\theta\big]+2g_A^e g_A^f\beta\cos\theta\Big] \\ &\quad +\chi_2(s)\Big[{g_V^f}^2({g_V^e}^2+{g_A^e}^2)\big[1+\cos^2\theta+(1-\beta^2)\sin^2\theta\big] \\ &\quad\quad +{g_A^f}^2({g_V^e}^2+{g_A^e}^2)\beta^2(1+\cos^2\theta)+8g_A^e g_V^e g_A^f g_V^f\beta\cos\theta\Big]\Big\} \qquad (8.55)\end{aligned}$$

where

$$\begin{aligned}\chi_1(s) &= Re(F(s)) = \frac{1}{\sin^2\theta_W\cos^2\theta_W}\,\frac{s(s-M_Z^2)}{(s-M_Z^2)^2+\Gamma_Z^2M_Z^2} \\ \chi_2(s) &= |F(s)|^2 = \frac{1}{\sin^4\theta_W\cos^4\theta_W}\,\frac{s^2}{(s-M_Z^2)^2+\Gamma_Z^2M_Z^2} \qquad (8.56)\end{aligned}$$

To get the total cross section we have to integrate in the solid angle Ω. We obtain,

$$\begin{aligned}\sigma &= \frac{2\pi\alpha^2}{3s}\,\beta\Big\{3-\beta^2+2\chi_1(s)g_V^e g_V^f(3-\beta^2) \\ &\quad +\chi_2(s)\big[{g_V^f}^2({g_V^e}^2+{g_A^e}^2)(3-\beta^2)+2{g_A^f}^2({g_V^e}^2+{g_A^e}^2)\beta^2\big]\Big\} \qquad (8.57)\end{aligned}$$

while in pure QED we got Eq. (5.56),

$$\sigma_{\rm QED} = \frac{2\pi\alpha^2}{3s}\,\beta(3-\beta^2) \qquad (8.58)$$

As we said before, the expressions in Eqs. (8.55) and (8.57) are written in such a way that they can be applied to any fermion f (except for the e). For the μ^- we have

$$\begin{aligned}Q_f &= -1 \\ g_A^f &= -\frac{1}{4} \\ g_V^f &= -\frac{1}{4}+\sin^2\theta_W \qquad (8.59)\end{aligned}$$

In Fig.8.7 we compare the result of Eq. (8.57) with the QED approximation in Eq. (8.58). We see that for $\sqrt{s}\ll M_Z$ the approximation is good, but for $\sqrt{s}\simeq$ 40 GeV the differences become very large, especially for $\sqrt{s}\simeq M_Z$.

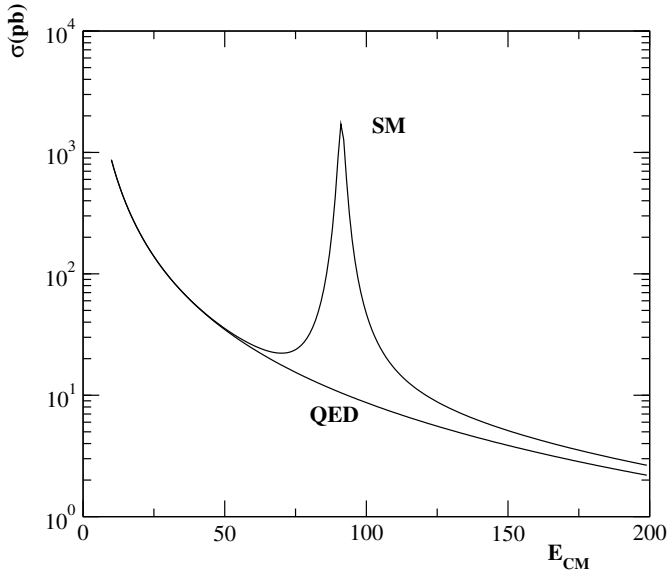


Figure 8.7: Comparison of the cross sections for the SM and for QED.

8.4 The decay of the μ^-

We are going to finish these simple examples of calculations in the SM with the decay of the muon. Historically this process was very important for the acceptance of an effective and incomplete theory of the weak interactions, the so-called *Fermi theory*. Today the process is understood in the SM of the electroweak interactions through the diagram in Fig.8.8, that show that the interaction with W boson is the

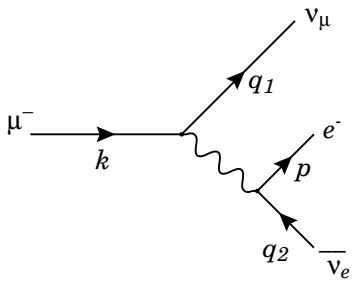


Figure 8.8: Diagram for muon decay in the SM.

key to explain the decay. In fact this boson was proposed even before there was a consistent theory and much before its discovery.

The amplitude corresponding to the diagram of Fig.8.8 is,

$$i\,\mathcal{M} = \left(\frac{-ig}{\sqrt{2}}\right)^2 \overline{u}(q_1)\gamma^\mu \left(\frac{1-\gamma_5}{2}\right) u(k) \frac{-i\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{M_W^2}\right)}{q^2 - M_W^2 + iM_W\Gamma_W} \overline{u}(p)\gamma^\nu \left(\frac{1-\gamma_5}{2}\right) v(q_2)$$

$$=i\frac{g^2}{8}\overline{u}(q_1)\gamma^\mu\left(1-\gamma_5\right)u(k)\overline{u}(p)\gamma^\nu\left(1-\gamma_5\right)v(q_2)\frac{g_{\mu\nu}-\frac{q_\mu q_\nu}{M_W^2}}{q^2-M_W^2+iM_W\Gamma_W} \tag{8.60}$$

where $q=k-q_1$. Once we know the amplitude, the decay width is obtained through the usual rules, that is,

$$\Gamma=\int\frac{1}{2m_\mu}\overline{|\mathcal{M}|^2}(2\pi)^4\delta^4(k-p-q_1-q_2)\frac{d^3p}{(2\pi)^3 2p^0}\prod_{i=1}^{2}\frac{d^3q_1}{(2\pi)^3 2q_i^0} \tag{8.61}$$

where we have used the kinematics shown in the figure. Although apparently simple, it turns out that there a difficulty because of the phase space of three particles, something we have not seen yet in all the other examples. However one can make an approximation that will simplify all the calculations. This approximation results from the observation that $q^2<q^2_{\rm max}=m_\mu^2\ll M_W^2$. Therefore we can neglect the terms $q_\mu q_\nu/M_W^2$ in the numerator and q^2 in the denominator of Eq. (8.60). This approximation corresponds in fact to the effective Fermi theory and is equivalent to collapse the propagator in one point,

$$\frac{g_{\mu\nu}-\frac{q_\mu q_\nu}{M_W^2}}{q^2-M_W^2}\to-\frac{g_{\mu\nu}}{M_W^2} \tag{8.62}$$

In this approximation the amplitude becomes

$$\mathcal{M}=-\frac{g^2}{8M_W^2}\,\overline{u}(q_1)\gamma^\mu\left(1-\gamma_5\right)u(k)\overline{u}(p)\gamma_\mu\left(1-\gamma_5\right)v(q_2) \tag{8.63}$$

and the average over spin of the square of the amplitude is then

$$\begin{aligned}\overline{|\mathcal{M}|^2} &= \frac{1}{2}\sum_{\rm spins}|\mathcal{M}|^2\\ &= \frac{g^4}{128M_W^4}\,{\rm Tr}\left[\not{q}_1\gamma^\mu(1-\gamma_5)(\not{k}+m_\mu)\gamma^\nu(1-\gamma_5)\right]\\ &\qquad\qquad {\rm Tr}\left[(\not{p}+m_e)\gamma_\mu(1-\gamma_5)\not{q}_2\gamma_\nu(1-\gamma_5)\right]\\ &= 2\left(\frac{g^4}{M_W^4}\right)\,k\cdot q_2\;p\cdot q_1\end{aligned} \tag{8.64}$$

We get for the width,

$$\Gamma=\left(\frac{g}{M_W}\right)^4\frac{1}{(2\pi)^5 m_\mu}\int\frac{d^3p}{2p^0}\int\frac{d^3q_1}{2q_1^0}\frac{d^3q_2}{2q_2^0}\delta^4(k-p-q_1-q_2)\;k\cdot q_2\;p\cdot q_1 \tag{8.65}$$

Now the phase space integration presents some difficulties, as we have nine integrations and only one δ^4. However we can make use of the fact that the second integral in Eq. (8.65) is Lorentz invariant and can therefore be evaluated in the more

convenient reference frame. This frame is the CM frame of the system of the two neutrinos, that are not detected in the decay. If we define,

$$I_{\alpha\beta} = \int \frac{d^3q_1}{2q_1^0}\ \frac{d^3q_2}{2q_2^0}\ \delta^4(\Delta - q_1 - q_2) q_{1\alpha} q_{2\beta} \tag{8.66}$$

the previous argument about Lorentz invariance shows that we should have

$$I_{\alpha\beta} = A g_{\alpha\beta} + B \Delta_\alpha \Delta_\beta \tag{8.67}$$

where A and B are functions only of the invariant Δ^2. To determine these invariants we note that

$$g^{\alpha\beta} I_{\alpha\beta} = 4A + B\Delta^2 \tag{8.68}$$

and

$$\Delta^\alpha \Delta^\beta I_{\alpha\beta} = A\Delta^2 + B\Delta^4 \tag{8.69}$$

Now the integral in Eqs. (8.68) and (8.69) are easily calculated if we note that ($q_1^2 = q_2^2 = 0$)

$$g^{\alpha\beta} q_{1\alpha} q_{2\beta} = q_1 \cdot q_2 = \frac{1}{2}\Delta^2 \tag{8.70}$$

and

$$\Delta^\alpha \Delta^\beta q_{1\alpha} q_{2\beta} = (q_1 \cdot \Delta)(q_2 \cdot \Delta) = (q_1 \cdot q_2)^2 = \frac{1}{4}\Delta^4 \tag{8.71}$$

Therefore

$$g^{\alpha\beta} I_{\alpha\beta} = \frac{1}{2}\Delta^2 I \tag{8.72}$$

and

$$\Delta^\alpha \Delta^\beta I_{\alpha\beta} = \frac{1}{4}\Delta^4 I \tag{8.73}$$

and to finish we have only the evaluate the scalar integral, I, given by

$$\begin{aligned} I &= \int \frac{d^3q_1}{2q_1^0}\ \frac{d^3q_2}{2q_2^0}\ \delta^4(\Delta - q_1 - q_2) \\ &= \int \frac{d^3q_1}{2q_1^0} \frac{1}{2q_2^0} \delta(\Delta^0 - q_1^0 - q_2^0) \end{aligned} \tag{8.74}$$

In the CM frame of the pair $\overline{\nu}_e\ \nu_\mu$ we have

$$\begin{aligned} q_1^0 &= q_2^0 \\ d^3q_1 &= (q_1^0)^2\ dq_1^0\ d\Omega \end{aligned} \tag{8.75}$$

and therefore

$$I = \frac{1}{4}\int dq_1^0 d\Omega \delta(\Delta^0 - q_1^0 - q_2^0)$$

$$
\begin{aligned}
&= \frac{1}{8}\int d\Omega \\
&= \frac{\pi}{2}
\end{aligned}
\tag{8.76}
$$

Using now Eqs. (8.68), (8.69) and (8.74) we get

$$
\begin{cases}
\dfrac{1}{2}\Delta^2\dfrac{\pi}{2} = 4A + B\Delta^2 \\
\dfrac{1}{4}\Delta^4\dfrac{\pi}{2} = A\Delta^2 + B\Delta^4
\end{cases}
\tag{8.77}
$$

from which we get

$$
\begin{cases}
A = \dfrac{\pi}{24}\Delta^2 \\
B = \dfrac{\pi}{12}
\end{cases}
\tag{8.78}
$$

and finally

$$
I_{\alpha\beta} = \frac{\pi}{24}\left(g_{\alpha\beta}\Delta^2 + 2\Delta_\alpha\Delta_\beta\right) \tag{8.79}
$$

Using this result and noticing that in our case $\Delta = k - p$, we get[10]

$$
\begin{aligned}
\Gamma &= \left(\frac{g}{M_W}\right)^4 \frac{1}{(2\pi)^5 m_\mu}\int \frac{d^3p}{2p^0}\frac{\pi}{24}\left[3k\cdot p(m_\mu^2 + m_e^2) - 4(k\cdot p)^2 - 2m_\mu^2 m_e^2\right] \\
&= \left(\frac{g}{M_W}\right)^4 \frac{1}{384\pi^3}\int_{m_e}^{\frac{m_\mu^2+m_e^2}{2m_\mu}} dE_e\sqrt{E_e^2 - m_e^2}\left[3E_e(m_\mu^2 + m_e^2) - 4E_e^2 m_\mu - 2m_\mu m_e^2\right] \\
&= \left(\frac{g}{M_W}\right)^4 \frac{m_\mu^5}{384\pi^3}\left[\frac{1}{16}(1-x^2)(1-7x^2-7x^4+x^6) - \frac{3}{2}x^4\ln x\right]
\end{aligned}
\tag{8.80}
$$

where $x = \frac{m_e}{m_\mu}$. It is usual to write this expression in terms of the Fermi constant,

$$
\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} \tag{8.81}
$$

We get then

$$
\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3}\left[(1-x^2)(1-7x^2-7x^4+x^6) - 24x^4\ln x\right] \tag{8.82}
$$

Using now $m_\mu = 105.7$ MeV, $m_e = 0.51$ MeV, $G_F = 1.166\times 10^{-11}$ MeV^{-2} we get

$$
\Gamma = 2.95\times 10^{-16}\ \text{MeV} = 4.48\times 10^5\ \text{s}^{-1} \tag{8.83}
$$

and therefore

$$
\tau = \frac{1}{\Gamma} = 2.2\times 10^{-6}\ \text{s} \tag{8.84}
$$

This ends our study of the decay of the muon and this chapter on simple examples of processes in the SM.

[10] See Prob. 8.8 for the determination of the integration limits em E_e.

Software

Code 8.1 Spinor Products for $Z \to f\overline{f}$

```
(*******************    SpinorProductsZfF.m      ****************)
(* Program to Calculate the Helicity Amplitudes Z-> f fbar decay,
   Eq. (6.39) of ITC 2020

 Last Version: 04/05/2020
 Author: Jorge C. Romao
 email: jorge.romao@tecnico.ulisboa.pt
*)
Remove["Global`*"]

(* Definitions *)
dp[s_]:= (1 + s DiracMatrix[5])/2

U[p_,s_]:= dp[s] . Spinor[p,0]

UBar[p_,s_]:=  SpinorUBar[p,0] . dp[-s]

UBarGammaUGammaU=Function[{p1,p2,si,i},If[i==1, 2 U[p2,si],
If[i==2, 2 U[p1,-si]]]];

UBarGammaUGammaUBar=Function[{p1,p2,si,i},If[i==1, UBar[p1,si],
If[i==2, UBar[p2,-si]]]];

(* Amplitudes *)

gvga= gm dp[-1] + gp dp[1]

M1[s1_,s2_]:= -gZ Sum[(UBar[q1,s1] . UBarGammaUGammaU[r1,r2,-1,i1]
UBarGammaUGammaUBar[r1,r2,-1,i1] . gvga . U[q2,s2]),{i1,1,2}]

res1[s1_,s2_]:=DiracSimplify[M1[s1,s2],DiracSubstitute67->True]

vlist={q1,q2,r1,r2}

simp1=Table[Spinor[vlist[[i]],0] . Spinor[vlist[[j]],0] ->
MySP[vlist[[i]],vlist[[j]]] + MySPc[vlist[[j]],vlist[[i]]],
{i,1,4},{j,1,4}] /. {MySP[p_, p_] -> 0, MySPc[q_, q_] -> 0}

simp2=Table[Spinor[vlist[[i]],0] . DiracMatrix[5] . Spinor[vlist[[j]],0]
-> -MySP[vlist[[i]],vlist[[j]]] + MySPc[vlist[[j]],vlist[[i]]],{i,1,4},
{j,1,4}] /. {MySP[p_, p_] -> 0, MySPc[q_, q_] -> 0}


simp=Flatten[{simp1,simp2}];


M[s1_,s2_]:=Expand[(res1[s1,s2] /. simp)
/. {MySP[a_,b_]-> sp[a,b],MySPc[a_,b_]-> spc[a,b]}]

(*******************    End SpinorProductsZfF.m    ***************)
```

Code 8.2 Spinor Products for $Z \to f\overline{f}$ with mass

```
(******************  SpinorProductsZfFMass.m   ******************)
(* Program to Calculate the Helicity Amplitudes Z-> f fbar decay
   with massive fermions, Eq. (6.45) of ITC

   Compatible with FeynCalc 9.3
```

```
 Last Version: 04/05/2020
 Author: Jorge C. Romao
 email: jorge.romao@tecnico.ulisboa.pt
*)
Remove["Global`*"]

(* Definitions *)
dp[s_]:= (1 + s DiracMatrix[5])/2

U[p_,s_]:= dp[s] . Spinor[p,0]

UBar[p_,s_]:=  SpinorUBar[p,0] . dp[-s]

UBarGammaUGammaU=Function[{p1,p2,si,i},If[i==1, 2 U[p2,si],If[i==2,
2 U[p1,-si]]]];

UBarGammaUGammaUBar=Function[{p1,p2,si,i},If[i==1,UBar[p1,si],If[i==2,
UBar[p2,-si]]]];

gvga= gm dp[-1] + gp dp[1]
```

```
(* Fermions with masss *)

Um[p1_,p2_,m_,s_]:=(If[s==1,MySP[p1,p2],MySPc[p2,p1]]/m U[p1,s] +
U[p2,-s])
Vm[p1_,p2_,m_,s_]:=(If[s==1,MySP[p1,p2],MySPc[p2,p1]]/m U[p1,s] -
U[p2,-s])

UmBar[p1_,p2_,m_,s_]:=(If[s==1,MySPc[p1,p2],MySP[p2,p1]]/m UBar[p1,s]
                       + UBar[p2,-s])
VmBar[p1_,p2_,m_,s_]:=(If[s==1,MySPc[p1,p2],MySP[p2,p1]]/m UBar[p1,s]
                       - UBar[p2,-s])
(* Amplitudes *)

M1[s1_,s2_]:=-gZ Sum[(UmBar[w1,w2,m,s1] . UBarGammaUGammaU[r1,r2,-1,i1]
UBarGammaUGammaUBar[r1,r2,-1,i1] . gvga . Vm[t1,t2,m,s2]),{i1,1,2}];

M1c[s1_,s2_]:=-gZ Sum[(VmBar[t1,t2,m,s2] . UBarGammaUGammaU[r1,r2,-1,i1]
UBarGammaUGammaUBar[r1,r2,-1,i1] . gvga . Um[w1,w2,m,s1]),{i1,1,2}];

res[s1_,s2_]:=DiracSimplify[M1[s1,s2],DiracSubstitute67-> True];
resc[s1_,s2_]:=DiracSimplify[M1c[s1,s2],DiracSubstitute67->True];

vlist={w1,w2,t1,t2,r1,r2}

simp1=Table[Spinor[vlist[[i]],0] . Spinor[vlist[[j]],0] ->
MySP[vlist[[i]],vlist[[j]]] + MySPc[vlist[[j]],vlist[[i]]],{i,1,6},
{j,1,6}] /. {MySP[p_, p_] -> 0, MySPc[q_, q_] -> 0}

simp2=Table[Spinor[vlist[[i]],0].DiracMatrix[5].Spinor[vlist[[j]],0]->
-MySP[vlist[[i]],vlist[[j]]] + MySPc[vlist[[j]],vlist[[i]]],{i,1,6},
{j,1,6}] /. {MySP[p_, p_] -> 0, MySPc[q_, q_] -> 0}

simp=Flatten[{simp1,simp2}];

Maux[s1_,s2_]:=Expand[res[s1,s2] /. simp]
Mauxc[s1_,s2_]:=Expand[resc[s1,s2] /. simp]

M[s1_,s2_]:= Expand[Maux[s1,s2] /. {MySP[p_, q_] -> sp[p, q],
  MySPc[p_, q_] -> spc[p, q]}]
Mc[s1_,s2_]:=Expand[Mauxc[s1,s2] /.{MySP[p_, q_] -> sp[p, q],
  MySPc[p_, q_] -> spc[p, q]} ]
```

```
(* Change to True to write OutputFortran *)

If[False,
ANS=Expand[M[1,1]*Mc[1,1]+M[-1,1]*Mc[-1,1]+M[1,-1]*Mc[1,-1]+
    M[-1,-1]*Mc[-1,-1]];
simp4={sp[p_,q_] spc[p_,q_]->2 dot[p,q]};

ANS=ANS /. simp4;

simp5={dot[t1,t2]->m^2/2,dot[t2,t1]->m^2/2,dot[w1,w2]->m^2/2,
  dot[w2,w1]->m^2/2};

ANS=ANS /. simp5 ;
ANS=ANS /. simp4 ;
ANS=ANS /. simp4 ;
ANS=ANS /. simp4 ;
ANS=ANS /. simp5 ;

SetOptions[$Output,PageWidth->65];
mpp=M[1,1];
mpm=M[1,-1];
mmp=M[-1,1];
mmm=M[-1,-1];

stmp=OpenWrite["ZfF.f",FormatType -> FortranForm];
Write[stmp,"MPP=",mpp];
Write[stmp,"MPM=",mpm];
Write[stmp,"MMP=",mmp];
Write[stmp,"MMM=",mmm];
Close[stmp];
]
(************************ End Program  ************************)
```

Problems

8.1 Consider the two decays of the Z^0 boson,

$$Z^0 \to \nu\overline{\nu} \tag{8.85}$$

$$Z^0 \to e^- e^+ \ . \tag{8.86}$$

Show that

$$\frac{\Gamma(Z^0 \to \nu\overline{\nu})}{\Gamma(Z^0 \to e^- e^)} \simeq 2 \ . \tag{8.87}$$

8.2 Derive Eq. (8.26),

$$\begin{aligned} T_1 &= \mathrm{Tr}\left[(\not{q}_1 + m_f)\gamma_\mu \left(g_V^f - g_A^f\gamma_5\right)(\not{q}_2 - m_f)\gamma_\nu \left(g_V^f - g_A^f\gamma_5\right)\right] \\ &= 4\left[\left(g_V^{f\,2} + g_A^{f\,2}\right)(q_{1\mu}q_{2\nu} + q_{1\nu}q_{2\mu} - g_{\mu\nu}\ q_1\cdot q_2) - g_{\mu\nu}\ m_f^2\left(g_V^{f\,2} - g_A^{f\,2}\right)\right. \\ &\qquad \left. -2i\epsilon^{\alpha\beta}{}_{\mu\nu}q_{1\alpha}q_{2\beta}\ g_V^f g_A^f\right] \end{aligned} \tag{8.88}$$

8.3 Neglecting all fermion masses show that

$$BR(Z^0 \to e^-\ e^+) \equiv \frac{\Gamma(Z^0 \to e^-\ e^+)}{\Gamma_Z} \simeq 3.4\% \tag{8.89}$$

where $\Gamma_Z \equiv \Gamma(Z^0 \to \text{tudo})$.

8.4 Consider the process $e^+e^- \to \nu_e\overline{\nu}_e$. Neglect the fermion masses.

a) Draw the diagrams that contribute in lowest order to this process.

b) Write the amplitude that corresponds to the dominant diagram for $\sqrt{s} \simeq M_z$.

c) Show that for $\sqrt{s} \simeq M_Z$ we have

$$\frac{\sigma(e^+e^- \to \nu_e\overline{\nu}_e)}{\sigma(e^+e^- \to e^+e^-)} \simeq 2 \tag{8.90}$$

8.5 Consider the decay $W^- \to e^- \, \overline{\nu}_e$.

a) Determine the velocity of the electron in the rest frame of the W.

b) Write the expression for the amplitude for this process.

c) Neglecting the electron mass, determine the expression for the decay width. Compare with the experimental result $\Gamma(W^- \to e^- \, \overline{\nu}_e) = 229$ MeV.

8.6 Determine the *branching ratio* $BR(W^- \to e^- \nu)$ defined as

$$BR(W^- \to e^- \nu) \equiv \frac{\Gamma(W^- \to e^- \nu)}{\Gamma(W^- \to \text{All})} \tag{8.91}$$

where $\Gamma(W^- \to \text{All}) = \Gamma_W = 2.0$ GeV.

8.7 Verify Eqs. (8.52), (8.53) and (8.54).

8.8 Show that in the muon decay, the energies of the electron in the rest frame of the muon, are in the interval,

$$E_e \in \left[m_e, \frac{m_\mu^2 + m_e^2}{2m_\mu} \right] \tag{8.92}$$

8.9 This problem is meant to illustrate the gauge invariance of the electromagnetic current, which is true not only in QED but also in the SM. For the following processes;

$e^+ + e^- \to \nu_\mu + \overline{\nu}_\mu + \gamma$	$Z^0 \to e^- e^+ \gamma$	$\mu^+ + \mu^- \to \nu_e + \overline{\nu}_e + \gamma$
$\nu_\mu + e^- \to \nu_\mu + e^- + \gamma$	$e^- + e^+ \to \mu^+ + \mu^- + \gamma$	$e^- + e^+ \to \nu_e + \overline{\nu}_e + \gamma$
$W^- \to e^- + \overline{\nu}_e + \gamma$	$\nu_\mu + e^- \to \mu^- + \nu_e + \gamma$	$t \to b + W^+ + \gamma$

a) Draw the diagrams that contribute in lowest order. Do not forget that lepton number is conserved in the SM (neglecting neutrino mixing).

b) Write the expression for the amplitude of each process and verify its gauge invariance, that is, if

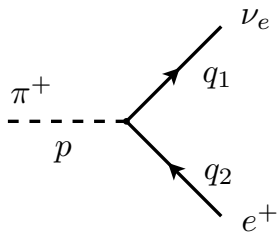
$$\mathcal{M} = \varepsilon^\mu(k) V_\mu$$

then

$$k^\mu V_\mu = 0$$

where k^μ is the 4-momentum of the photon.

8.10 Although the $\pi^{\pm}$ meson (pion) is not an elementary particle, as it is made of quark-antiquark pairs, for some processes it is a good approximation the assume that is a point like particle with an effective interaction. Then the vertex responsible for the process $\pi^+ \to e^+ + \nu_e$ is



$$\frac{G_F}{\sqrt{2}} V_{ud}^* f_\pi \gamma^\mu (1-\gamma_5)\, p_\mu$$

a) Write the amplitude for the process.

b) Write an expression for the ratio R defined by

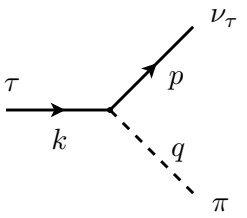
$$R = \frac{\Gamma(\pi^+ \to e^+ \nu_e)}{\Gamma(\pi^+ \to \mu^+ \nu_\mu)} \tag{8.93}$$

as a function of m_e, m_μ and m_π. Compare your value with the experimental result, $R_{\text{exp}} = 1.23 \times 10^{-4}$.

c) Knowing that the lifetime of the π^+ is, $\tau_\pi = 2.6 \times 10^{-8}$ s, and that $V_{ud} = 0.974$, determine the constant f_π.

d) The result from b) may seem strange, as the decay width of the electron channel is much smaller than that of the muon channel, although the available energy (phase space) is much larger. Show that $R = 0$ in the limit $m_e = 0$. Explain this result.

8.11 Consider the process $\tau^- \to \pi^- + \nu_\tau$. The effective vertex is given by



$$\frac{G_F}{\sqrt{2}} V_{ud}^* f_\pi\, q^\mu\, \gamma_\mu (1-\gamma_5)$$

a) Write the amplitude for the process.

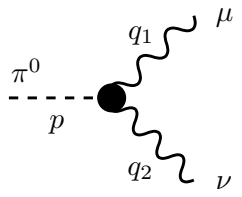
b) Determine the decay width, $\Gamma(\tau^- \to \pi^- + \nu_\tau)$. Consider that the ν_τ is massless, but do not neglect the masses of the other particles.

c) Knowing the the lifetime of the τ is, $\tau_\tau = 2.9 \times 10^{-13} s$, determine the *Branching Ratio* for that channel. Take $f_\pi = 131$ MeV and $V_{ud} = 0.974$. `Note:` The definition of the pion decay constant, f_π differs in the literature by factors of $\sqrt{2}$. Our definition is that of Griffiths [46] and corresponds to the vertex given in the figure.

8.12 The main decay of the π^0 is in two photons with a *Branching Ratio* given by $BR(\pi^0 \to \gamma\gamma) = 98.8\%$. This process is an higher order process (loop) as the π^0 being neutral has no direct coupling to the photon. We can parameterize this coupling through an effective Lagrangian,

$$\mathcal{L} = g_{\pi^0}\ \pi^0 \epsilon^{\alpha\mu\beta\nu} F_{\alpha\mu} F_{\beta\nu}$$

which gives and effective vertex,



$$-8i\, g_{\pi^0}\ \epsilon^{\alpha\mu\beta\nu} q_{1\alpha} q_{2\beta}$$

a) Write the expression for the amplitude.

b) Show that the width is given by,

$$\Gamma = \frac{g_{\pi^0}^2 m_\pi^3}{\pi}$$

c) Knowing that the lifetime of the π^0 is, $\tau_{\pi^0} = 8.4 \times 10^{-17}$ s, determine the constant g_{π^0}.

8.13 Consider the muon decay, $\mu^- \to e^- + \overline{\nu}_e + \nu_\mu$. Evaluate without approximations the decay width and compare with the result obtained in section 8.4.

8.14 When we neglect the lepton masses and consider that the CM energy, $\sqrt{s}$, is much smaller than the masses of the W and Z bosons, the total cross sections for the processes shown in the table below,

Process	λ_i
$\nu_\mu + e^- \to \mu^- + \nu_e$	1
$\overline{\nu}_e + e^- \to \mu^- + \overline{\nu}_\mu$	$\frac{1}{3}$
$\nu_\mu + e^- \to \nu_\mu + e^-$	$\frac{32}{3}\ \left[\left(g_V^{\nu\,2} + g_A^{\nu\,2}\right)\left(g_V^{e\,2} + g_A^{e\,2}\right) + 2 g_V^\nu g_A^\nu g_V^e g_A^e\right]$
$\overline{\nu}_\mu + e^- \to \overline{\nu}_\mu + e^-$	
$\mu^- + e^+ \to \nu_\mu + \overline{\nu}_e$	
$\nu_e + e^- \to \nu_e + e^-$	

can all be written as

$$\sigma_i = \frac{\lambda_i}{\pi} G_F^2 \, s$$

a) Show that this is true, verifying the values already in the table and filling the other entries.

b) Show that

$$\frac{\sigma(\nu_\mu + e^- \to \nu_\mu + e^-)}{\sigma(\overline{\nu}_\mu + e^- \to \overline{\nu}_\mu + e^-)} = \frac{3L_e^2 + R_e^2}{L_e^2 + 3R_e^2}$$

where,

$$L_e = g_V^e + g_A^e, \qquad R_e = g_V^e - g_A^e$$

c) Define $R(x) = \sigma(\nu_\mu e^- \to \nu_\mu e^-)/\sigma(\overline{\nu}_\mu + e^- \to \overline{\nu}_\mu + e^-)$ where $x = \sin^2\theta_W$. Verify that $R(0.25) = 1$.

d) Consider the process $\nu_\mu + e^- \to \nu_\mu + e^-$ in the SM. Reproduce the plot in Fig. 8.9.

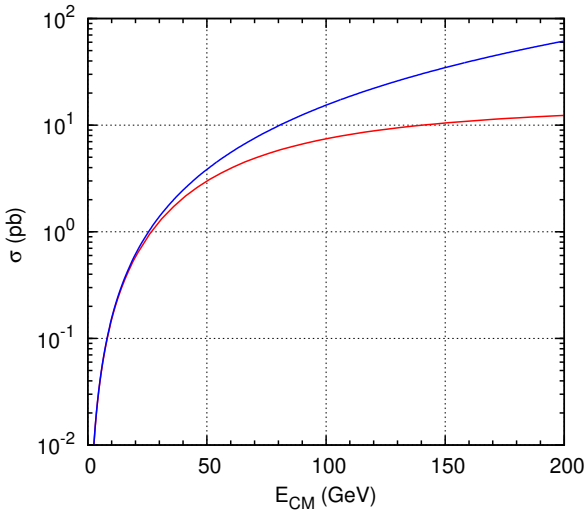


Figure 8.9: Comparison of the approximate result for low energy (blue curve) with the exact result (red curve) for the process $\nu_\mu + e^- \to \nu_\mu + e^-$.

8.15 In Problem 8.14 we studied the process, $\nu_\mu(p_1) + e^-(p_2) \to \nu_\mu(p_3) + e^-(p_4)$. The results, for low and mid range energies shown in Fig. 8.9 are not clear about the high energy behaviour of the cross section. This process, that has only the t-channel Z exchange diagram, allows us to study this behaviour without worrying about the interferences with other diagrams. Neglect the fermion masses in this problem.

a) To solve this problem one has to find an explicit expression for $\mathcal{M}(s,\theta)$. To be able to get that one needs to have explicit expressions for the spinors, or apply the *spinor product* technique and use the explicit form of Eq. (9.83) as it was explained in section 6.3.1. Show that there are only two non-vanishing helicity amplitudes, $\mathcal{M}(--;--)$ and $\mathcal{M}(-+;-+)$ with the notation $\mathcal{M}(h_{\nu_\mu}, h_e; h_{\nu_\mu}, h_e)$. Find the expression for these amplitudes in terms of the spinors products, and then as functions of s and θ.

b) Show that these amplitudes for very energies, $\sqrt{s} \gg M_Z$, tends to a constant, except for the exceptional angle $\theta = 0$, where θ is the scattering angle of the ν_μ. This is the required behaviour for an amplitude to respect unitarity (see the discussion in the book of Jiri Horejsi [71]).

c) In the discussion of unitarity [12, 71] one defines the partial wave expansion (due to Jacob and Wick [72]),

$$\sigma = \sum_J \frac{16\pi}{s}(2J+1)|a_J(s)|^2\,, \tag{8.94}$$

where the partial wave, a_J, is defined by

$$a_J = \frac{1}{32\pi}\int_{-1}^{1} \mathcal{M}(s,\theta)P_J(\cos\theta)d(\cos\theta)\,, \tag{8.95}$$

and the P_J are the Legendre polynomials. For the two helicity combinations, find a_0 and a_1 and show that they diverge logarithmically in the high energy limit, $\sqrt{s} \gg M_Z$. Note that

$$\sigma_{\rm T}(s) = \frac{1}{2}\left[\sigma(--;--) + \sigma(-+;-+)\right] \tag{8.96}$$

in an obvious notation (see appendix E of Ref. [71]).

d) Show that, despite the behaviour of the amplitudes, the cross section tends to a constant as $\sqrt{s} \gg M_Z$. Find this constant, that is,

$$\begin{aligned}\lim_{\sqrt{s}\gg M_Z} \sigma_{\rm T}(s) \equiv &\sigma_{\rm HE} = \frac{g^4}{128\pi M_Z^2 c_W^4}(1 - 4s_W^2 + 8s_W^4) \\ = &\frac{G_F^2}{4\pi}M_Z^2(1 - 4s_W^2 + 8s_W^4)\,.\end{aligned} \tag{8.97}$$

e) Verify numerically with `CalcHEP`. Make sure to use the same constants.

f) Evaluate $a_J(--;--)$ and $a_J(-+;-+)$ for $J = 0, 1, \ldots, 9$. Using these results evaluate numerically

$$\sigma^{\rm J_{max}} = \frac{1}{2}\left[\sigma^{\rm J_{max}}(--;--) + \sigma^{\rm J_{max}}(-+;-+)\right]\,, \tag{8.98}$$

where, for instance,

$$\sigma^{\mathrm{J_{max}}}(--;--) = \sum_{J=0}^{\mathrm{J_{max}}} \frac{16\pi}{s}(2J+1)|a_J(--;--)|^2\,, \tag{8.99}$$

Show that you reproduce the plot of Fig. 8.10, where we compare σ_{T} with $\sigma^{\mathrm{J_{max}}}$ for $J^{\max} = 0, 1, \ldots, 9$.

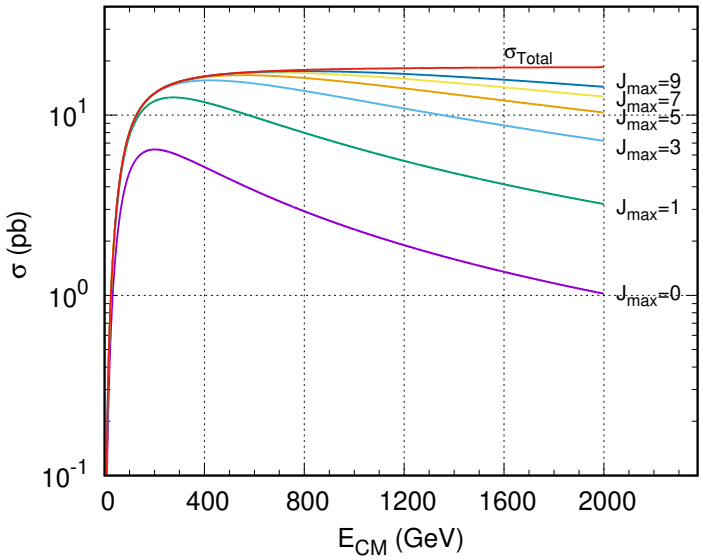


Figure 8.10: Comparison of the total cross section with the partial wave expansion of Eq. (8.94).

8.16 Consider the process $\nu_\mu + e^- \to \mu^- + \nu_e$ in the SM, but assume that the coupling of the W with the leptons is modified to

$$-i\frac{g}{\sqrt{2}}\,\gamma^\mu\frac{(1-\gamma_5)}{2} \to -i\frac{g}{\sqrt{2}}\,\gamma^\mu\frac{(1-a\,\gamma_5)}{2}$$

where, $a = 1-\epsilon$, and ϵ is a small number.

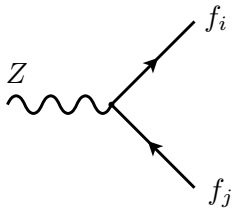
a) Write the amplitude for the process.

b) Consider that the CM energy is much smaller that the W mass. Write that the expression for the amplitude in this approximation.

c) Evaluate the differential cross section $d\sigma/d\Omega$ in the CM frame, in the limit where we neglect all the fermions masses. The angles in $d\Omega$ are those corresponding to the direction of the μ^- in the CM frame with respect to the direction of the incident ν_μ.

d) Determine the total cross section σ in the CM. What would be the necessary precision in the measurement of σ in order to have a 5% error in the determination of ϵ?

8.17 Consider a more general theory than the SM, where we have n fermions f_i^- and their antiparticles, f_i^+, where $i = 1, 2, \ldots n$. The general interaction with the Z boson is given by,



$$\frac{ig}{\cos\theta_W}\,\gamma_\mu\,(O_L P_L + O_R P_R)$$

a) Write the amplitude for the process $Z \to f_i^- f_j^+$.

b) Show that we get for the width,

$$\begin{aligned}\Gamma =& \frac{M_Z}{24\pi}\,\frac{g^2}{\cos^2\theta_W}\,\sqrt{\left[1-(x_i+x_j)^2\right]\left[1-(x_i-x_j)^2\right]} \\ & \times\left\{(O_L^2+O_R^2)\left[1-\frac{1}{2}(x_i^2+x_j^2)-\frac{1}{2}(x_i^2-x_j^2)^2\right]+6O_L O_R x_i x_j\right\}\end{aligned}$$

where $x_i = m_i/M_Z$.

c) Show that in the case of $f_i^- = f_j^- = e$ (electron) the expression reduces to the known result. Identify O_L and O_R for this case.

8.18 Consider the process $e^+e^- \to f\overline{f}$ (with $f \neq e^-$).

a) Evaluate the differential cross section $d\sigma/d\Omega$.

b) Find an expression for the front-backwards asymmetry A_{FB} defined by,

$$A_{FB} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B}$$

where

$$\sigma_F = 2\pi\int_0^1 d(\cos\theta)\frac{d\sigma}{d\Omega} \qquad \sigma_B = 2\pi\int_{-1}^0 d(\cos\theta)\frac{d\sigma}{d\Omega}$$

c) Show that for $\sqrt{s} = M_Z$ the A_{FB} can be written as

$$A_{FB} = \frac{3}{4}\frac{4g_V^e g_A^e g_V^\mu g_A^\mu}{[(g_V^e)^2+(g_A^e)^2]\,[(g_V^\mu)^2+(g_A^\mu)^2]} = \frac{3}{4}\frac{[(c_L^e)^2-(c_R^e)^2]\,[(c_L^\mu)^2-(c_R^\mu)^2]}{[(c_L^e)^2+(c_R^e)^2]\,[(c_L^\mu)^2+(c_R^\mu)^2]}\,,$$

where

$$c_L = g_V + g_A, \quad , c_R = g_V - g_A.$$

d) Make a plot of A_{FB} as a function of $\sqrt{s}$ in the interval [70,110] GeV, for the μ lepton and for quarks of type u and d.

e) Look in the literature for the corresponding experimental values and compare the results.

8.19 Consider the process $e^-e^+ \to ZH$

a) Evaluate the total cross section as a function of the mass of the H boson for a CM energy $\sqrt{s} = 500$ GeV. Make a plot of $\sigma(e^-e^+ \to ZH)$ as a function of M_H. (Note: this problem was specially relevant before the discovery of the Higgs boson at 125 GeV.)

b) Consider now the process $e^-e^+ \to H\ell_i\overline{\ell}_i$. Show that

$$\sum_i^3 \sigma(e^-e^+ \to H\ell_i\overline{\ell}_i) \simeq \sigma(e^-e^+ \to HZ) \times BR(Z \to \text{leptões}) \tag{8.100}$$

To understand this result make a plot of

$$\frac{d\sigma(e^-e^+ \to H\ell_i\overline{\ell}_i)}{dE_H} \tag{8.101}$$

as a function of E_H.

8.20 Consider the process $e^-e^+ \to Z^0Z^0$.

a) Write the amplitude for the process.

b) Determine the differential cross section.

c) Evaluate the total cross section.

d) Make a plot of the total cross section as a function of the CM energy $\sqrt{s}$, for values from the threshold to 250 GeV.

8.21 Consider the process $e^- + e^+ \to \nu_e + \overline{\nu}_e + H$.

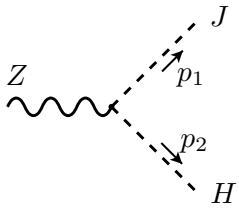
a) Show that the differential cross section for this process can be written in the form,

$$\frac{d\sigma}{dE_H\, d\cos\theta} = \frac{G_F^3 M_Z^8 p_H}{\sqrt{2}\pi^3 s}\ (X_Z + X_W + X_I)$$

where $\sqrt{s}$ is the center of mass energy and E_H, $p_H = \sqrt{E_H^2 - M_H^2}$ and θ are, respectively, the energy, the linear momentum and the polar angle of the Higgs boson in that frame. In the previous expression X_Z, X_W and X_I are, respectively, the contributions from the diagrams with the Z, with the W and their interference. Determine X_Z, X_W and X_I.

b) Make a plot of the cross section $\sigma(e^-e^+ \to \nu_e \overline{\nu}_e H)$ as a function of the Higgs boson mass, M_H, for $\sqrt{s} = 192$ GeV, and for M_H between 70 GeV and 140 GeV. Compare in the same plot the contribution from the Z, the W and their interference. Comment on the results. Neglect the masses of the leptons and the width of the W and Z whenever it is possible.

8.22 Consider the process $Z \to H + J$ in a BSM theory that, besides the SM particles, has a scalar boson H and a massless pseudo-scalar J, with the coupling,



$$\frac{g}{2\cos\theta_W}\,(p_1 - p_2)^\mu$$

a) Write the amplitude for this process.

b) Evaluate the decay width as a function of the masses M_Z and M_H.

c) Show that in the limit $M_H \ll M_Z$ the contribution of this process to the width of the Z is equivalent to 1/2 of one neutrino family.

Note: Before the discovery of the SM Higgs boson with a mass of 125 GeV, this process was used to put constraints in models with spontaneous flavour violation [73].

8.23 Consider the process $e^-e^+ \to W^-W^+$.

a) Draw the diagrams that contribute to this process in lowest order.

b) Write the expression for the amplitude.

c) Evaluate the cross section in the CM frame.

d) Show that the behaviour of the cross section is determined by cancellations that have origin in the fact that the SM is a gauge theory. For this, show that,

$$\lim_{s \gg M_W^2} \sigma_{\nu-\text{exchange}} = \frac{\pi\alpha^2 s}{96\sin^4\theta_W M_W^4}$$

while

$$\lim_{s \gg M_W^2} \sigma_{\text{total}} = \frac{\pi\alpha^2}{2\sin^4\theta_W}\,\frac{1}{s}\,\ln\left(\frac{s}{m_W^2}\right)$$

Make a plot of the two cross sections as a function of $\sqrt{s}$.

8.24 Consider the process, $\nu_e + d \to e^- + u$, in the SM.

a) Draw the diagrams that contribute in lowest order to the process.

b) Write the amplitude for this process.

c) Evaluate, in the CM frame, the differential cross section, $d\sigma/d\Omega$, in the limit where we can neglect all fermion masses. The solid angle Ω corresponds to that of the electron with respect to the direction of the ν_e.

d) Evaluate the total cross section, $\sigma(s)$, and show that,

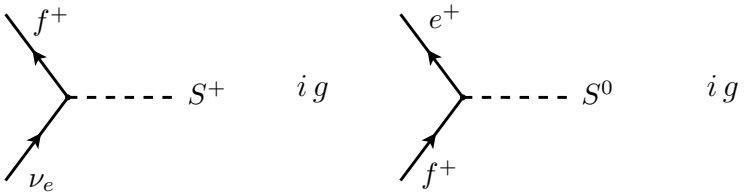
$$\lim_{\sqrt{s}\gg M_W} \sigma(s) = \sigma_0 \left(\frac{s}{m_W^2}\right)^{\alpha}$$

Determine the constants σ_0 and α.

8.25 Consider the process

$$\nu_e + e^+ \to S^+ + S^0$$

in a theory with the following vertices,



where S^+, S^0 are scalars (spin 0), and f^+ it is a fermion (spin 1/2) with mass m_f. ν_e and e^+ are, respectively, the electron neutrino and the positron.

a) Draw the diagrams that contribute to process in lowest order.

b) Write the amplitude for the process.

c) Evaluate in the CM frame the differential cross section, $d\sigma/d\Omega$, in the limit where one can neglect all initial and final state masses. Ω corresponds the solid angle of the S^+ with respect to the direction of the ν_e.

d) Determine the dominant term in the total cross section, $\sigma(s)$, when $\sqrt{s} \gg m_f$. How does it behave with respect to $\sqrt{s}$?
Hint: use the following results,

$$\begin{aligned}
\int_{-1}^{1} dx \, \frac{1}{(1+2\varepsilon-x)^2} &= \frac{1}{2\varepsilon} - \frac{1}{2} + \mathcal{O}(\varepsilon) \\
\int_{-1}^{1} dx \, \frac{x}{(1+2\varepsilon-x)^2} &= \frac{1}{2\varepsilon} + \left(\ln\varepsilon + \frac{1}{2}\right) + \mathcal{O}(\varepsilon)
\end{aligned}$$

$$\int_{-1}^{1} dx\, \frac{x^2}{(1+2\varepsilon-x)^2} \;=\; \frac{1}{2\varepsilon} + \left(2\ln\varepsilon + \frac{7}{2}\right) + \mathcal{O}(\varepsilon)$$

where $\varepsilon \ll 1$.

8.26 Consider the process $e^+ + e^- \to \phi + \gamma$ in a theory described by the following Lagrangian,

$$\mathcal{L} = \mathcal{L}_{\rm QED} + \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m_\phi^2\phi^2 - \beta\,\overline{\psi}\gamma_5\psi\,\phi$$

where ϕ is a neutral scalar field and ψ is the electron. Besides the Feynman rules for QED, we have those of the figure,

a) Draw the diagrams that contribute to the process in lowest order.

b) Write the amplitude for the process.

c) Show that the amplitude is gauge invariant, that is, if $\mathcal{M} \equiv \epsilon^\mu(k)\,\mathcal{M}_\mu$ where k is the 4-momentum of the photon, then we have, $k^\mu\mathcal{M}_\mu = 0$.

8.27 Consider now the process $\phi \to e^+ + e^-$ for the theory described in Prob. 8.26.

a) Write the amplitude for the process.

b) Evaluate the decay width $\Gamma(\phi \to e^+ + e^-)$ as a function of the parameters of the model.

c) Imagine the one measures $m_\phi = 1.8$ GeV and a lifetime $\tau_\phi = 8.5 \times 10^{-23}$ s. What is the value of β? Take $m_e = 0.511$ MeV

8.28 Consider the following process for Higgs production at a future collider,

$$e^-(p_1) + e^+(p_2) \to Z(q_1) + Z(q_2) + H(k)$$

a) Determine the cross section in the CM frame as a function of the masses of the particles and the CM energy, $\sqrt{s}$.

b) Make a plot of the cross section for three values of the Higgs boson mass, $M_H = \{90, 120, 150\}$ for $\sqrt{s} \in \Big[M_H + 2M_Z + 10, 2000\Big]$ GeV. **Note:** These values were before the Higgs boson was discovered, of course.

c) Look in the literature for data on this process and compare with your results and with the plot in the left panel of Fig. 8.28.

d) Show that the interferences are crucial for the decreasing of the cross section with the CM energy, therefore not violating the unitarity limit. For this reproduce the plot in the right panel of Fig. 8.28, where we did not consider the interference contribution.

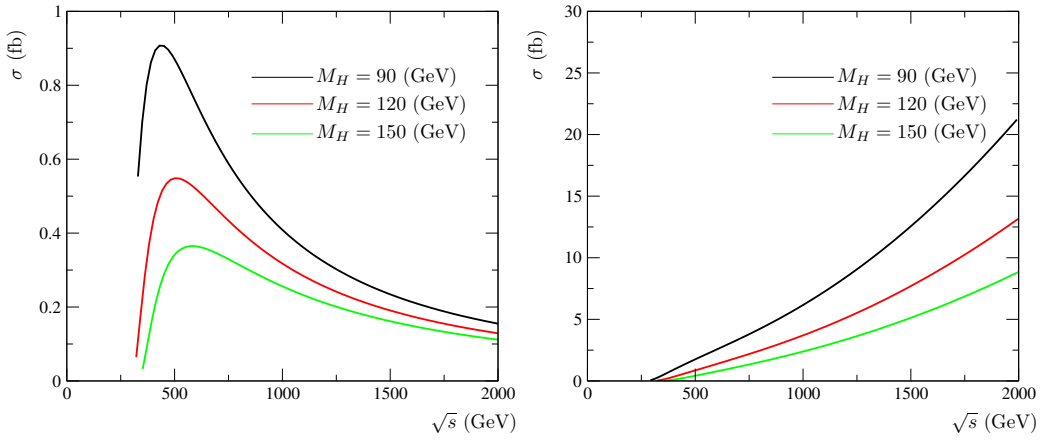


Figure 8.11: Left panel: Cross section for $\sigma(e^+e^- \to HZZ)$ with all the diagrams; Right panel without the interference diagrams

`Hints:`

1. Neglect the electron and positron masses.

2. Do not forget that you have two identical particles in the final state.

3. In d) we argued that the signs are crucial to get the correct result. Use `FeynCalc` to do the traces. It is useful to note that we can write (with appropriate Γ_i):
$$\mathcal{M}_i = \overline{v}(p_2)\Gamma_i u(p_1)$$

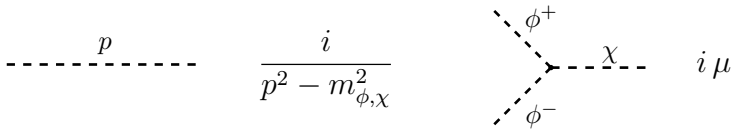
4. In the plots show the cross sections in fentobarn (`fb`).

5. To do the numerical integration it can be useful to look for similar examples in my site [37].

8.29 Consider the decay $\chi \to \phi^+ + \phi^-$ in the model described by the following Lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\chi\,\partial^\mu\chi + \partial_\mu\phi^+\,\partial^\mu\phi^- \; - \frac{1}{2}m_\chi^2\,\chi^2 - m_\phi^2\,\phi^+\phi^- + \mu\,\phi^+\phi^-\,\chi$$

where χ is a neutral scalar field, and $\phi^\pm$ é is a charged scalar field. This charge corresponds to some internal symmetry and it is not the electric charge, therefore there is no interaction with photons. The constant μ has dimensions of mass. The propagators and the only vertex are,



In the vertex all particles are entering the diagram. Notice that $\phi^\pm$ entering corresponds to $\phi^\mp$ leaving the diagram.

a) Write the amplitude for the process.

b) Evaluate the decay width $\Gamma(\chi \to \phi^+ + \phi^-)$ in terms of the parameters of the model.

c) What is the lifetime, in seconds, knowing that $m_\chi = 5$ GeV, $m_\chi = 1$ GeV and $\mu = 10$ GeV.

8.30 Consider the process $\phi^- + \chi \to \phi^- + \chi$ in the model described in Prob. 8.29.

a) Draw the diagrams that contribute to the process in lowest order.

b) Write the amplitude for the process.

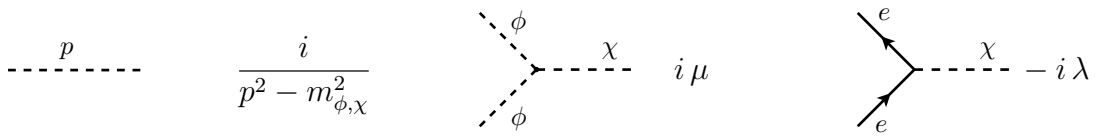
c) Consider that $\sqrt{s} \gg m_\phi, m_\chi$ and that therefore it is a good approximation to take $m_\phi = m_\chi = 0$. in these conditions evaluate the differential cross section, as a function of the CM energy $\sqrt{s}$) and scattering angle θ.

d) In the conditions of c) evaluated the total cross section in the CM frame for $\theta^{\min} < \theta < \pi - \theta^{\min}$. What would happen if $\theta^{\min} = 0$? Would this be a real problem?

8.31 Consider the process $e^+ + e^- \to \phi + \phi + \gamma$ in a theory given by the following Lagrangian,

$$\mathcal{L} = \mathcal{L}_{\rm QED} + \frac{1}{2}\partial_\mu\chi\,\partial^\mu\chi + \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi \; - \frac{1}{2}m_\chi^2\,\chi^2 - \frac{1}{2}m_\phi^2\,\phi^2 + \frac{\mu}{2}\,\phi^2\,\chi - \lambda\,\overline{\psi}\psi\,\chi$$

where χ and ϕ are neutral scalar fields and ψ is the electron. The constant μ has mass dimension in our natural units system and λ is dimensionless. Besides QED the new propagators and vertices are,



a) Draw the diagrams contributing in lowest order to the process.

b) Write the amplitude for the process.

c) Show that the amplitude is gauge invariant, that is, if $\mathcal{M} \equiv \epsilon^\mu(k)\,\mathcal{M}_\mu$ where k is the 4-momentum of the photon, we have then, $k^\mu \mathcal{M}_\mu = 0$.

8.32 Consider the decay $\chi \to e^+ + e^-$ in the model described in Prob. 8.31.

a) Write the amplitude for the process.

b) Evaluate the decay width $\Gamma(\chi \to e^+ + e^-)$ as a function of the parameters of the model.

c) Imagine that one measures $m_\chi = 1.8$ GeV and a lifetime of $\tau_\chi = 1.3 \times 10^{-25}$ s. What is the value of λ? Take $m_e = 0.511$ MeV.

8.33 Consider the process $e^- + \chi \to e^- + \chi$ in the model described in Prob. 8.31.

a) Draw the diagrams that contribute in lowest order to the process.

b) Write the expression for the amplitude.

c) Consider that $\sqrt{s} \gg m_e, m_\chi$ and that therefore it is a good approximation to take $m_e = m_\chi = 0$. In these conditions evaluate the differential cross section in the CM frame, as a function of the CM energy, $\sqrt{s}$ and scattering angle θ.

8.34 Consider the following process,

$$e^-(p_1) + e^+(p_2) \to \nu_e(p_3) + \overline{\nu}_e(p_4)$$

Neglect the masses of all fermions.

a) Use the program `qgraf` [41] to verify that there are two diagrams in lowest order and that there is a relative minus sign between them.

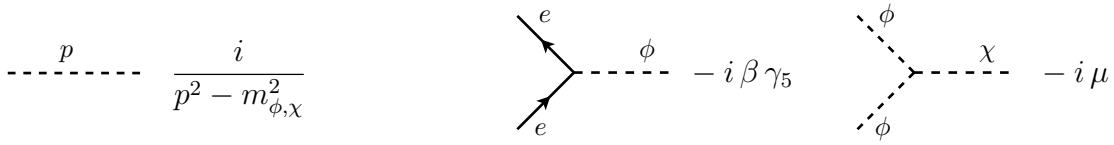
b) Use the technique of the *spinor products* (helicity amplitudes), to write the amplitudes for the process.

c) Using the results from b) evaluate the total cross section in the CM frame, for $\sqrt{s} \in [40, 300]$GeV. Make the corresponding plot, with the cross section in picobarn (pb).

8.35 Consider the process $e^- + \phi \to e^- + \gamma$ in a theory described by the following Lagrangian,

$$\mathcal{L} = \mathcal{L}_{\rm QED} + \frac{1}{2}\partial_\mu\phi\ \partial^\mu\phi\ + \frac{1}{2}\partial_\mu\chi\ \partial^\mu\chi\ - \frac{1}{2}m_\phi^2\ \phi^2 - \frac{1}{2}m_\chi^2\ \chi^2 - \frac{1}{2}\mu\ \phi^2\chi - \beta\ \overline{\psi}\gamma_5\psi\,\phi$$

where ϕ is a neutral pseudo-scalar field, χ is a neutral scalar field and ψ is the electron. The constant β is dimensionless in our natural system of units, and the constant μ has dimensions of a mass. Besides QED we have the following propagators and new vertices,



a) Draw the diagrams that contribute in lowest order to the process.

b) Write the amplitude for this process.

c) Show that the amplitude is gauge invariant, that is, if $\mathcal{M} \equiv \epsilon^\mu(k)\,\mathcal{M}_\mu$ where k is the 4-momentum of the photon, then we have $k^\mu\mathcal{M}_\mu = 0$.

8.36 Consider the process,

$$\nu_e(p_1) + e^-(p_2) + \to \nu_e(p_3) + e^-(p_4)$$

a) Use the program `qgraf` [41] to verify that in lowest order there are two diagrams with a relative minus sign.

b) Use the technique of the *spinor products* (helicity amplitudes), to write the amplitudes for the process.

c) Using the results from b) evaluate the total cross section in the CM frame, for $\sqrt{s} \in [100, 500]$GeV. Make the corresponding plot, with the cross section in picobarn (pb).

d) Compare the exact result evaluate here with the approximate result of Prob. 8.14. Make a plot of the two results in the relevant energy range.

8.37 Consider the process $H \to e^- + \overline{\nu}_e + W^+$. Neglect the fermion masses, but not the mass and widths of the gauge bosons.

a) Draw the diagrams for this process in lowest order and write the amplitude.

b) Using the technique of your choice, evaluate the decay width. Make a plot of the decay width as a function of the Higgs boson mass, M_H, in the interval, $M_H \in [100, 200]$ GeV. Compare, in the same plot, with the width, $\Gamma(H \to b\bar{b})$. Verify that you reproduce the plot shown Fig. 8.12.

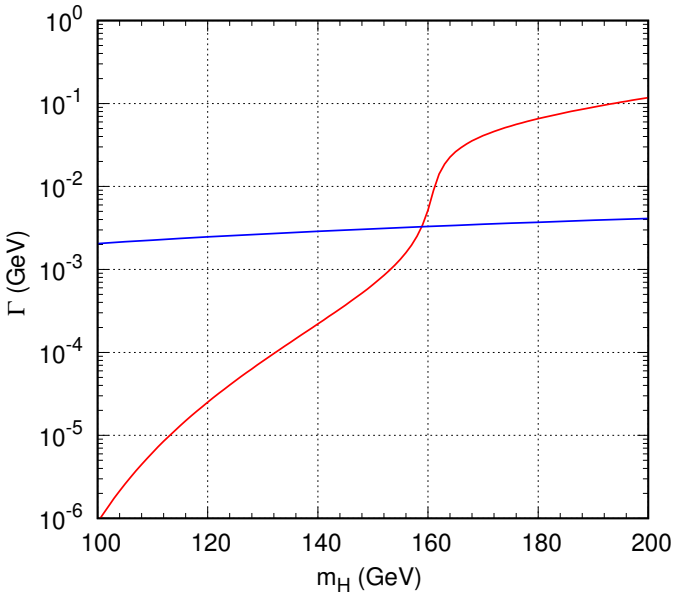
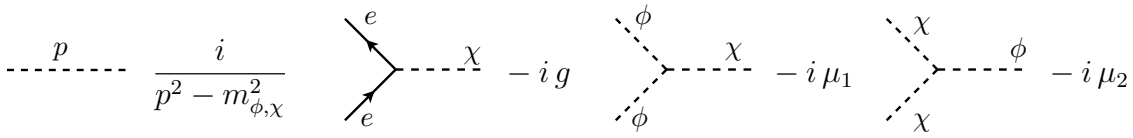


Figure 8.12: Comparison of the partial widths, $H \to e^- + \overline{\nu}_e + W^+$ (red), and $H \to b + \overline{b}$ (blue).

8.38 Consider the process $e^- + \chi \to e^- + \gamma$ in a theory described by the following Lagrangian,

$$\mathcal{L} = \mathcal{L}_{\rm QED} + \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}\partial_\mu\chi\,\partial^\mu\chi - \frac{1}{2}m_\phi^2\,\phi^2 - \frac{1}{2}m_\chi^2\,\chi^2 - \frac{1}{2}\mu_1\,\phi^2\chi - \frac{1}{2}\mu_2\,\phi\chi^2 - g\,\overline{\psi}\psi\,\chi$$

where ϕ and χ are neutral scalar fields and ψ is the electron. The constant g is dimensionless and the constants μ_1, μ_2 have dimension of mass in our system of units. Besides QED we have the new propagator and vertices,



a) Draw the diagrams that contribute to the process in lowest order.

b) Write the amplitude for the process.

c) Show that the amplitude is gauge invariant, that is, if $\mathcal{M} \equiv \epsilon^{\mu}(k)\,\mathcal{M}_{\mu}$ where k is the 4-momentum of the photon, then we have $k^{\mu}\mathcal{M}_{\mu} = 0$.

8.39 Consider the process $e^- + e^+ \to \phi + \chi$ in the model described in Prob. 8.38.

a) Draw the diagrams that contribute to the process in lowest order.

b) Write the amplitude for the process.

c) Consider that $\sqrt{s} \gg m_e, m_\phi$ and that therefore is a good approximation to take $m_e = m_\phi = m_\chi = 0$. In these conditions evaluate the differential cross section $d\sigma/d\Omega$ in the CM frame as a function of the CM energy $\sqrt{s}$ and scattering angle θ.

d) Knowing that for $\sqrt{s} = 100$ GeV one has measured the cross section, $\sigma(e^- + e^+ \to \phi + \chi) = 4$ pb, determine, in GeV, the product $g\mu_2$.

8.40 Consider the decay of the top quark, $t \to b + W^+$ in the SM. In the calculations neglect the mass of the bottom quark, b.

a) Write the amplitude for the process.

b) What is the velocity of the W^+ boson in the rest frame of the top?

c) Evaluate the expression for the decay width, $\Gamma(t \to b + W^+)$, in terms of the model parameters.

d) Knowing that the polarization vector for the longitudinal polarization of the W boson, in a frame where it moves with velocity $\vec{\beta}$ is given by $\varepsilon_L^{\mu} = (\gamma\beta, \gamma\vec{\beta}/\beta)$, show that the fraction of the decays for which the W^+ is longitudinally polarized is,

$$F_L = \frac{m_t^2}{m_t^2 + 2M_W^2}$$

8.41 Consider the process $\nu_\mu + e^- \to \mu^- + \nu_e$ in the SM.

a) Assume that all the energies are much smaller that the W boson mass. Write the expression for the amplitude in this approximation.

b) Evaluate the differential cross section $d\sigma/d\Omega$ in the CM frame, in the limit where all fermion masses are neglected. The angles in the solid angle $d\Omega$ are those, in the CM, that the direction of the μ^- with respect to the direction of the incident ν_μ.

c) Evaluate the total cross section in the CM. Express the result in picobarn for $\sqrt{s} = 5$ GeV.

8.42 With this problem we want to show the crucial role that the Higgs boson has in the consistency of the Standard Model. For this consider the following process in the CM frame,

$$W_L^-(p_1) + W_L^+(p_2) \to W_L^-(q_1) + W_L^+(q_2) \tag{8.102}$$

where the momenta are indicated and the subscript L means that the $W^\pm$ bosons are longitudinally polarized.

a) Knowing that in the rest frame where, $p^\mu = (M_W, 0, 0, 0)$, the longitudinal polarization vector is, $\varepsilon_L^\mu(p) = (0, 0, 0, 1)$, satisfying $\varepsilon_L(p) \cdot \varepsilon_L(p) = -1$ and $\varepsilon_L(p) \cdot p = 0$, show that in the frame where the W moves with velocity $\vec{\beta}$, this vector is,

$$\varepsilon_L^\mu(p) = (\gamma\beta, \gamma\hat{\beta})$$

where, as usually, $\vec{\beta} = \vec{p}/E$, $\gamma^{-1} = \sqrt{1-\beta^2}$ and $\hat{\beta} = \vec{\beta}/\beta$. Verify that the invariant relations $\varepsilon_L(p) \cdot \varepsilon_L(p) = -1$ and $\varepsilon_L(p) \cdot p = 0$ are kept.

b) For processes with two particles in the initial and final state, one can show that unitarity (probability conservation) has the consequence that the total amplitude for the process, in the high energy limit, $s \gg M_W^2$, has to be constant or decrease with the energy. Show that this means that the total cross section decreases with energy.

c) Write the amplitudes for the process in Eq. (8.102). Write the amplitudes in the form,

$$\mathcal{M} = \mathcal{M}^s_{\gamma+Z} + \mathcal{M}^t_{\gamma+Z} + \mathcal{M}_{4W} + \mathcal{M}^{s+t}_H \tag{8.103}$$

where, in an obvious notation, $\mathcal{M}^s_{\gamma+Z}$, is the sum of the diagrams for the s channel for γ and Z^0 and similarly for the others.

d) Consider now the high energy limit. For this define a dimensionless variable, $x = s/(4M_W^2)$ and show that the various amplitudes can be written in the form ($x \gg 1$),

$$\mathcal{M}_i = A_i x^2 + B_i x + C_i + \mathcal{O}(1/x)$$

e) In agreement with the result of b) we should have

$$\sum_i A_i = 0, \quad \sum_i B_i = 0$$

where the sum is over all the diagrams. Show that this happens to $\sum_i A_i = 0$, that originate only in the gauge couplings, but that for $\sum_i B_i = 0$ the diagrams for the Higgs bosons, with the couplings coming from the Higgs mechanism,

are crucial, showing in this way the fundamental role of the the Higgs boson in the consistency of the theory.

`Hint:` To check that you are in the good path, we leave here some intermediate results,

$$A^s_{\gamma+Z} = -4g^2\cos\theta, \quad B^t_{\gamma+Z} = g^2\left(-\frac{3}{2}+\frac{15}{2}\cos\theta\right)$$

where θ is the scattering angle, in the `CM` frame, between the momentum $\vec{q}_1$ and the incident momentum $\vec{p}_1$.

8.43 Consider the main production process of the Higgs boson in a linear collider,

$$e^-e^+ \to ZH$$

a) Evaluate the total cross section in the CM frame as a function of the masses of the Z and H bosons and the CM energy $\sqrt{s}$. Neglect the electron mass. You can use any method at your disposal.

b) Make a plot of the total cross section for $\sqrt{s} \in \Big[M_H + M_Z, 1000\Big]$ GeV and for three values of the Higgs boson mass, $M_H = 110, 125, 140$ GeV.

c) Consider now that the Z^0 boson is longitudinally polarized (see Prob. 8.42). Evaluate the total cross section σ_L. Make a plot for $M_H = 125$ GeV and for $\sqrt{s} \in \Big[M_H + M_Z, 1000\Big]$ GeV, of the ratio

$$R = \frac{\sigma_L}{\sigma_L + \sigma_T}$$

Discuss the result in the limit when $\sqrt{s} \gg M_Z, M_H$. Note that you do not need to evaluate the cross section, σ_T, for the transverse polarization, as $\sigma_{\text{Total}} = \sigma_L + \sigma_T$ and σ_{Total} is the result of a).

Chapter 9

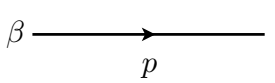
Radiative Corrections

9.1 QED Renormalization at one-loop

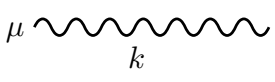
We will consider the theory described by the Lagrangian

$$\mathcal{L}_{\rm QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial\cdot A)^2 + \overline{\psi}(i\partial\!\!\!/ + eA\!\!\!/ - m)\psi \ . \tag{9.1}$$

The free propagators are

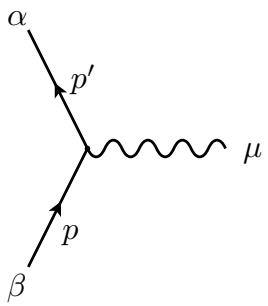


$$\left(\frac{i}{p\!\!\!/ - m + i\varepsilon}\right)_{\beta\alpha} \equiv S^0_{F\beta\alpha}(p) \tag{9.2}$$



$$\begin{aligned} & -i\left[\frac{g_{\mu\nu}}{k^2+i\varepsilon} - (1-\xi)\frac{k_\mu k_\nu}{(k^2+i\varepsilon)^2}\right] \\ = \ & -i\left\{\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2+i\varepsilon} + \xi\frac{k_\mu k_\nu}{k^4}\right\} \\ \equiv \ & G^0_{F\mu\nu}(k) \end{aligned} \tag{9.3}$$

and the vertex



$$+ie(\gamma_\mu)_{\beta\alpha} \qquad e = |e| > 0 \tag{9.4}$$

We will now consider the one-loop corrections to the propagators and to the vertex. We will work in the Feynman gauge ($\xi = 1$).

9.1.1 Vacuum Polarization

In first order the contribution to the photon propagator is given by the diagram of Fig. 9.1 that we write in the form,

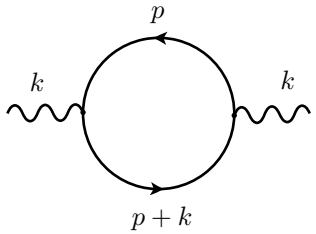


Figure 9.1: Vacuum polarization.

$$G^{(1)}_{\mu\nu}(k) \equiv G^0_{\mu\mu'}(k)\; i\,\Pi^{\mu'\nu'}(k) G^0_{\nu'\nu}(k) \tag{9.5}$$

where

$$\begin{aligned} i\,\Pi_{\mu\nu}(k) &= -(+ie)^2 \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}[\gamma_\mu(\not{p}+m)\gamma_\nu(\not{p}+\not{k}+m)]}{(p^2-m^2+i\varepsilon)((p+k)^2-m^2+i\varepsilon)} \\ &= -4e^2 \int \frac{d^4p}{(2\pi)^4} \frac{[2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2 + p\cdot k - m^2)}{(p^2-m^2+i\varepsilon)((p+k)^2-m^2+i\varepsilon)} \end{aligned} \tag{9.6}$$

Simple power counting indicates that this integral is quadratically divergent for large values of the internal loop momenta. In fact the divergence is milder, only logarithmic. The integral being divergent we have first to regularize it and then to define a renormalization procedure to cancel the infinities. For this purpose we will use the method of dimensional regularization. For a value of d small enough the integral converges. If we define $\epsilon = 4 - d$, in the end we will have a divergent result

in the limit $\epsilon \to 0$. We get therefore[1]

$$\begin{aligned} i\,\Pi_{\mu\nu}(k,\epsilon) &= -4e^2\,\mu^\epsilon \int \frac{d^dp}{(2\pi)^d} \frac{[2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2 + p\cdot k - m^2)]}{(p^2 - m^2 + i\varepsilon)((p+k)^2 - m^2 + i\varepsilon)} \\ &= -4e^2\,\mu^\epsilon \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p,k)}{(p^2 - m^2 + i\varepsilon)((p+k)^2 - m^2 + i\varepsilon)} \end{aligned} \tag{9.7}$$

where

$$N_{\mu\nu}(p,k) = 2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2 + p\cdot k - m^2) \tag{9.8}$$

To evaluate this integral we first use the Feynman parameterization to rewrite the denominator as a single term. For that we use (see Appendix D.3),

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2} \tag{9.9}$$

S to get

$$\begin{aligned} i\,\Pi_{\mu\nu}(k,\epsilon) &= -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p,k)}{[x(p+k)^2 - xm^2 + (1-x)(p^2-m^2) + i\varepsilon]^2} \\ &= -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p,k)}{[p^2 + 2k\cdot px + xk^2 - m^2 + i\varepsilon]^2} \\ &= -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p,k)}{[(p+kx)^2 + k^2x(1-x) - m^2 + i\varepsilon]^2} \end{aligned} \tag{9.10}$$

For dimension d sufficiently small this integral converges and we can change variables

$$p \to p - kx \tag{9.11}$$

We then get

$$i\,\Pi_{\mu\nu}(k,\epsilon) = -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p-kx,k)}{[p^2 - C + i\epsilon]^2} \tag{9.12}$$

where

$$C = m^2 - k^2x(1-x) \tag{9.13}$$

$N_{\mu\nu}$ is a polynomial of second degree in the loop momenta as can be seen from Eq. (9.8). However as the denominator in Eq. (9.12) only depends on p^2 is it easy to show that

$$\int \frac{d^dp}{(2\pi)^d} \frac{p^\mu}{[p^2 - C + i\epsilon]^2} = 0$$

[1]Where μ is a parameter with dimensions of a mass that is introduced to ensure the correct dimensions of the coupling constant in dimension d, that is, $[e] = \frac{4-d}{2} = \frac{\epsilon}{2}$. We take then $e \to e\mu^{\frac{\epsilon}{2}}$. For more details see the Appendix D.1.

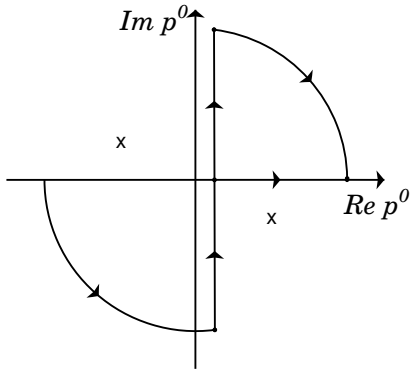


Figure 9.2: Wick rotation.

$$\int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu}{\left[p^2 - C + i\epsilon\right]^2} = \frac{1}{d} g^{\mu\nu} \int \frac{d^d p}{(2\pi)^d} \frac{p^2}{\left[p^2 - C + i\epsilon\right]^2} \tag{9.14}$$

This means that we only have to calculate integrals of the form

$$\begin{aligned} I_{r,m} &= \int \frac{d^d p}{(2\pi)^d} \frac{(p^2)^r}{\left[p^2 - C + i\epsilon\right]^m} \\ &= \int \frac{d^{d-1} p}{(2\pi)^d} \int dp^0 \frac{(p^2)^r}{\left[p^2 - C + i\epsilon\right]^m} \end{aligned} \tag{9.15}$$

To make this integration we will use integration in the plane of the complex variable p^0 as described in Fig. 9.2. The deformation of the contour corresponds to the so called Wick rotation,

$$p^0 \to i p_E^0 \qquad ; \qquad \int_{-\infty}^{+\infty} \to i \int_{-\infty}^{+\infty} dp_E^0 \tag{9.16}$$

and $p^2 = (p^0)^2 - |\vec{p}|^2 = -(p_E^0)^2 - |\vec{p}|^2 \equiv -p_E^2$, where $p_E = (p_E^0, \vec{p})$ is an Euclidean vector, that is

$$p_E^2 = (p_E^0)^2 + |\vec{p}|^2 \tag{9.17}$$

We can then write (see the Appendix for more details),

$$I_{r,m} = i(-1)^{r-m} \int \frac{d^d p_E}{(2\pi)^d} \frac{p_E^{2r}}{\left[p_E^2 + C\right]^m} \tag{9.18}$$

where we do not need the $i\epsilon$ anymore because the denominator is positive definite[2]($C > 0$). To proceed with the evaluation of $I_{r,m}$ we write,

$$\int d^d p_E = \int d\overline{p}\ \overline{p}^{\ d-1}\, d\Omega_{d-1} \tag{9.19}$$

[2]The case when $C < 0$ is obtained by analytical continuation of the final result.

where $\overline{p} = \sqrt{(p_E^0)^2 + |\vec{p}|^2}$ is the length of of vector p_E in the Euclidean space with d dimensions and $d\Omega_{d-1}$ is the solid angle that generalizes spherical coordinates. We can show (see Appendix) that

$$\int d\Omega_{d-1} = 2\,\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \tag{9.20}$$

The $\overline{p}$ integral is done using the result,

$$\int_0^\infty dx\ \frac{x^p}{(x^2+C)^m} = \frac{\Gamma\left(\frac{p+1}{2}\right) C^{\frac{1}{2}(p-2m+1)} \Gamma\left(-\frac{p}{2}+m-\frac{1}{2}\right)}{2\,\Gamma(m)} \tag{9.21}$$

and we finally get

$$I_{r,m} = iC^{r-m+\frac{d}{2}}\ \frac{(-1)^{r-m}}{(4\pi)^{\frac{d}{2}}}\ \frac{\Gamma(r+\frac{d}{2})}{\Gamma(\frac{d}{2})}\ \frac{\Gamma(m-r-\frac{d}{2})}{\Gamma(m)} \tag{9.22}$$

Note that the integral representation of $I_{r,m}$, Eq. (9.15) is only valid for $d < 2(m-r)$ to ensure the convergence of the integral when $\overline{p} \to \infty$. However the final form of Eq. (9.22) can be analytically continued for all the values of d except for those where the function $\Gamma(m-r-d/2)$ has poles, which are (see section D.2),

$$m - r - \frac{d}{2} \neq 0, -1, -2, \ldots \tag{9.23}$$

For the application to dimensional regularization it is convenient to write Eq. (9.22) after making the substitution $d = 4-\epsilon$. We get

$$I_{r,m} = i\frac{(-1)^{r-m}}{(4\pi)^2}\ \left(\frac{4\pi}{C}\right)^{\frac{\epsilon}{2}}\ C^{2+r-m}\ \frac{\Gamma(2+r-\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})}\ \frac{\Gamma(m-r-2+\frac{\epsilon}{2})}{\Gamma(m)} \tag{9.24}$$

that has poles for $m - r - 2 \leq 0$ (see section D.2).

We now go back to calculate $\Pi_{\mu\nu}$. First we notice that after the change of variables of Eq. (9.11) we get, neglecting terms that vanish due to Eq. (9.14),

$$N_{\mu\nu}(p-kx,k) = 2p_\mu p_\nu + 2x^2 k_\mu k_\nu - 2xk_\mu k_\nu - g_{\mu\nu}\left(p^2 + x^2k^2 - xk^2 - m^2\right) \tag{9.25}$$

and therefore

$$\begin{aligned}\mathcal{N}_{\mu\nu} &\equiv \mu^\epsilon \int \frac{d^dp}{(2\pi)^d}\ \frac{N_{\mu\nu}(p-kx,k)}{\left[p^2 - C + i\epsilon\right]^2} \\ &= \left(\frac{2}{d}-1\right) g_{\mu\nu}\mu^\epsilon I_{1,2} + \Big[-2x(1-x)k_\mu k_\nu + x(1-x)k^2 g_{\mu\nu} + g_{\mu\nu}m^2\Big]\mu^\epsilon I_{0,2}\end{aligned} \tag{9.26}$$

Using now Eq. (9.24) we can write

$$\mu^\epsilon I_{0,2} \ = \ \frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\frac{\epsilon}{2}}\ \frac{\Gamma(\frac{\epsilon}{2})}{\Gamma(2)}$$

$$= \frac{i}{16\pi^2}\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right) + \mathcal{O}(\epsilon) \tag{9.27}$$

where we have used the expansion of the Γ function, Eq. (D.14),

$$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon) \tag{9.28}$$

γ being the Euler constant and we have defined,

$$\Delta_\epsilon = \frac{2}{\epsilon} - \gamma + \ln 4\pi \tag{9.29}$$

In a similar way

$$\begin{aligned}\mu^\epsilon I_{1,2} &= -\frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\frac{\epsilon}{2}} C\,\frac{\Gamma(3-\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})}\,\frac{\Gamma(-1+\frac{\epsilon}{2})}{\Gamma(2)} \\ &= \frac{i}{16\pi^2}C\left(1 + 2\Delta_\epsilon - 2\ln\frac{C}{\mu^2}\right) + \mathcal{O}(\epsilon)\end{aligned} \tag{9.30}$$

Due to the existence of a pole in $1/\epsilon$ in the previous equations we have to expand all quantities up to $\mathcal{O}(\epsilon)$. This means for instance, that

$$\frac{2}{d} - 1 = \frac{2}{4-\epsilon} - 1 = -\frac{1}{2} + \frac{1}{8}\epsilon + \mathcal{O}(\epsilon^2) \tag{9.31}$$

Substituting back into Eq. (9.26), and using Eq. (9.13), we obtain

$$\begin{aligned}\mathcal{N}_{\mu\nu} &= g_{\mu\nu}\left[-\frac{1}{2} + \frac{1}{8}\epsilon + \mathcal{O}(\epsilon^2)\right]\left[\frac{i}{16\pi^2}C\left(1 + 2\Delta_\epsilon - 2\ln\frac{C}{\mu^2}\right) + \mathcal{O}(\epsilon)\right] \\ &\quad + \left[-2x(1-x)k_\mu k_\nu + x(1-x)k^2 g_{\mu\nu} + g_{\mu\nu}m^2\right]\left[\frac{i}{16\pi^2}\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right) + \mathcal{O}(\epsilon)\right] \\ &= -\frac{i}{16\pi^2}k_\mu k_\nu\left[\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right)2x(1-x)\right] \\ &\quad + \frac{i}{16\pi^2}g_{\mu\nu}k^2\left[\Delta_\epsilon\Big(x(1-x) + x(1-x)\Big) + \ln\frac{C}{\mu^2}\Big(-x(1-x) - x(1-x)\Big)\right. \\ &\qquad\qquad \left. + x(1-x)\left(\frac{1}{2} - \frac{1}{2}\right)\right] \\ &\quad + \frac{i}{16\pi^2}g_{\mu\nu}m^2\left[\Delta_\epsilon(-1+1) + \ln\frac{C}{\mu^2}(1-1) + (-\frac{1}{2} + \frac{1}{2})\right]\end{aligned} \tag{9.32}$$

and finally

$$\mathcal{N}_{\mu\nu} = \frac{i}{16\pi^2}\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right)\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)2x(1-x) \tag{9.33}$$

Now using Eq. (9.7) we get

$$\begin{aligned}\Pi_{\mu\nu}(k) &= -4e^2\frac{1}{16\pi^2}\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\int_0^1 dx\; 2x(1-x)\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right)\\ &= -\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\Pi(k^2,\epsilon)\end{aligned} \tag{9.34}$$

where

$$\Pi(k^2,\epsilon) \equiv \frac{2\alpha}{\pi}\int_0^1 dx\; x(1-x)\left[\Delta_\epsilon - \ln\frac{m^2 - x(1-x)k^2}{\mu^2}\right] \tag{9.35}$$

This expression clearly diverges as $\epsilon \to 0$. Before we show how to renormalize it let us discuss the meaning of $\Pi_{\mu\nu}(k)$. The full photon propagator is given by the series represented in Fig. 9.3, where

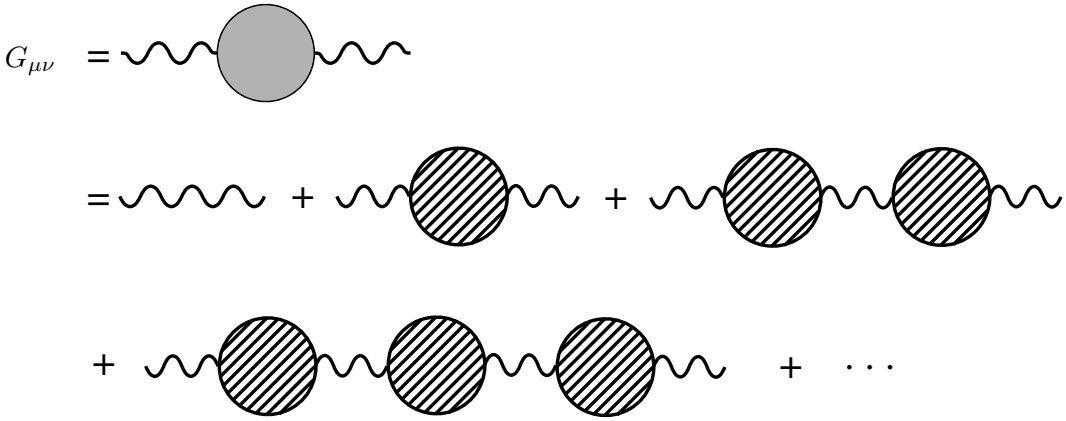
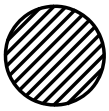


Figure 9.3: Full photon propagator.



$$\equiv i\,\Pi_{\mu\nu}(k) = \begin{array}{l}\text{sum of all one-particle irreducible}\\ (\textit{proper})\text{ diagrams to } \textit{all} \text{ orders}\end{array} \tag{9.36}$$

In lowest order we have the contribution represented in Fig. 9.4, which is what we

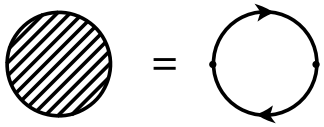


Figure 9.4: Lowest order contribution.

have just calculated. To continue it is convenient to rewrite the free propagator of

the photon (in an arbitrary gauge ξ) in the following form

$$\begin{aligned} iG^0_{\mu\nu} &= \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2} + \xi\frac{k_\mu k_\nu}{k^4} = P^T_{\mu\nu}\frac{1}{k^2} + \xi\frac{k_\mu k_\nu}{k^4} \\ &\equiv iG^{0T}_{\mu\nu} + iG^{0L}_{\mu\nu} \end{aligned} \tag{9.37}$$

where we have introduced the transversal projector $P^T_{\mu\nu}$ defined by

$$P^T_{\mu\nu} = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \tag{9.38}$$

obviously satisfying the relations,

$$\begin{cases} k^\mu P^T_{\mu\nu} = 0 \\ P^{T\,\nu}_{\mu} P^T_{\nu\rho} = P^T_{\mu\rho} \end{cases} \tag{9.39}$$

The full photon propagator can also in general be written separating its transversal an longitudinal parts

$$G_{\mu\nu} = G^T_{\mu\nu} + G^L_{\mu\nu} \tag{9.40}$$

where $G^T_{\mu\nu}$ satisfies

$$G^T_{\mu\nu} = P^T_{\mu\nu} G_{\mu\nu} \tag{9.41}$$

Eq. (9.34) means that, to first order, the vacuum polarization tensor is transversal, that is

$$i\,\Pi_{\mu\nu}(k) = -ik^2 P^T_{\mu\nu}\,\Pi(k) \tag{9.42}$$

This result is in fact valid to all orders of perturbation theory, a result that can be shown using the Ward-Takahashi identities. This means that the longitudinal part of the photon propagator is not renormalized,

$$G^L_{\mu\nu} = G^{0L}_{\mu\nu} \tag{9.43}$$

For the transversal part we obtain from Fig. 9.3,

$$\begin{aligned} iG^T_{\mu\nu} &= P^T_{\mu\nu}\frac{1}{k^2} + P^T_{\mu\mu'}\frac{1}{k^2}(-i)k^2 P^{T\mu'\nu'}\Pi(k^2)(-i)P^T_{\nu'\nu}\frac{1}{k^2} \\ &\quad + P^T_{\mu\rho}\frac{1}{k^2}(-i)k^2 P^{T\rho\lambda}\,\Pi(k^2)(-i)P^T_{\lambda\tau}\frac{1}{k^2}(-i)k^2 P^{T\tau\sigma}\,\Pi(k^2)(-i)P^T_{\sigma\nu}\frac{1}{k^2} + \cdots \\ &= P^T_{\mu\nu}\frac{1}{k^2}\left[1 - \Pi(k^2) + \Pi^2(k^2) + \cdots\right] \end{aligned} \tag{9.44}$$

which gives, after summing the geometric series,

$$iG^T_{\mu\nu} = P^T_{\mu\nu}\frac{1}{k^2\big[1+\Pi(k^2)\big]} \tag{9.45}$$

All that we have done up to this point is formal because the function $\Pi(k)$ diverges. The most satisfying way to solve this problem is the following. The initial Lagrangian from which we started has been obtained from the classical theory and nothing tell us that it should be exactly the same in quantum theory. In fact, as we have just seen, the normalization of the wave functions is changed when we calculate *one-loop* corrections, and the same happens to the physical parameters of the theory, the charge and the mass. Therefore we can think that the correct Lagrangian is obtained by adding corrections to the classical Lagrangian, order by order in perturbation theory, so that we keep the definitions of charge and mass as well as the normalization of the wave functions. The terms that we add to the Lagrangian are called *counterterms*[3]. The total Lagrangian is then,

$$\mathcal{L}_{\text{total}} = \mathcal{L}(e, m, ...) + \Delta\mathcal{L} \tag{9.46}$$

Counterterms are defined from the normalization conditions that we impose on the fields and other parameters of the theory. In QED we have at our disposal the normalization of the electron and photon fields and of the two physical parameters, the electric charge and the electron mass. The normalization conditions are, to a large extent, arbitrary. It is however convenient to keep the expressions as close as possible to the free field case, that is, without radiative corrections. We define therefore the normalization of the photon field as,

$$\lim_{k\to 0} k^2 iG^{RT}_{\mu\nu} = 1\cdot P^T_{\mu\nu} \tag{9.47}$$

where $G^{RT}_{\mu\nu}$ is the renormalized propagator (the transversal part) obtained from the Lagrangian $\mathcal{L}_{\text{total}}$. The justification for this definition comes from the following argument. Consider the Coulomb scattering to all orders of perturbation theory. We have then the situation described in Fig. 9.5.

Using the Ward-Takahashi identities one can show that the last three diagrams cancel in the limit $q = p' - p \to 0$. Then the normalization condition, Eq. (9.47), means that we have the situation described in Fig. 9.6, that is, the experimental value of the electric charge is determined in the limit $q \to 0$ of the Coulomb scattering.

The counterterm Lagrangian has to have the same form as the classical Lagrangian to respect the symmetries of the theory. For the photon field it is traditional to write

$$\Delta\mathcal{L} = -\frac{1}{4}(Z_3 - 1)F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\delta Z_3\, F_{\mu\nu}F^{\mu\nu} \tag{9.48}$$

corresponding to the following Feynman rule

[3]This interpretation in terms of quantum corrections makes sense. In fact we can show that an expansion in powers of the coupling constant can be interpreted as an expansion in $\hbar^L$, where L is the number of the loops in the expansion term.

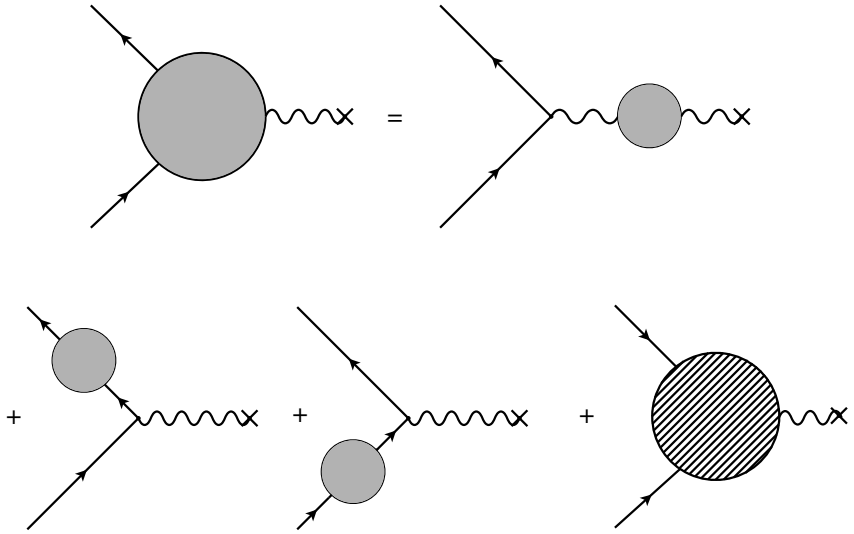


Figure 9.5: Corrections to Coulomb scattering.

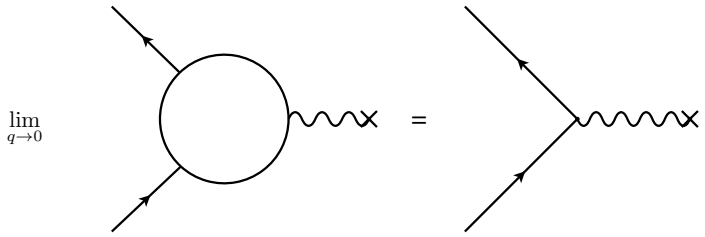


Figure 9.6: Definition of the electric charge.

$$\mu \overset{k}{\sim\!\!\times\!\!\sim} \overset{k}{\sim\!\!\times\!\!\sim} \nu \quad -i\,\delta Z_3 k^2 \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \tag{9.49}$$

We have then

$$\begin{aligned} i\Pi_{\mu\nu} &= i\Pi^{loop}_{\mu\nu} - i\,\delta Z_3 k^2 \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \\ &= -i\,\left(\Pi(k,\epsilon) + \delta Z_3\right) k^2\, P^T_{\mu\nu} \end{aligned} \tag{9.50}$$

Therefore we should make the substitution

$$\Pi(k,\epsilon) \to \Pi(k,\epsilon) + \delta Z_3 \tag{9.51}$$

in the photon propagator. We obtain,

$$iG^T_{\mu\nu} = P^T_{\mu\nu}\,\frac{1}{k^2}\,\frac{1}{1+\Pi(k,\epsilon)+\delta Z_3} \tag{9.52}$$

The normalization condition, Eq. (9.47), implies

$$\Pi(0,\epsilon) + \delta Z_3 = 0 \tag{9.53}$$

from which one determines the constant δZ_3. We get

$$\begin{aligned} \delta Z_3 &= -\Pi(0,\epsilon) = -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\left[\Delta_\epsilon - \ln\frac{m^2}{\mu^2}\right] \\ &= -\frac{\alpha}{3\pi}\left[\Delta_\epsilon - \ln\frac{m^2}{\mu^2}\right] \end{aligned} \tag{9.54}$$

The renormalized photon propagator can then be written as [4]

$$iG_{\mu\nu}(k) = \frac{P^T_{\mu\nu}}{k^2[1+\Pi(k,\epsilon)-\Pi(0,\epsilon)]} + i\,G^L_{\mu\nu} \tag{9.55}$$

The *finite* radiative corrections are given through the function

$$\begin{aligned} \Pi^R(k^2) \equiv& \Pi(k^2,\epsilon) - \Pi(0,\epsilon) \\ =& -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\left[\frac{m^2 - x(1-x)k^2}{m^2}\right] \\ =& -\frac{\alpha}{3\pi}\left\{\frac{1}{3} + 2\left(1+\frac{2m^2}{k^2}\right)\left[\left(\frac{4m^2}{k^2}-1\right)^{1/2}\cot^{-1}\left(\frac{4m^2}{k^2}-1\right)^{1/2} - 1\right]\right\} \end{aligned} \tag{9.56}$$

where the last equation is valid for $k^2 < 4m^2$. For $k^2 \ll m^2$ we get,

$$\Pi^R(k^2) = \frac{\alpha}{15\pi}\frac{k^2}{m^2}\,. \tag{9.57}$$

For values $k^2 > 4m^2$ the result for $\Pi^R(k^2)$ can be obtained from Eq. (9.56) by analytical continuation. Using $(k^2 > 4m^2)$

$$\cot^{-1} iz = i\left(-\tanh^{-1} z + \frac{i\pi}{2}\right) \tag{9.58}$$

and

$$\left(\frac{4m^2}{k^2}-1\right)^{1/2} \to i\sqrt{1-\frac{4m^2}{k^2}} \tag{9.59}$$

we get

$$\Pi^R(k^2) = -\frac{\alpha}{3\pi}\left\{\frac{1}{3} + 2\left(1+\frac{2m^2}{k^2}\right)\left[-1 + \sqrt{1-\frac{4m^2}{k^2}}\tanh^{-1}\left(1-\frac{4m^2}{k^2}\right)\right.\right. \tag{9.60}$$

[4] Notice that the photon mass is not renormalized, that is the pole of the photon propagator remains at $k^2 = 0$.

$$-i\frac{\pi}{2}\sqrt{1-\frac{4m^2}{k^2}}\Bigg]\Bigg\} \tag{9.61}$$

The imaginary part of Π^R is given by

$$Im\ \Pi^R(k^2) = \frac{\alpha}{3}\left(1+\frac{2m^2}{k^2}\right)\sqrt{1-\frac{4m^2}{k^2}}\theta\left(1-\frac{4m^2}{k^2}\right) \tag{9.62}$$

and it is related to the pair production that can occur [5] for $k^2 > 4m^2$.

For future reference let us consider the case when $k^2 < 0$, that occurs when the photon is exchanged in the t-channel. This case can more easily done returning to the original expression in Eq. (9.56) and making the identification,

$$k^2 \equiv -Q^2, \quad \sinh^2\varphi = \frac{Q^2}{4m^2} \tag{9.63}$$

We then get

$$\begin{aligned}\Pi^R(-Q^2) &= -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\left[1+x(1-x)4\sinh^2\varphi\right] \\ &= -\frac{\alpha}{\pi}\left[\left(1-\frac{\coth^2\varphi}{3}\right)(\varphi\coth\varphi-1)+\frac{1}{9}\right]\end{aligned} \tag{9.64}$$

For the case $Q^2 \gg m^2$ the expression simplifies and reduces to,

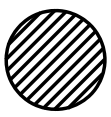
$$\Pi^R(-Q^2) = -\frac{\alpha}{3\pi}\left[\ln\frac{Q^2}{m^2}-\frac{5}{3}\right]\ . \tag{9.65}$$

9.1.2 Self-energy of the electron

The electron full propagator is given by the diagrammatic series of Fig. 9.7, which can be written as,

$$\begin{aligned}S(p) &= S^0(p) + S^0(p)\Big(-i\,\Sigma(p)\Big)\,S^0(p)+\cdots \\ &= S^0(p)\Big[1-i\,\Sigma(p)S(p)\Big]\end{aligned} \tag{9.66}$$

that we can represent in diagrammatic form as,
where we have identified

 $\equiv -i\,\Sigma(p)$ (9.67)

[5] For $k^2 > 4m^2$ there is the possibility of producing one pair e^+e^-. Therefore on top of a virtual process (vacuum polarization) there is a real process (pair production).

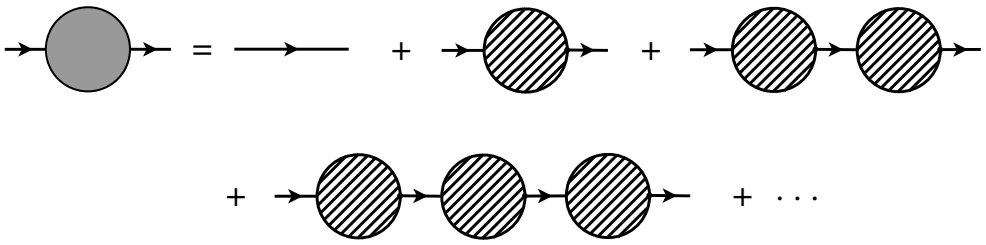


Figure 9.7: Full electron propagator

Multiplying on the left with $S_0^{-1}(p)$ and on the right with $S^{-1}(p)$ we get

$$S_0^{-1}(p) = S^{-1}(p) - i\,\Sigma(p) \tag{9.68}$$

which we can rewrite as

$$S^{-1}(p) = S_0^{-1}(p) + i\,\Sigma(p) \tag{9.69}$$

Using the expression for the free field propagator,

$$S_0(p) = \frac{i}{\not{p} - m} \Longrightarrow S_0^{-1}(p) = -i(\not{p} - m) \tag{9.70}$$

we can then write

$$\begin{aligned} S^{-1}(p) &= S_0^{-1}(p) + i\Sigma(p) \\ &= -i\left[\not{p} - (m + \Sigma(p))\right] \end{aligned} \tag{9.71}$$

We conclude that it is enough to calculate $\Sigma(p)$ to all orders of perturbation theory to obtain the full electron propagator. The name *self-energy* given to $\Sigma(p)$ comes from the fact that, as can be seen in Eq. (9.71), it comes as an additional (momentum dependent) contribution to the mass.

In lowest order there is only the diagram of Fig. 9.8 contributing to $\Sigma(p)$ and therefore we get,

$$-i\Sigma(p) = (+ie)^2 \int \frac{d^4k}{(2\pi)^4}(-i)\frac{g_{\mu\nu}}{k^2 - \lambda^2 + i\varepsilon}\gamma^\mu \frac{i}{\not{p} + \not{k} - m + i\varepsilon}\gamma^\nu \tag{9.72}$$

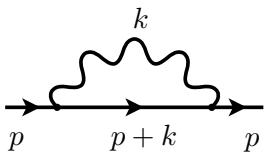


Figure 9.8: Self-energy of the electron in lowest order.

where we have chosen the Feynman gauge ($\xi = 1$) for the photon propagator and we have introduced a small mass for the photon λ, in order to control the infrared divergences (IR) that will appear when $k^2 \to 0$ (see below). Using dimensional regularization and the results of the Dirac algebra in dimension d,

$$\begin{aligned} \gamma_\mu(\not p + \not k)\gamma^\mu &= -(\not p + \not k)\gamma_\mu\gamma^\mu + 2(\not p + \not k) = -(d-2)(\not p + \not k) \\ m\gamma_\mu\gamma^\mu &= m\,d \end{aligned} \tag{9.73}$$

we get

$$\begin{aligned} -i\,\Sigma(p) &= -\mu^\epsilon e^2 \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-\lambda^2+i\varepsilon}\,\gamma_\mu\,\frac{\not p + \not k + m}{(p+k)^2-m^2+i\varepsilon}\,\gamma^\mu \\ &= -\mu^\epsilon e^2 \int \frac{d^dk}{(2\pi)^d}\frac{-(d-2)(\not p+\not k)+m\,d}{[k^2-\lambda^2+i\varepsilon]\,[(p+k)^2-m^2+i\varepsilon]} \\ &= -\mu^\epsilon e^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{-(d-2)(\not p+\not k)+m\,d}{\left[(k^2-\lambda^2)\,(1-x)+x(p+k)^2-xm^2+i\varepsilon\right]^2} \\ &= -\mu^\epsilon e^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{-(d-2)(\not p+\not k)+m\,d}{\left[(k+px)^2+p^2x(1-x)-\lambda^2(1-x)-xm^2+i\varepsilon\right]^2} \\ &= -\mu^\epsilon e^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{-(d-2)\,[\not p(1-x)+\not k]+m\,d}{\left[k^2+p^2x(1-x)-\lambda^2(1-x)-xm^2+i\varepsilon\right]^2} \\ &= -\mu^\epsilon e^2 \int_0^1 dx\,\Big[-(d-2)\not p(1-x)+m\,d\Big]\,I_{0,2} \end{aligned} \tag{9.74}$$

where[6]

$$I_{0,2} = \frac{i}{16\pi^2}\left[\Delta_\epsilon - \ln\left[-p^2x(1-x)+m^2x+\lambda^2(1-x)\right]\right] \tag{9.75}$$

The contribution from the *loop* in Fig. 9.8 to the electron self-energy $\Sigma(p)$ can then be written in the form,

$$\Sigma(p)^{loop} = A(p^2) + B(p^2)\not p \tag{9.76}$$

with

$$\begin{aligned} A &= e^2\mu^\epsilon(4-\epsilon)m\frac{1}{16\pi^2}\int_0^1 dx\,\left[\Delta_\epsilon - \ln\left[-p^2x(1-x)+m^2x+\lambda^2(1-x)\right]\right] \\ B &= -e^2\mu^\epsilon(2-\epsilon)\frac{1}{16\pi^2}\int_0^1 dx\,(1-x)\left[\Delta_\epsilon \right. \\ &\qquad \left. - \ln\left[-p^2x(1-x)+m^2x+\lambda^2(1-x)\right]\right] \end{aligned} \tag{9.77}$$

[6]The linear term in k vanishes.

Using now the expansions

$$\begin{aligned}
\mu^\epsilon(4-\epsilon) &= 4\left[1+\epsilon\left(\ln\mu-\frac{1}{4}\right)+\mathcal{O}(\epsilon^2)\right] \\
\mu^\epsilon(4-\epsilon)\Delta_\epsilon &= 4\left[\Delta_\epsilon+2\left(\ln\mu-\frac{1}{4}\right)+\mathcal{O}(\epsilon)\right] \\
\mu^\epsilon(2-\epsilon) &= 2\left[1+\epsilon\left(\ln\mu-\frac{1}{2}\right)+\mathcal{O}(\epsilon^2)\right] \\
\mu^\epsilon(2-\epsilon)\Delta_\epsilon &= 2\left[\Delta_\epsilon+2\left(\ln\mu-\frac{1}{2}\right)+\mathcal{O}(\epsilon)\right]
\end{aligned} \tag{9.78}$$

we can finally write,

$$A(p^2)=\frac{4\,e^2m}{16\pi^2}\int_0^1 dx\,\left[\Delta_\epsilon-\frac{1}{2}-\ln\left[\frac{-p^2x(1-x)+m^2x+\lambda^2(1-x)}{\mu^2}\right]\right] \tag{9.79}$$

and

$$B(p^2)=-\frac{2\,e^2}{16\pi^2}\int_0^1 dx\,(1-x)\left[\Delta_\epsilon-1-\ln\left[\frac{-p^2x(1-x)+m^2x+\lambda^2(1-x)}{\mu^2}\right]\right] \tag{9.80}$$

To continue with the renormalization program we have to introduce the counterterm Lagrangian and define the normalization conditions. We have

$$\Delta\mathcal{L}=i\,(Z_2-1)\,\overline{\psi}\gamma^\mu\partial_\mu\psi-(Z_2-1)\,m\,\overline{\psi}\psi+Z_2\delta m\,\overline{\psi}\psi+(Z_1-1)e\,\overline{\psi}\gamma^\mu\psi A_\mu \tag{9.81}$$

and therefore we get for the self-energy

$$-i\,\Sigma(p)=-i\,\Sigma^{loop}(p)+i\,(\not{p}-m)\,\delta Z_2+i\,\delta m \tag{9.82}$$

Contrary to the case of the photon we see that we have two constants to determine. In the *on-shell* renormalization scheme that is normally used in QED the two constants are obtained by requiring that the pole of the propagator corresponds to the physical mass (hence the name of *on-shell* renormalization), and that the residue of the pole of the renormalized electron propagator has the same value as the free field propagator. This implies,

$$\begin{aligned}
\Sigma(\not{p}=m)=0 \;&\rightarrow\; \delta m=\Sigma^{loop}(\not{p}=m) \\
\left.\frac{\partial\Sigma}{\partial\not{p}}\right|_{\not{p}=m}=0 \;&\rightarrow\; \delta Z_2=\left.\frac{\partial\Sigma^{loop}}{\partial\not{p}}\right|_{\not{p}=m}
\end{aligned} \tag{9.83}$$

We then get for δm,

$$\delta m \;=\; A(m^2)+m\,B(m^2)$$

$$
\begin{aligned}
&= \frac{2\,me^2}{16\pi^2}\int_0^1 dx\,\left\{\left[2\Delta_\epsilon - 1 - 2\ln\left(\frac{m^2x^2+\lambda^2(1-x)}{\mu^2}\right)\right]\right. \\
&\qquad \left. -(1-x)\left[\Delta_\epsilon - 1 - \ln\left(\frac{m^2x^2+\lambda^2(1-x)}{\mu^2}\right)\right]\right\} \\
&= \frac{2\,m\,e^2}{16\pi^2}\left[\frac{3}{2}\Delta_\epsilon - \frac{1}{2} - \int_0^1 dx\,(1+x)\ln\left(\frac{m^2x^2+\lambda^2(1-x)}{\mu^2}\right)\right] \\
&= \frac{3\alpha m}{4\pi}\left[\Delta_\epsilon - \frac{1}{3} - \frac{2}{3}\int_0^1 dx\,(1+x)\ln\left(\frac{m^2x^2}{\mu^2}\right)\right] \qquad (9.84)
\end{aligned}
$$

where in the last step in Eq. (9.84) we have taken the limit $\lambda \to 0$ because the integral does not diverge in that limit[7]. In a similar way we get for δZ_2,

$$
\delta Z_2 = \left.\frac{\partial \Sigma^{loop}}{\partial \not{p}}\right|_{\not{p}=m} = \left.\frac{\partial A}{\partial \not{p}}\right|_{\not{p}=m} + B + m\left.\frac{\partial B}{\partial \not{p}}\right|_{\not{p}=m} \qquad (9.85)
$$

where

$$
\begin{aligned}
\left.\frac{\partial A}{\partial \not{p}}\right|_{\not{p}=m} &= \frac{4\,e^2\,m^2}{16\pi^2}\int_0^1 dx\,\frac{2(1-x)x}{-m^2x(1-x)+m^2x+\lambda^2(1-x)} \\
&= \frac{2\,\alpha\,m^2}{\pi}\int_0^1 dx\,\frac{(1-x)x}{m^2x^2+\lambda^2(1-x)} \\
B &= -\frac{\alpha}{2\pi}\int_0^1 dx\,(1-x)\left[\Delta_\epsilon - 1 - \ln\left(\frac{m^2x^2+\lambda^2(1-x)}{\mu^2}\right)\right] \\
m\left.\frac{\partial B}{\partial \not{p}}\right|_{\not{p}=m} &= -\frac{\alpha}{2\pi}\,m^2\int_0^1 dx\,\frac{2x(1-x)^2}{m^2x^2+\lambda^2(1-x)} \qquad (9.86)
\end{aligned}
$$

Substituting Eq. (9.86) in Eq. (9.85) we get,

$$
\begin{aligned}
\delta Z_2 &= -\frac{\alpha}{2\pi}\left[\frac{1}{2}\Delta_\epsilon - \frac{1}{2} - \int_0^1 dx\,(1-x)\ln\left(\frac{m^2x^2}{\mu^2}\right) - 2\int_0^1 dx\,\frac{(1+x)(1-x)xm^2}{m^2x^2+\lambda^2(1-x)}\right] \\
&= \frac{\alpha}{4\pi}\left[-\Delta_\epsilon - 4 + \ln\frac{m^2}{\mu^2} - 2\ln\frac{\lambda^2}{m^2}\right] \qquad (9.87)
\end{aligned}
$$

where we have taken the $\lambda \to 0$ limit in all cases that was possible. It is clear that the final result in Eq. (9.87) diverges in that limit, therefore implying that Z_2 is IR divergent. This is not a problem for the theory because δZ_2 is not a physical parameter. We will see in section 9.3.4 that the IR diverges cancel for real processes. If we had taken a general gauge ($\xi \neq 1$) we would find out that δm would not be changed but that Z_2 would show a gauge dependence. Again, in physical processes this should cancel in the end.

[7] δm is not IR divergent.

9.1.3 The Vertex

The diagram contributing to the QED vertex at one-loop is the one shown in Fig. 9.9. In the Feynman gauge ($\xi = 1$) this gives a contribution,

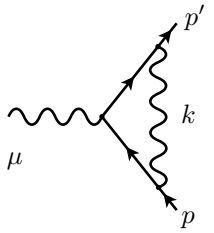


Figure 9.9: The QED vertex.

$$
\begin{aligned}
ie\,\mu^{\epsilon/2}\Lambda_\mu^{loop}(p',p) &= (ie\,\mu^{\epsilon/2})^3 \int \frac{d^dk}{(2\pi)^d}(-i)\frac{g_{\rho\sigma}}{k^2-\lambda^2+i\varepsilon} \\
&\quad \gamma^\sigma \frac{i[(\not{p}'+\not{k})+m]}{(p'+k)^2-m^2+i\varepsilon}\gamma_\mu \frac{i[(\not{p}+\not{k})+m]}{(p+k)^2-m^2+i\varepsilon}\gamma^\rho \qquad (9.88)
\end{aligned}
$$

where Λ_μ is related to the full vertex Γ_μ through the relation

$$
\begin{aligned}
i\Gamma_\mu &= ie\,(\gamma_\mu + \Lambda_\mu^{loop} + \gamma_\mu \delta Z_1) \\
&= ie\,\left(\gamma_\mu + \Lambda_\mu^R\right) \qquad (9.89)
\end{aligned}
$$

The integral that defines $\Lambda_\mu^{loop}(p',p)$ is divergent. As before we expect to solve this problem by regularizing the integral, introducing counterterms and normalization conditions. The counterterm has the same form as the vertex and is already included in Eq. (9.89). The normalization constant is determined by requiring that in the limit $q = p' - p \to 0$ the vertex reproduces the tree level vertex because this is what is consistent with the definition of the electric charge in the $q \to 0$ limit of the Coulomb scattering. Also this should be defined for on-shell electrons. We have therefore that the normalization condition gives,

$$
\overline{u}(p)\left(\Lambda_\mu^{loop} + \gamma_\mu \delta Z_1\right) u(p)\big|_{\not{p}=m} = 0 \qquad (9.90)
$$

If we are interested only in calculating δZ_1 and in showing that the divergences can be removed with the normalization condition then the problem is simpler. It can be done in two ways.

1st method

We use the fact that δZ_1 is to be calculated on-shell and for $p = p'$. Then

$$i\Lambda_\mu^{loop}(p,p) = e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-\lambda^2+i\varepsilon}\gamma_\rho\frac{1}{\not p+\not k-m+i\varepsilon}\gamma_\mu\frac{1}{\not p+\not k-m+i\varepsilon}\gamma^\rho \tag{9.91}$$

However we have

$$\frac{1}{\not p+\not k-m+i\varepsilon}\gamma_\mu\frac{1}{\not p+\not k-m+i\varepsilon} = -\frac{\partial}{\partial p^\mu}\frac{1}{\not p+\not k-m+i\varepsilon} \tag{9.92}$$

and therefore

$$\begin{aligned} i\Lambda_\mu^{loop}(p,p) &= -e^2\mu^\epsilon\frac{\partial}{\partial p^\mu}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-\lambda^2+i\varepsilon}\gamma_\rho\frac{\not p+\not k+m}{(p+k)^2-m^2+i\varepsilon}\gamma^\rho & (9.93)\\ &= -i\frac{\partial}{\partial p^\mu}\Sigma^{loop}(p) & (9.94)\end{aligned}$$

We conclude then, that $\Lambda_\mu^{loop}(p,p)$ is related to the self-energy of the electron[8],

$$\Lambda_\mu^{loop}(p,p) = -\frac{\partial}{\partial p^\mu}\Sigma^{loop} \tag{9.95}$$

On-shell we have

$$\Lambda_\mu^{loop}(p,p)\Big|_{\not p=m} = -\frac{\partial\Sigma^{loop}}{\partial p^\mu}\Big|_{\not p=m} = -\delta Z_2\gamma_\mu \tag{9.96}$$

and the normalization condition, Eq. (9.90), gives

$$\delta Z_1 = \delta Z_2 \tag{9.97}$$

As we have already calculated δZ_2 in Eq. (9.87), then δZ_1 is determined.

2nd method

In this second method we do not rely in the Ward identity but just calculate the integrals for the vertex in Eq. (9.88). For the moment we do not put $p' = p$ but we will assume that the vertex form factors are to be evaluated for on-shell spinors. Then we have

$$\begin{aligned} i\,\overline{u}(p')\Lambda_\mu^{loop}u(p) &= e^2\mu^\epsilon\int\frac{d^dk}{(2\pi)^d}\,\frac{\overline{u}(p)\gamma_\rho\left[\not p'+\not k+m)\right]\gamma_\mu\left[\not p+\not k+m)\right]\gamma^\rho u(p)}{D_0D_1D_2} \\ &= e^2\mu^\epsilon\int\frac{d^dk}{(2\pi)^d}\,\frac{\mathcal{N}_\mu}{D_0D_1D_2} & (9.98)\end{aligned}$$

[8]This result is one of the forms of the Ward-Takahashi identity.

where

$$\mathcal{N}_\mu = \overline{u}(p)\Big[(-2+d)k^2\gamma_\mu + 4p\cdot p'\gamma_\mu + 4(p+p')\cdot k\,\gamma_\mu + 4m\,k_\mu - 4\not{k}\,(p+p')_\mu + 2(2-d)\not{k}\not{k}k_\mu\Big]\,u(p) \tag{9.99}$$

$$D_0 = k^2 - \lambda^2 + i\epsilon \tag{9.100}$$

$$D_1 = (k+p')^2 - m^2 + i\epsilon \tag{9.101}$$

$$D_2 = (k+p)^2 - m^2 + i\epsilon \tag{9.102}$$

Now we use the results of section D.5.3 to do the momentum integrals. We have for our case

$$r_1^\mu = p'^\mu \quad ; \quad r_2^\mu = p^\mu \tag{9.103}$$

$$P^\mu = x_1 p'^\mu + x_2 p^\mu \tag{9.104}$$

$$C = (x_1+x_2)^2 m^2 - x_1 x_2\, q^2 + (1-x_1-x_2)\lambda^2 \tag{9.105}$$

where

$$q = p' - p\ . \tag{9.106}$$

We get,

$$\begin{aligned} i\,\overline{u}(p')\Lambda_\mu^{loop}u(p) &= i\,\frac{\alpha}{4\pi}\Gamma(3)\int_0^1 dx_1\int_0^{1-x_1} dx_2\ \frac{1}{2C} \\ &\quad\Big\{\overline{u}(p')\gamma_\mu u(p)\Big[-(-2+d)(x_1^2m^2 + x_2^2m^2 + 2x_1x_2p'\cdot p) - 4p'\cdot p \\ &\quad +4(p+p')\cdot(x_1p' + x_2p) + \frac{(2-d)^2}{2}\,C\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right)\Big] \\ &\quad +\overline{u}(p)u(p)m\ \Big[4(x_1p' + x_2p)_\mu - 4(p'+p)_\mu(x_1+x_2) \\ &\quad -2(2-d)(x_1+x_2)(x_1p' + x_2p)_\mu\Big]\Big\} \end{aligned} \tag{9.107}$$

$$= i\,\overline{u}(p)\left[G(q^2)\,\gamma_\mu + H(q^2)\,(p+p')_\mu\right]u(p) \tag{9.108}$$

where we have defined[9],

$$G(q^2) \equiv \frac{\alpha}{4\pi}\left[\Delta_\epsilon - 2 - 2\int_0^1 dx_1\int_0^{1-x_1} dx_2\ \ln\frac{(x_1+x_2)^2m^2 - x_1x_2q^2 + (1-x_1-x_2)\lambda^2}{\mu^2}\right.$$

[9]To obtain Eq. (9.113) one has to show that the symmetry of the integrals in $x_1 \leftrightarrow x_2$ implies that the coefficient of p is equal to the coefficient of p'. To see this define

$$H = \int_0^1 dx_1\int_0^{1-x_1} f(x_1,x_2) \tag{9.109}$$

$$+\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \left(\frac{-2(x_1+x_2)^2 m^2 - x_1 x_2 q^2 - 4m^2 + 2q^2}{(x_1+x_2)^2 m^2 - x_1 x_2 q^2 + (1-x_1-x_2)\lambda^2} + \frac{2(x_1+x_2)(4m^2-q^2)}{(x_1+x_2)^2 m^2 - x_1 x_2 q^2 + (1-x_1-x_2)\lambda^2} \right) \Bigg] \tag{9.112}$$

$$H(q^2) \equiv \frac{\alpha}{4\pi} \left[\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \, \frac{-2m\,(x_1+x_2) + 2m\,(x_1+x_2)^2}{(x_1+x_2)^2 m^2 - x_1 x_2 q^2 + (1-x_1-x_2)\lambda^2} \right] \tag{9.113}$$

Now, using the definition of Eq. (9.89), we get for the renormalized vertex,

$$\overline{u}(p')\Lambda^R_\mu(p',p)u(p) = \overline{u}(p') \left[\left(G(q^2) + \delta Z_1 \right) \gamma_\mu + H(q^2)\,(p+p')_\mu \right] u(p) \tag{9.114}$$

As δZ_1 is calculated in the limit of $q = p' - p \to 0$ it is convenient to use the Gordon identity to get rid of the $(p'+p)^\mu$ term. We have

$$\overline{u}(p')\,(p'+p)_\mu\,u(p) = \overline{u}(p') \left[2m\gamma_\mu - i\sigma_{\mu\nu}\,q^\nu \right] u(p) \tag{9.115}$$

and therefore,

$$\begin{aligned} \overline{u}(p')\Lambda^R_\mu(p',p)u(p) &= \overline{u}(p') \left[\left(G(q^2) + 2mH(q^2) + \delta Z_1 \right) \gamma_\mu - i\,H(q^2)\,\sigma_{\mu\nu}\,q^\nu \right] u(p) \\ &= \overline{u}(p') \left[\gamma_\mu F_1(q^2) + \frac{i}{2m}\sigma_{\mu\nu}q^\nu F_2(q^2) \right] u(p) \end{aligned} \tag{9.116}$$

where we have introduced the usual notation for the vertex form factors,

$$F_1(q^2) \equiv G(q^2) + 2mH(q^2) + \delta Z_1 \tag{9.117}$$

$$F_2(q^2) \equiv -2mH(q^2) \tag{9.118}$$

The normalization condition of Eq. (9.90) implies $F_1(0) = 0$, that is,

$$\delta Z_1 = -G(0) - 2m\,H(0) \tag{9.119}$$

We have therefore to calculate $G(0)$ and $H(0)$. In this limit the integrals of Eqs. (9.112) and (9.113) are much simpler. We get (we change variables $x_1 + x_2 \to y$),

$$G(0) = \frac{\alpha}{4\pi} \left[\Delta_\epsilon - 2 - 2\int_0^1 dx_1 \int_{x_1}^1 dy\, \ln \frac{y^2 m^2 + (1-y)\lambda^2}{\mu^2} \right.$$

Then use

$$f(x_1,x_2) = \frac{1}{2}\left[f(x_1,x_2) + f(x_2,x_1)\right] \frac{1}{2}\left[f(x_1,x_2) - f(x_2,x_1)\right] \tag{9.110}$$

to show that

$$H = \int_0^1 dx_1 \int_0^{1-x_1} \frac{1}{2}\left(f(x_1,x_2) + f(x_2,x_1)\right) \ . \tag{9.111}$$

Also notice that you can put $d = 4$ in this term because H is not divergent.

$$+\int_0^1 dx_1 \int_{x_1}^1 dy\, \frac{-2y^2m^2 - 4m^2 + 8ym^2}{y^2m^2 + (1-y)\lambda^2}\Bigg] \tag{9.120}$$

$$H(0) = \frac{\alpha}{4\pi} \int_0^1 dx_1 \int_{x_1}^1 dy\, \frac{-2m\,y + 2m\,y^2}{y^2m^2 + (1-y)\lambda^2} \tag{9.121}$$

Now using

$$\int_0^1 dx_1 \int_{x_1}^1 dy\, \ln \frac{y^2m^2 + (1-y)\lambda^2}{\mu^2} = \frac{1}{2}\left(\ln \frac{m^2}{\mu^2} - 1\right) \tag{9.122}$$

$$\int_0^1 dx_1 \int_{x_1}^1 dy\, \frac{-2y^2m^2 - 4m^2 + 8ym^2}{y^2m^2 + (1-y)\lambda^2} = 7 + 2\ln \frac{\lambda^2}{m^2} \tag{9.123}$$

$$\int_0^1 dx_1 \int_{x_1}^1 dy\, \frac{-2m\,y + 2m\,y^2}{y^2m^2 + (1-y)\lambda^2} = -\frac{1}{m} \tag{9.124}$$

(where we took the limit $\lambda \to 0$ if possible) we get,

$$G(0) = \frac{\alpha}{4\pi}\left[\Delta_\epsilon + 6 - \ln \frac{m^2}{\mu^2} + 2\ln \frac{\lambda^2}{m^2}\right] \tag{9.125}$$

$$H(0) = -\frac{\alpha}{4\pi}\frac{1}{m} \tag{9.126}$$

Substituting the previous expressions in Eq. (9.119) we get finally,

$$\delta Z_1 = \frac{\alpha}{4\pi}\left[-\Delta_\epsilon - 4 + \ln \frac{m^2}{\mu^2} - 2\ln \frac{\lambda^2}{m^2}\right] \tag{9.127}$$

in agreement with Eq. (9.87) and Eq. (9.97). The general form of the form factors $F_i(q^2)$, for $q^2 \neq 0$, is quite complicated. We give here only the result for $q^2 < 0$ (in the Appendix C of Ref. [12] one can find a general formula for numerical evaluation of these functions),

$$F_1(q^2) = \frac{\alpha}{4\pi}\left\{\left(2\ln \frac{\lambda^2}{m^2} + 4\right)(\theta \coth\theta - 1) - \theta\, \tanh\frac{\theta}{2} - 8\coth\theta \int_0^{\theta/2} \beta \tanh\beta d\beta\right\}$$

$$F_2(q^2) = \frac{\alpha}{2\pi}\frac{\theta}{\sinh\theta} \tag{9.128}$$

where

$$\sinh^2\frac{\theta}{2} = -\frac{q^2}{4m^2}. \tag{9.129}$$

In the limit of zero transferred momenta ($q = p' - p = 0$) we get

$$\begin{cases} F_1(0) = 0 \\ F_2(0) = \dfrac{\alpha}{2\pi} \end{cases} \tag{9.130}$$

a result that we will use in section 9.3.3 while discussing the anomalous magnetic moment of the electron.

9.2 Counterterms and power counting

All that we have shown in the previous sections can be interpreted as follows. The initial Lagrangian $\mathcal{L}(e, m, \cdots)$ has been obtained from a correspondence between classical and quantum theory. It is then natural that the initial Lagrangian has to be modified by quantum corrections. The total Lagrangian is then given by,

$$\mathcal{L}_{\text{total}} = \mathcal{L}(e, m, \cdots) + \Delta\mathcal{L} \tag{9.131}$$

and

$$\Delta\mathcal{L} = \Delta\mathcal{L}^{(1)} + \Delta\mathcal{L}^{[2]} + \cdots \tag{9.132}$$

where $\Delta\mathcal{L}^{[i]}$ is the $i^{\text{th}} - loops$ correction. This also correspond to order $\hbar^i$ as counting in terms of loops is equivalent to counting in terms of $\hbar$[10]. This interpretation is quite attractive because in the limit $\hbar \to 0$ the total Lagrangian reduces to the classical one. With the Lagrangian $\mathcal{L}_{tot}$ we can then obtain finite results, although $\mathcal{L}_{tot}$ is divergent because of the counter-terms in $\Delta\mathcal{L}$.

With this language the results up to the first order in $\hbar$ can be written as,

$$\begin{aligned}
\mathcal{L}(e, m, \cdots) &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\lambda^2}{2}A^\mu A_\mu - \frac{1}{2\xi}(\partial \cdot A)^2 \\
&\quad + i\overline{\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi - e\overline{\psi}A\!\!\!/\psi \qquad (9.133) \\
\Delta\mathcal{L}^{(1)} &= -\frac{1}{4}(Z_3 - 1)F_{\mu\nu}F^{\mu\nu} + (Z_2 - 1)(i\overline{\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi) \\
&\quad + Z_2\delta m\overline{\psi}\psi - e(Z_1 - 1)\overline{\psi}A\!\!\!/\psi \qquad (9.134)
\end{aligned}$$

The Lagrangian

$$\begin{aligned}
\mathcal{L}_{\text{total}} &= -\frac{1}{4}Z_3F_{\mu\nu}F^{\mu\nu} + \frac{\lambda^2}{2}A_\mu A^\mu - \frac{1}{2\xi}(\partial \cdot A)^2 \\
&\quad + Z_2(i\overline{\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi + \delta m\overline{\psi}\psi) \\
&\quad - eZ_1\overline{\psi}A\!\!\!/\psi \qquad (9.135)
\end{aligned}$$

will give the renormalized Green's functions up the the order $\hbar$.

In fact, we have only shown that the two-point and three-point Green's functions (self-energies and vertex) were finite. It is important to verify that all the Green's

[10] $\hbar^{E-1+L} = \hbar^{\frac{E}{2}+\frac{V}{2}}$. We have the following relations $L = I - V + 1$; $3V = E + 2I$ (this only for QED).

functions, with an arbitrary number of external legs are finite, as we have already used all our freedom in the renormalization of those Green's functions. This leads us to the so-called *power counting*.

Let us consider a Feynman diagram G, with L *loops*, I_B bosonic and I_F fermionic internal lines. If there are vertices with derivatives, δ_v is the number of derivatives in that vertex. We define then the *superficial degree of divergence* of the diagram (note that $L = I_B + I_F + 1 - V$) by,

$$\begin{aligned} \omega(G) &= 4L + \sum_v \delta_v - I_F - 2I_B \\ &= 4 + 3I_F + 2I_B + \sum_v (\delta_v - 4) \end{aligned} \tag{9.136}$$

For large values of the momenta the diagram will be divergent as

$$\Lambda^{\omega}(G) \quad \text{if} \quad \omega(G) > 0 \tag{9.137}$$

and as

$$\ln \Lambda \quad \text{if} \quad \omega(G) = 0 \tag{9.138}$$

where Λ is a *cutoff*. The origin of the different terms can be seen in the following correspondence,

$$\begin{aligned} \int \frac{d^4q}{(2\pi)^4} \ (\text{for each loop}) &\rightarrow 4L \\ \partial_\mu \Leftrightarrow k_\mu &\rightarrow \delta_v \\ \frac{i}{\not{q} - m} &\rightarrow -I_F \\ \frac{i}{q^2 - m^2} &\rightarrow -2I_B \end{aligned} \tag{9.139}$$

The expression for $\omega(G)$ is more useful when expressed in terms of the number of external legs and of the dimensionality of the vertices of the theory. Let ω_v be the dimension, in terms of mass, of the vertex v, that is,

$$\omega_v = \delta_v + \#\text{campos bosónicos} \ + \ \frac{3}{2}\#\text{campos fermiónicos} \tag{9.140}$$

Then, if we denote by $f_v(b_v)$ the number of fermionic (bosonic) internal lines that join at the vertex v, we can write,

$$\sum_v \omega_v = \sum_v (\delta_v + \frac{3}{2} f_v + b_v) + \frac{3}{2} E_F + E_B \tag{9.141}$$

where $E_F(E_B)$ are the total number of *external* fermionic (bosonic) lines of the diagram. As we have,

$$I_F = \frac{1}{2} \sum_v f_v$$

$$I_B = \frac{1}{2}\sum_v b_v \tag{9.142}$$

we get

$$\sum_v \omega_v = \sum_v \delta_v + 3I_F + 2I_B + \frac{3}{2}E_F + E_B \tag{9.143}$$

Substituting in the expression for $\omega(G)$ we get finally,

$$\begin{aligned}\omega(G) =& 4 - \frac{3}{2}E_F - E_B + \sum_v(\omega_v - 4) \\ =& 4 - \frac{3}{2}E_F - E_B - \sum_v [g_v]\end{aligned} \tag{9.144}$$

where $[g_v]$ denotes the dimension in terms of mass of the coupling constant of vertex v, satisfying,

$$\omega_v + [g_v] = 4 \ . \tag{9.145}$$

From the previous expression for the superficial degree of divergence, Eq. (9.144), we can then classify theories in three classes,

i) Non-renormalizable Theories

They have at least one vertex with $\omega_v > 4$ (or $[g_v] < 0$). The superficial degree of divergence increases with the number of vertices, that is, with the order of perturbation theory. For an order high enough all the Green functions will diverge.

ii) Renormalizable Theories

All the vertices have $\omega_v \leq 4$ and at least one has $\omega_v = 4$. If all vertices have $\omega_v = 4$ then

$$\omega(G) = 4 - \frac{3}{2}E_F - E_B \tag{9.146}$$

and all the diagrams contributing to a given Green function have the same degree of divergence. Only a *finite* number of Green functions are divergent.

iii) Super-Renormalizable Theories

All the vertices have $\omega_v < 4$. Only a finite number of diagrams are divergent.[11]

Coming back to our question of knowing which are the divergent diagrams in `QED`, we can now summarize the situation in Table 9.1. All the other diagrams are superficially convergent. We have therefore a situation where there are only a finite

[11] The effective degree of divergence it is sometimes smaller than the superficial degree because of symmetries of the theory. This is what happens for gauge theories like `QED` (see Table 9.1).

E_F	E_B	$\omega(G)$	Effective degree of divergence
0	2	2	0 (Current Conservation (CC))
0	3		0 (Furry's Theorem)
0	4	0	Convergent (CC)
2	0	1	0 (Current Conservation)
2	1	0	0

Table 9.1: Superficial and effective degree of divergence in QED.

number of divergent diagrams, exactly the ones that we considered before. This analysis shows that, up to order $\hbar$, the Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{total}} &= -\frac{1}{4}Z_3 F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\lambda^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial\cdot A)^2 \\ &\quad + Z_2(i\overline{\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi + \delta m\overline{\psi}\psi) \\ &\quad - eZ_1\overline{\psi}A\!\!\!/\psi \end{aligned} \tag{9.147}$$

gives Green functions that are finite and renormalized with an arbitrary number of external legs. It remains to be shown that this Lagrangian is still valid up an arbitrary order in $\hbar$, with the only modification that the renormalization constants Z_1, Z_2, Z_3 e δm are now given by power series,

$$Z_1 = Z_1^{(1)} + Z_1^{(2)} + \cdots \tag{9.148}$$

The previous Lagrangian, Eq. (9.147), allows for another interpretation that it is also useful. The fields $A, \overline{\psi}$ and ψ are the renormalized fields that give the residues equal to 1 for the poles of the propagators and the constants e, m are the physical electric charge and mass of the electron. Let us define the non-renormalized fields $\psi_0, \overline{\psi}_0$ and A_0 and the *bare* (cutoff dependent) μ_0^2, m_0 through the definitions,

$$\begin{aligned} &\psi_0 = \sqrt{Z_2}\,\psi \qquad && m_0 = m - \delta m \\ &\overline{\psi}_0 = \sqrt{Z_2}\,\overline{\psi} \qquad && \lambda_0^2 = Z_3^{-1}\lambda^2 \\ &A_0 = \sqrt{Z_3}\,A \qquad && e_0 = Z_1 Z_2^{-1}\sqrt{Z_3^{-1}}e = \tfrac{1}{\sqrt{Z_3}}e \\ &\xi_0 = Z_3\xi \end{aligned} \tag{9.149}$$

Then the Lagrangian written in terms of the bare quantities is identical to the original Lagrangian[12]

$$\mathcal{L} = -\frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} + \frac{1}{2}\lambda_0^2\, A_{0\mu}A_0^\mu - \frac{1}{2\xi_0}(\partial\cdot A_0)^2$$

[12]The terms $\frac{\lambda^2}{2}A^2 = \frac{\lambda_0^2}{2}A_0^2$ and $\frac{1}{2\xi}(\partial\cdot A)^2 = \frac{1}{2\xi_0}(\partial\cdot A_0)^2$ are not renormalized. This a consequence of the Ward-Takahashi identities for QED. The Ward identity $Z_1 = Z_2$ is crucial for the equality $e_0 A_0 = eA$ giving a meaning to the electric charge independently of the renormalization scheme.

$$+i(\overline{\psi}_0\partial\!\!\!/\psi_0 - m_0\overline{\psi}_0\psi_0) - e_0\overline{\psi}_0 A\!\!\!/_0\psi_0 \tag{9.150}$$

Finally we notice that the bare Green functions are related to the renormalized ones by

$$\begin{aligned} &G_0^{n,\ell}(p_1,\cdots p_{2n},k_1,\cdots k_\ell,\mu_0,m_0,\ell_0,\xi_0,\Lambda) \\ &\qquad = Z_2^n(\Lambda)Z_3^{\ell/2}G_R^{n,\ell}(p_1,\cdots p_{2n},k_1\cdots k_\ell,\mu,m,e,\xi) \end{aligned} \tag{9.151}$$

where $p_1\cdots p_{2n}$ ($k_1\cdots k_\ell$) are the fermion (boson) momenta.

9.3 Finite contributions from RC to physical processes

The renormalization procedure described in the previous sections might appear, at first sight, as an artificial technique to hide the infinities. In fact this is not true, there are physical consequences of the renormalization procedure that can be verified experimentally, thus confirming the whole process. In this section we are going to look at few of these consequences.

9.3.1 Variation of α with the energy scale

Let us consider again, Coulomb scattering described in Fig. 9.5. The Ward-Takahashi, $Z_1 = Z_2$, valid in all orders in perturbation theory, ensures that the electromagnetic interaction is modified in the following way,

$$\frac{e^2}{4\pi}\frac{1}{q^2} \to \frac{e^2}{4\pi}\frac{1}{q^2\left[1+\Pi^R(q^2)\right]} \tag{9.152}$$

with the renormalization condition $\Pi^R(0) = 0$, that embodies the definition of the electric charge. Introducing $e^2 = 4\pi\alpha$ we can interpret Eq. (9.152) by saying that the electric charge depends on the scale, that is,

$$\alpha(q^2) = \frac{\alpha(0)}{1+\Pi^R(q^2)} \tag{9.153}$$

For the case $q^2 = -Q^2$, and in lowest order, we get

$$\begin{aligned} \alpha(Q^2) &= \alpha(0)\left[1-\Pi^R(-Q^2)\right] \\ &= \alpha(0) + \frac{\alpha(0)^2}{3\pi}\left[\ln\left(\frac{Q^2}{m^2}\right) - \frac{5}{3}\right] \end{aligned} \tag{9.154}$$

where we have used Eq. (9.65). The variation of the inverse of α, with the energy scale, is represented for QED in left panel of Fig. 9.10. We see that α^{-1} goes from

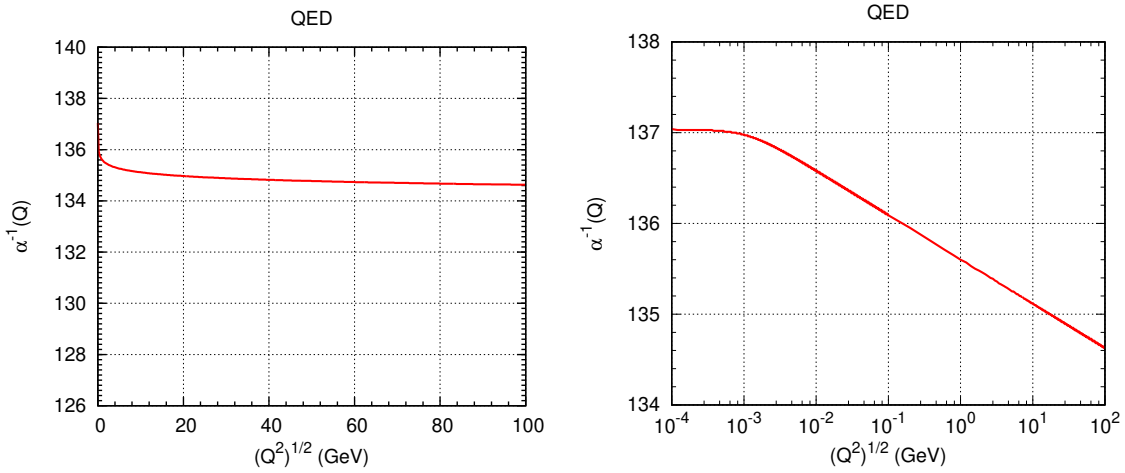


Figure 9.10: Variation of α^{-1} with energy for the case of QED. In the left pane in a linear scale while in the right panel we use a logarithmic scale.

the value 137.36 at $Q^2 = 0$ to the value 134.7 at $Q^2 = 91^2$ GeV^2. The variation is very fast for small values of Q^2. This can be seen in the right panel of Fig. 9.10 that uses a logarithmic scale for the same data. If we had included all the SM correction the value would be 128 at the scale M_Z. This effect was experimentally verified at the LEP accelerator at CERN, and it is represented in the left panel of Fig. 9.11. In the region of low k^2 the plot has an irregular behaviour. This is due to the opening of different channels and the development of an imaginary part for Π^R. In the right panel of Fig 9.11 we show this region in more detail. For a more detailed explanation see Complement 9.2.

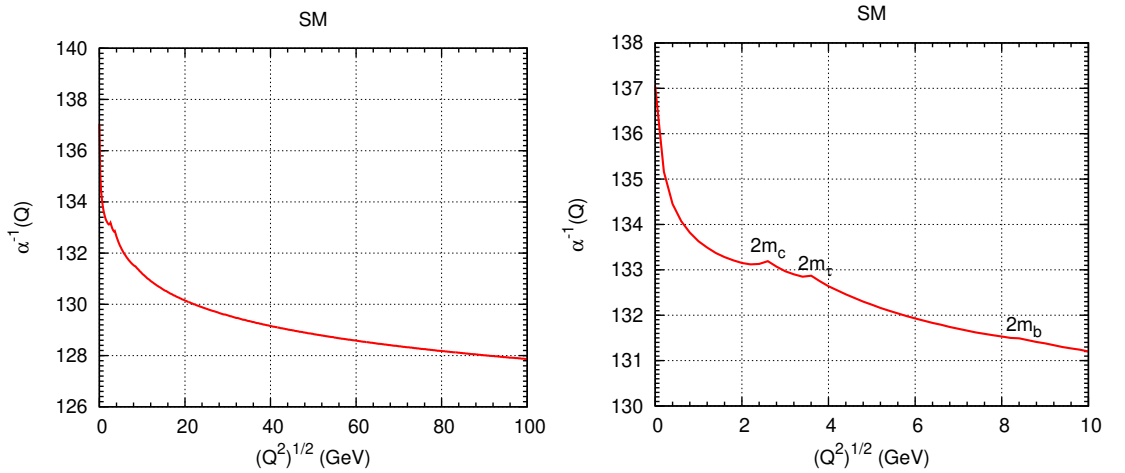


Figure 9.11: Variation of α^{-1} with energy in the SM. In the right panel we show in more detail the variation of α^{-1} in the range where the opening of various channels in the SM occurs.

We just saw that the effective coupling constant grows with the energy scale. This is a characteristic of abelian theories. For non- abelian theories, like QCD that

can have the opposite behaviour, called asymptotic freedom, that is, the coupling constant decreases with energy, making it possible to do perturbation theory in that regime.

9.3.2 Lamb shift

The difference A splitting in energy between the states $2S_{1/2}$ and $2P_{1/2}$ in the hydrogen atom was observed experimentally by Lamb e Retherford [74]. It is a very complicated problem, as there are several corrections that contribute to the final result. Here we are going just to show one of them, that is the modification of the Coulomb potential due to the vacuum polarization. For $k^2 \ll m^2$ the photon propagator is modified to,

$$\frac{-g_{\mu\nu}}{k^2} \to \frac{-g_{\mu\nu}}{k^2}\left(1 - \frac{\alpha}{15\pi}\frac{k^2}{m^2}\right) \tag{9.155}$$

where we have used the result of Eq. (9.57). Doing the inverse Fourier transform, we get the modified Coulomb potential,

$$V(r) = \frac{e}{4\pi r} + \frac{e\alpha}{15\pi m^2}\delta^3(\vec{r}) \tag{9.156}$$

The appearance of the delta function means that the new term only affects the states with $l = 0$ and therefore the sates $2S_{1/2}$ and $2P_{1/2}$ get split. However this effect is not enough the explain the magnitude of the splitting. For a complete treatment see the book of Itzykson and Zuber [32].

9.3.3 Anomalous electron magnetic moment

We will show here, for the case of the electron anomalous moment, how the finite part of the radiative corrections can be compared with experiment, given credibility to the renormalization program. In fact we will just consider the first order, while to compare with the present experimental limit one has to go to fourth order in QED and to include also the weak and QCD corrections. The electron magnetic moment is given by

$$\vec{\mu} = \frac{e}{2m} g \frac{\vec{\sigma}}{2} \tag{9.157}$$

where $e = -|e|$ for the electron. One of the biggest achievements of the Dirac equation was precisely to predict the value $g = 2$. Experimentally we know that g is close to, but not exactly, 2. It is usual to define this difference as the anomalous magnetic moment. More precisely,

$$g = 2(1 + a) \tag{9.158}$$

or

$$a = \frac{g}{2} - 1 \tag{9.159}$$

Our task is to calculate a from the radiative corrections that we have computed in the previous sections. To do that let us start to show how a value $a \neq 0$ will appear in non relativistic quantum mechanics. Schrödinger's equation for a charged particle in an exterior field is,

$$i\frac{\partial \varphi}{\partial t} = \left[\frac{(\vec{p} - e\vec{A})^2}{2m} + e\phi - \frac{e}{2m}(1+a)\vec{\sigma}\cdot\vec{B}\right]\varphi \tag{9.160}$$

Now we consider that the external field is a magnetic field $\vec{B} = \vec{\nabla}\times\vec{A}$. Then keeping only terms first order in e we get

$$\begin{aligned} H &= \frac{p^2}{2m} - e\frac{\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}}{2m} - \frac{e}{2m}(1+a)\vec{\sigma}\cdot\vec{\nabla}\times\vec{A} \\ &\equiv H_0 + H_{int} \end{aligned} \tag{9.161}$$

With this interaction Hamiltonian we calculate the transition amplitude between two electron states of momenta p and p'. We get

$$\begin{aligned} \langle p'|\, H_{int}\,|p\rangle &= -\frac{e}{2m}\int \frac{d^3x}{(2\pi)^3}\chi^\dagger e^{-i\vec{p}'\cdot\vec{x}}\left[\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p} + (1+a)\vec{\sigma}\times\vec{\nabla}\cdot\vec{A}\right]e^{i\vec{p}\cdot x}\chi \\ &= -\frac{e}{2m}\int \frac{d^3x}{(2\pi)^3}\chi^\dagger\left[(\vec{p}'+\vec{p})\cdot\vec{A} + i(1+a)\sigma^i\epsilon^{ijk}q^jA^k\right]e^{-i\vec{q}\cdot x}\chi \\ &= -\frac{e}{2m}\chi^\dagger\left[(p'+p)^k + i(1+a)\sigma^i\epsilon^{ijk}q^j\right]A^k(q)\chi \end{aligned} \tag{9.162}$$

This is the result that we want to compare with the non relativistic limit of the renormalized vertex. The amplitude is given by,

$$\begin{aligned} A &= e\overline{u}(p')(\gamma_\mu + \Lambda^R_\mu)u(p)A^\mu(q) \\ &= e\overline{u}(p')\left[\gamma_\mu(1+F_1(q^2)) + \frac{i}{2m}\sigma_{\mu\nu}q^\nu F_2(q^2)\right]u(p)A^\mu(q) \\ &= \frac{e}{2m}\overline{u}(p')\Big\{(p'+p)_\mu\left[1+F_1(q^2)\right] + i\sigma_{\mu\nu}q^\nu\left[1+F_1(q^2)+F_2(q^2)\right]\Big\}u(p)A^\mu(q) \end{aligned} \tag{9.163}$$

where we have used Gordon's identity. For an external magnetic field $\vec{B} = \vec{\nabla}\times\vec{A}$ and in the limit $q^2 \to 0$ this expression reduces to

$$\begin{aligned} A &= \frac{e}{2m}\overline{u}(p')\left\{(p'+p)_k\big[1+F_1(0)\big] + i\sigma_{kj}q^j\big[1+F_1(0)+F_2(0)\big]\right\}u(p)A^k(q) \\ &= \frac{e}{2m}\overline{u}(p')\left[-(p'+p)^k + i\Sigma^i\epsilon^{kij}q^j\left(1+\frac{\alpha}{2\pi}\right)\right]u(p)A^k(q) \end{aligned} \tag{9.164}$$

where we have used the results of Eq. (9.130),

$$\begin{cases} F_1(0) = 0 \\ F_2(0) = \dfrac{\alpha}{2\pi} \end{cases} \tag{9.165}$$

Using the explicit form for the spinors u

$$u(p) = \begin{pmatrix} \chi \\ \\ \dfrac{\vec{\sigma} \cdot (\vec{p} - e\vec{A})}{2m} \chi \end{pmatrix} \tag{9.166}$$

we can write in the non relativistic limit,

$$A = -\frac{e}{2m}\chi^\dagger \left[(p' + p)^k + i \left(1 + \frac{\alpha}{2\pi}\right) \sigma^i \epsilon^{ijk} q^j \right] \chi A^k \tag{9.167}$$

which after comparing with Eq. (9.162) leads to

$$a^e_{th} = \frac{\alpha}{2\pi} \tag{9.168}$$

This result obtained for the first time by Schwinger and experimentally confirmed, was very important in the acceptance of the renormalization program of Feynman, Dyson and Schwinger for QED.

9.3.4 Cancellation of IR divergences in Coulomb scattering

In this section we will show how the IR divergences cancel in physical processes. We will take as an example the Coulomb scattering from a fixed nucleus. This is better done if we start from first principles. Coulomb scattering corresponds to the diagram of Fig. 9.12, which gives the following matrix element for the S matrix,

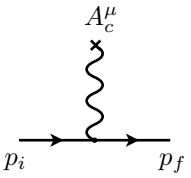


Figure 9.12: Lowest order diagram to Coulomb scattering.

$$S_{fi} = iZe^2(2\pi)\delta(E_i - E_f)\frac{1}{|\vec{q}|^2}\ \overline{u}(p_f)\gamma^0 u(p_i) \tag{9.169}$$

We will now study the radiative corrections to this result in lowest order in perturbation theory. Due to the IR divergences it is convenient to introduce a mass λ for the photon. For a classical field, as we are considering, this means a screening. If we take,

$$A_c^0(x) = Ze\frac{e^{-\lambda|\vec{x}|}}{4\pi|\vec{x}|} \tag{9.170}$$

the Fourier transform will be,

$$A_c^0(q) = Ze\frac{1}{|\vec{q}|^2+\lambda^2} \tag{9.171}$$

that shows that the screening is equivalent to a mass for the photon. With these modifications we have,

$$S_{fi} = iZe^2(2\pi)\delta(E_f - E_i)\ \frac{i}{|\vec{q}|^2+\lambda^2}\ \overline{u}(p_f)\gamma^0 u(p_i) \tag{9.172}$$

We are interested in calculating the corrections up to order e^3 in the amplitude. To this contribute[13] the diagrams of Fig. 9.13. Diagram 1 is of order e^2 while

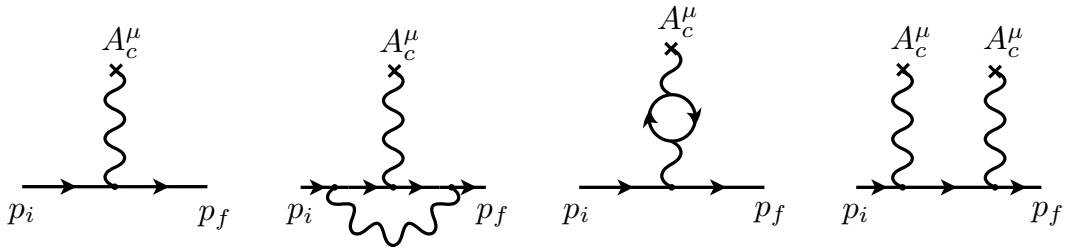


Figure 9.13: Coulomb scattering up to second order.

diagrams $2, 3, 4$ are of order e^4. Therefore the interference between 1 and $(2+3+4)$ is of order α^3 and should be added to the result of the bremsstrahlung in a Coulomb field. The contribution from $1+2+3$ can be easily obtained by noticing that

$$eA_c^\mu\gamma_\mu \to eA_c^\mu(\gamma_\mu + \Lambda_\mu^R + \Pi_{\mu\nu}^R G^{\nu\rho}\gamma_\rho) \tag{9.173}$$

where Λ_μ^R e $\Pi_{\mu\nu}^R$ have been calculated before. We get

$$\begin{aligned} S_{fi}^{(1+2+3)} &= iZe^2(2\pi)\delta(E_i-E_f)\frac{1}{|\vec{q}|^2+\lambda^2}\overline{u}(p_f)\gamma^0\left\{1+\frac{\alpha}{\pi}\left[-\frac{1}{2}\varphi\tanh\varphi\right.\right. \\ &\left(1+\ln\frac{\lambda}{m}\right)(2\varphi\coth 2\varphi-1) - 2\coth 2\varphi\int_0^\varphi \beta\tanh\beta d\beta \\ &\left.\left.+\left(1-\frac{\coth^2\varphi}{\beta}\right)(\varphi\coth\varphi-1)+\frac{1}{9}\right] - \frac{\not{q}}{2m}\frac{\alpha}{\pi}\frac{\varphi}{\sinh 2\varphi}\right\}u(p_i) \end{aligned} \tag{9.174}$$

where

[13]We do not have to consider the self-energies of the external legs of the electron because they are on-shell.

$$\frac{|\vec{q}|^2}{4m} = \sinh^2\varphi\,. \tag{9.175}$$

Finally the fourth diagram gives

$$\begin{aligned} S_{fi}^{(4)} &= (iZe)^2(e)^2 \int \frac{d^4k}{(2\pi)^4}\overline{u}(p_f)\left[\frac{2\pi\delta(E_f-k^0)}{(p_f-k)^2-\lambda^2}\gamma^0\frac{i}{\not k - m + i\varepsilon}\gamma^0\frac{2\pi\delta(k^0-E_i)}{(k-p_i)^2-\lambda^2}\right] \\ &= -2i\frac{Z^2\alpha^2}{\pi}2\pi\delta(E_f-E_i)\;\overline{u}(p_f)\left[m(I_1-I_2)+\gamma^0E_i(I_1+I_2)\right]u(p_i) \end{aligned} \tag{9.176}$$

with

$$I_1 = \int d^3k \frac{1}{[(\vec{p}_f-\vec{k})^2+\lambda^2][(\vec{p}_i-\vec{k})^2+\lambda^2][(\vec{p})^2-(\vec{k})^2+i\varepsilon]} \tag{9.177}$$

and

$$\frac{1}{2}(\vec{p}_i+\vec{p}_f)I_2 \equiv \int d^3k \frac{\vec{k}}{[(\vec{p}_f-\vec{k})^2+\lambda^2][(\vec{p}_i-\vec{k})^2+\lambda^2][(\vec{p})^2-(\vec{k})^2+i\varepsilon]}\,. \tag{9.178}$$

In the limit $\lambda \to 0$ it can be shown that

$$I_1 = \frac{\pi^2}{2ip^3\sin^2\theta/2}\ln\left(\frac{2p\sin(\theta/2)}{\lambda}\right) \tag{9.179}$$

$$I_2 = \frac{\pi^2}{2p^3\cos^2\theta/2}\left\{\frac{\pi}{2}\left[1-\frac{1}{\sin\theta/2}\right] - i\left[\frac{1}{\sin^2\theta/2}\ln\left(\frac{2p\sin\theta/2}{\lambda}\right)+\ln\frac{\lambda}{2p}\right]\right\} \tag{9.180}$$

With these expressions we get for the cross section

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{|\vec{q}|^2}\frac{1}{2}\sum_{pol}\left|\overline{u}(p_f)\Gamma u(p_i)\right|^2 \tag{9.181}$$

where

$$\Gamma = \gamma^0(1+A) + \gamma^0\frac{\not q}{2m}B + C \tag{9.182}$$

and

$$A = \frac{\alpha}{\pi}\left[\left(1+\ln\frac{\lambda}{m}\right)(2\varphi\coth 2\varphi - 1) - 2\coth 2\varphi\int_0^\varphi d\beta\beta\tanh\beta - \frac{\varphi}{2}\tanh\varphi\right.$$

$$+\left(1-\frac{1}{3}\coth^2\varphi\right)(\varphi\coth\varphi-1)+\frac{1}{9}\Big]-\frac{Z\alpha}{2\pi^2}|\vec{q}|^2E(I_1+I_2) \tag{9.183}$$

$$B = -\frac{\alpha}{\pi}\frac{\varphi}{\sinh 2\varphi} \tag{9.184}$$

$$C = -\frac{Z\alpha}{2\pi^2}m|\vec{q}|^2(I_1-I_2) \tag{9.185}$$

Therefore

$$\begin{aligned}\frac{1}{4}\sum_{pol}|\overline{u}(p_f)pu(p_i)|^2 &= \frac{1}{4}\mathrm{Tr}[\Gamma(\not{p}_i+m)\overline{\Gamma}(\not{p}_f+m)]\\ &= 2E^2(1-\beta^2\sin^2\theta/2)+2E^2 2B\beta^2\sin^2\frac{\theta}{2}\\ &+2E^2 2ReA\left(1-\beta^2\sin^2\frac{\theta}{2}\right)+2ReC(2mE)+O(\alpha^2)\end{aligned} \tag{9.186}$$

Notice that A, B e C are of order α. Therefore the final result is, up to order α^3:

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\text{elastic}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}}\Bigg\{1+\frac{2\alpha}{\pi}\Bigg[&\left(1+\ln\frac{\lambda}{m}\right)(2\varphi\coth\varphi-1)-\frac{\varphi}{2}\tanh\varphi\\ &-2\coth 2\varphi\int_0^{\varphi}d\beta\beta\tanh\beta+\left(-\frac{\coth^2\varphi}{3}\right)(\varphi\coth\varphi-1)+\frac{1}{9}\\ &-\frac{\varphi}{\sinh 2\varphi}\frac{B^2\sin^2\theta/2}{1-\beta^2\sin^2\theta/2}\Bigg]+Z\alpha\pi\frac{\beta\sin\frac{\theta}{2}[1-\sin\theta/2]}{1-\beta^2\sin^2\theta/2}\Bigg\}\end{aligned} \tag{9.187}$$

As we had said before the result is IR divergent in the limit $\lambda\to 0$. This divergence is not physical and can be removed in the following way. The detectors have an energy threshold, below which they can not detect. Therefore in the limit $\omega\to 0$ bremsstrahlung in a Coulomb field and Coulomb scattering can not be distinguished. This means that we have to add both results. If we consider an energy interval ΔE with $\lambda\le\Delta E\le E$ we get

$$\left[\frac{d\sigma}{d\Omega}(\Delta E)\right]_{BR}=\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}}\int_{\omega\le\Delta E}\frac{d^3k}{2\omega(2\pi)^3}e^2\left[\frac{2p_i\cdot p_f}{k_i\cdot p_i k\cdot p_f}-\frac{m^2}{(k\cdot p\cdot)^2}-\frac{m^2}{(k\cdot p_f)^2}\right] \tag{9.188}$$

Giving a mass to the photon (that is $\omega=(|\vec{k}|^2+\lambda^2)^{1/2}$) the integral can be done with the result,

$$\left[\frac{d\sigma}{d\Omega}(\Delta E)\right]_{BR}=\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}}\frac{2\alpha}{\pi}\left\{(2\varphi\coth 2\varphi-1)\ln\frac{2\Delta E}{\lambda}+\frac{1}{2\beta}\ln\frac{1+\beta}{1-\beta}\right.$$

$$-\frac{1}{2}\cosh 2\varphi \frac{1-\beta^2}{\beta \sin\theta/2}\int_{\cos\theta/2}^{1} d\xi \frac{1}{(1-\beta^2\xi^2)[\xi-\cos^2\theta/2]^{1/2}} \ln\frac{1+\beta\xi}{1-\beta\xi}\Bigg\} \tag{9.189}$$

The inclusive cross section can now be written as

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\Delta E) &= \left(\frac{d\sigma}{d\Omega}\right)_{\text{elastic}} + \left[\frac{d\sigma}{d\Omega}(\Delta E)\right]_{BR} \\ &= \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} (1-\delta_R+\delta_B) \end{aligned} \tag{9.190}$$

where δ_R and δ_B are complicated expressions that depend on the resolution of the detector ΔE but do not depend on λ that can be finally put to zero. One can show that in QED all the IR divergences can be treated in a similar way. One should note that the final effect of the bremsstrahlung is finite and can be important.

Complements

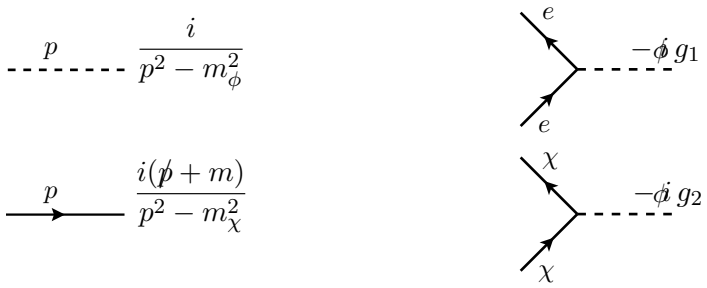
Complement 9.1 Renormalizable Theories

We said before that theories where all vertices had $\omega_v \leq 4$ are renormalizable. There is however one aspect that deserves clarification. *This is only true if the Lagrangian has all the possible interactions allowed by the symmetry.* To better understand this statement let us give an example.

Consider the theory described by the following Lagrangian,

$$\mathcal{L} = \mathcal{L}_{\rm QED} + \frac{1}{2}\partial_\mu\phi\ \partial^\mu\phi\ - \frac{1}{2}m_\phi^2\ \phi^2 + i\,\overline{\chi}\gamma^\mu\partial_\mu\chi - m_\chi\overline{\chi}\chi - g_1\,\overline{\psi}\psi\,\phi - g_2\,\overline{\chi}\chi\,\phi \qquad (9.191)$$

where ϕ is a neutral spin zero field, χ is a neutral spin 1/2 field, and ψ is the electron. The constants g_1, g_2 are dimensionless in our system of units. Besides QED the theory has the following propagators and vertices:



It is easy to verify that these new vertices have also $\omega_v = 4$. Does this mean that the theory is renormalizable? In fact it is not unless we modify the Lagrangian. Let us look at the vertex ϕ^3. It does not exist in the Lagrangian but it arises at one-loop, through the diagrams in Fig. 9.14. Now, if we look at the superficial degree of divergence

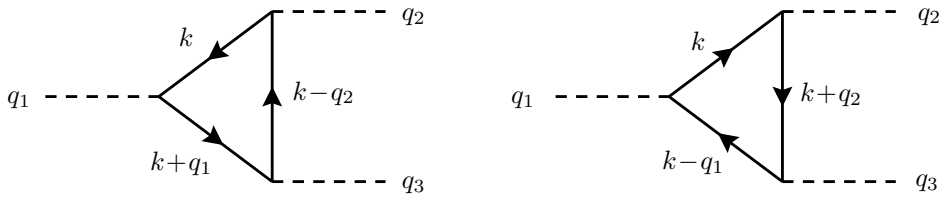


Figure 9.14: Diagrams for ϕ^3 at one loop level. The fermions in the loop are the electron and the neutral χ. All momenta are entering the diagrams.

of these diagrams we have $\omega(G) = 1$, that is they are divergent, unless they vanish by some symmetry reason (like the case of three photons in QED, Furry's theorem). Let us consider the diagram with the electron (the one with the χ will be similar). We have for the amplitude,

$$i\mathcal{M} = (-i\,g_1)^3 i^3 (-1)\left[\int \frac{d^4k}{(2\pi)^4}\frac{\mathrm{Tr}[(\not{k} + m_e)(\not{k} - \not{q}_2 + m_e)(\not{k} + \not{q}_1 + m_e)]}{[k^2 - m_e^2][(k+q_1)^2 - m_e^2][(k-q_2)^2 - m_e^2)]}\right.$$

$$+\int \frac{d^4k}{(2\pi)^4}\frac{\mathrm{Tr}[(\not{k}+m_e)(\not{k}-\not{q}_1+m_e)(\not{k}+\not{q}_2+m_e)]}{[k^2-m_e^2][(k-q_1)^2-m_e^2][(k+q_2)^2-m_e^2)]}\Bigg]$$

$$=-g_1^3\mu^{3/2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{\mathcal{N}}{[k^2-m_e^2][(k+q_1)^2-m_e^2][(k-q_2)^2-m_e^2)]} \tag{9.192}$$

where

$$\mathcal{N}=\mathrm{Tr}[(\not{k}+m_e)(\not{k}-\not{q}_2+m_e)(\not{k}+\not{q}_1+m_e)]+\mathrm{Tr}[(-\not{k}+m_e)(-\not{k}-\not{q}_1+m_e)(-\not{k}+\not{q}_2+m_e)] \tag{9.193}$$

and we have made, as usual, the change of variables $k\to -k$ in the second integral to reduce to a common denominator. A simple calculation gives

$$\mathcal{N}=8m_e\left[3k^2+2k\cdot q_1-2k\cdot q_2-q_1\cdot q_2+m_e^2\right] \tag{9.194}$$

showing that the result is divergent because of the term in k^2. Using `FeynCalc` we obtain

$$\begin{aligned}\mathcal{M}=&g_1^3m_e\mu^{3/2\epsilon}\left[8B_0(q_1^2,m_e^2,m_e^2)+8B_0(q_2^2,m_e^2,m_e^2)-40B_0(q_3^2,m_e^2,m_e^2)\right.\\&\left.-4\left(3q_1^2+3q_2^2-q_3^2+8m_e^2\right)C_0(q_1^2,q_2^2,q_3^2,m_e^2,m_e^2,m_e^2)\right]\end{aligned} \tag{9.195}$$

The Passarino-Veltman function C_0 is convergent, but B_0 diverges as

$$\mathrm{Div}[B_0]=\Delta_\epsilon=\frac{2}{\epsilon}-\gamma+\ln 4\pi \tag{9.196}$$

Therefore we have for the divergent part of the amplitude

$$\mathrm{Div}[\mathcal{M}]=-24g_1^3m_e\mu^{3/2\epsilon}\,\Delta_\epsilon \tag{9.197}$$

The diagram with the χ gives the same result with $g_1\to g_2$ and $m_e\to m_\chi$.
Now this is a problem because as there is no ϕ^3 in the Lagrangian, there will be no counterterm to absorb this infinity. The problem is that such a term is not forbidden by the symmetry and therefore should be added to the Lagrangian, along with the corresponding ϕ^4 term for the theory to be renormalizable. *Hence the rule that you should have in your Lagrangian all the the terms that are compatible with the symmetry of the theory*, otherwise you might get in trouble. For this case, the theory is renormalizable with the Lagrangian

$$\begin{aligned}\mathcal{L}=&\mathcal{L}_{\mathrm{QED}}+\frac{1}{2}\partial_\mu\phi\;\partial^\mu\phi\;-\frac{1}{2}m_\phi^2\;\phi^2+i\,\overline{\chi}\gamma^\mu\partial_\mu\chi-m_\chi\overline{\chi}\chi-g_1\,\overline{\psi}\psi\,\phi-g_2\,\overline{\chi}\chi\,\phi\\&-\frac{\mu}{3!}\phi^3-\frac{\lambda}{4!}\phi^4\;.\end{aligned} \tag{9.198}$$

Complement 9.2 Variation of α^{-1} with the energy in the SM

To get the correct result for the SM, it is necessary to consider all the charged particles that can circulate in the *loop*. Besides we have to see in which of k^2 we want to work. At LEP we had e^-e^+ collisions and therefore the main process corresponds top the exchange of a photon in the s-channel. Therefore we should have

$$\alpha^{-1}(k^2)=\alpha^{-1}(0)\;\left[1+\sum_f\Pi^R(k^2,m_f)\right] \tag{9.199}$$

where $\Pi^R(k^2, m_f)$ is given in Eqs. (9.56), (9.61) or (9.64), depending in the kinematic region. For LEP, $k^2 > 0$ and for all the fermions, except the top quarks, we should use Eq. (9.61), that is, Π^R is a complex function. To be able to make a plot for $0 < k^2 < M_Z$ it is better to use the expressions for $\Pi^R(k^2)$ written in terms of the Passarino-Veltman functions [75] as explained in Ref. [12]. We get

$$\Pi^R(k^2, m^2) = \frac{\alpha}{3\pi}\left[-\frac{1}{3} + \left(1 + \frac{2m^2}{k^2}\right)\left(B_0(k^2, m^2, m^2) - B_0(0, m^2, m^2)\right)\right] \tag{9.200}$$

that covers all the cases discussed before. These functions can be numerically evaluated with the help of the software `LoopTools` [76] that implements the code previously developed by Oldenborg [77]. The result is shown in Fig. 9.11.

$$\alpha^{-1}(k^2) = \alpha^{-1}(0)\left|1 + \sum_f \Pi^R(k^2, m_f)\right|. \tag{9.201}$$

Problems

9.1 In QED consider the *1-loop* process with 3 photons in the external lines. The amplitude for the process can be expressed as,

$$\mathcal{M} = \epsilon^{\mu}(k_1)\epsilon^{\nu}(k_2)\epsilon^{\rho}(k_3)\,\mathcal{M}_{\mu\nu\rho}$$

a) Draw the diagrams that contribute to the process.

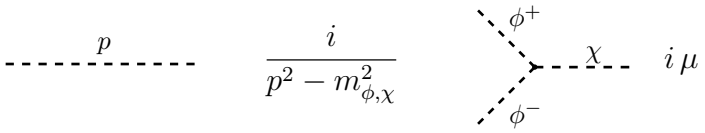
b) Write the expression for the amplitude $\mathcal{M}$.

c) Show that $\mathcal{M} = 0$.
Notes:
i) You do not have to do the integrations, not even the traces.
ii) Assume that you have some regularization so that you can change variables in the integrals.
iii) The result of Theorem 4.6 of section 4.4 is crucial.

9.2 Consider the theory described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\chi\,\partial^\mu\chi + \partial_\mu\phi^+\,\partial^\mu\phi^- \; - \frac{1}{2}m_\chi^2\,\chi^2 - m_\phi^2\,\phi^+\phi^- + \mu\,\phi^+\phi^-\,\chi$$

where χ is a spin 0 neutral scalar field and $\phi^\pm$ is a complex scalar field that has a charge as indicated. This charge corresponds to some internal symmetry and it is not the electric charge, so there is no interaction with photons. The constant μ has dimensions of a mass. The propagators and the only vertex are,



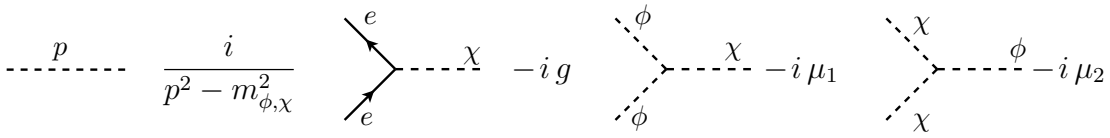
At the vertex the particles are entering the vertex. Note that a $\phi^\pm$ entering corresponds to a $\phi^\mp$ leaving the vertex. Draw the diagrams at one *loop* for the propagators of the fields ϕ e χ and for the vertex. Consider only the *One Particle Irreducible* diagrams, that is, those that do not split in two separated diagrams by the cut

of just one line in the diagram. Without evaluating discuss the divergence of the diagrams.

9.3 Consider the theory described by the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\rm QED} + \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}\partial_\mu\chi\,\partial^\mu\chi - \frac{1}{2}m_\phi^2\,\phi^2 - \frac{1}{2}m_\chi^2\,\chi^2 - \frac{1}{2}\mu_1\,\phi^2\chi - \frac{1}{2}\mu_2\,\phi\chi^2 - g\,\overline{\psi}\psi\,\chi$$

where ϕ and χ are neutral scalar fields and ψ is the electron. The constant g is dimensionless (in the system $\hbar = c = 1$) and the constants μ_1, μ_2 have dimensions of mass. Besides QED, the new propagators and vertices are,



Consider the radiative corrections at one *loop* in this model. In all the questions consider only diagrams one particle irreducible. You do not have to do any calculations.

a) Draw the diagrams for the self energy of the electron at one *loop*.

b) Draw the diagrams for the self energy of the scalar χ at one *loop*.

c) Draw the diagrams for the corrections to the vertex $\overline{\psi}\psi\chi$ at *loop*. Discuss the superficial degree of divergence.

d) Draw the one *loop* diagrams for the vertex $\overline{\psi}\psi\phi$. Discuss the superficial degree of divergence.

e) Is the theory renormalizable? Justify your answer.

Appendix A

Revisions: Special Relativity and Quantum Mechanics

Students taking their first QFT course usually arrive with a variety of backgrounds and skills. This appendix has several objectives. It is intended to set up the notation. It also intends to remind the reader who has learned these subjects, of what one is expected to know. This helps students recognize which topics they should brush up upon before continuing. Lastly, some students may not have seen these subjects treated with this notation before (for example, we have had some students previously unfamiliar with the 4-vector notation). For those, this provides a brief review, and we will include a number of references for further reading that they should use to become fully familiar with the prior material.

[**QE:**] Stands for "Quick Exercise", and it is supposed to be done as you read the text.

A.1 Special Relativity in 4-vector notation

Consider figure A.1, where the primed reference frame moves with velocity $\vec{v}$ with respect to the unprimed reference frame along the z axis. The Lorentz transformation

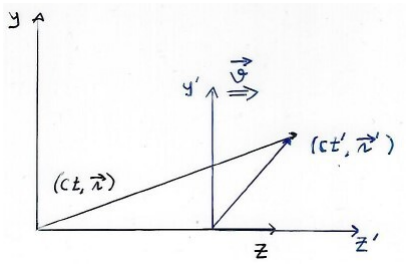


Figure A.1: Prime reference frame, moving with velocity v along the z direction with respect to the unprimed reference frame.

(LT) is

$$\left\{ \begin{array}{l} ct' = \gamma\,(ct - \beta z) \\ z' = \gamma\,(z - \beta\, ct) \end{array} \right. \iff \left\{ \begin{array}{l} t' = \gamma\left(t - \frac{v}{c^2} z\right) \\ z' = \gamma\,(z - vt) \end{array} \right. , \tag{A.1}$$

with $x' = x$ and $y' = y$, and where

$$\vec{\beta} = \beta\hat{\beta} = \vec{v}/c, \quad \beta = v/c, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}\,. \tag{A.2}$$

The transformation (A.1) may be written in matrix form

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \Lambda \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} . \tag{A.3}$$

In natural units ($c = 1$),

$$\Lambda = \begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix} . \tag{A.4}$$

We define a *contravariant* 4-vector in Minkowski space as any vector[1] which transforms as in (A.1). Examples include the position (in the generalized sense of time/3-vector) and momentum (in the generalized sense of energy/3-momentum) vectors as

$$\begin{aligned} x^\mu = (ct, \vec{r}) &==(x^0, x^1, x^2, x^3) = (x^0, \vec{x})\,, \\ p^\mu = (E/c\vec{p}) = \gamma m(c, \vec{v}) &==(p^0, p^1, p^2, p^3) = \gamma m(1, \vec{\beta})\,, \end{aligned} \tag{A.5}$$

where == holds in natural units, and a number of notations present in the literature have been shown. Greek letters (α, β, ...) take values $0, 1, 2, 3$, while Roman letters take only space values $1, 2, 3$. Because the LT is linear and homogeneous, we may write it as

$$x'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha}\, x^\alpha = \Lambda^\mu_{\ \alpha}\, x^\alpha\,. \tag{A.6}$$

We will use the convention that repeated indices are summed over. Another important 4-vector is the 4-current

$$J^\mu = (\rho, \vec{J})\,, \tag{A.7}$$

where ρ is the charge (or probability) density, and $\vec{J}$ is the charge (or probability) current.

[1] When there is no confusion, we will refer to 4-vectors simply as vectors.

We also define the *covariant* 4-vector

$$x_\mu = (x_0, x_1, x_2, x_3) = (x^0, -x^1, -x^2, -x^3) = (t, -\vec{x})\,. \tag{A.8}$$

Using the 4×4 metric tensor

$$g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)\,, \tag{A.9}$$

it is easy to show that

$$x_\mu = g_{\mu\nu} x^\nu\,. \tag{A.10}$$

Thus, the metric can be used to transform a contravariant vector into a covariant vector (so-named, lower the index) or vice-versa (raise the index). Moreover

$$g_\mu{}^\nu = g_{\mu\beta} g^{\beta\nu} = \delta_\mu{}^\nu \tag{A.11}$$

where $\delta_\mu{}^\nu$ is the Kronecker delta, vanishing when $\mu \neq \nu$ and yielding $+1$ when $\mu = \nu$.

Using (A.3)-(A.4), the transformation of a covariant matrix could be written in matrix form as

$$\begin{bmatrix} t' \\ -x' \\ -y' \\ -z' \end{bmatrix} = \Lambda_{\mathrm{aux}} \begin{bmatrix} t \\ -x \\ -y \\ -z \end{bmatrix}\,, \tag{A.12}$$

with

$$\Lambda_{\mathrm{aux}} = \begin{bmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{bmatrix} = \Lambda^{-1}\,. \tag{A.13}$$

Thus,

$$x'_\mu = \left(\Lambda^{-1}\right)^\beta_{\cdot\mu}\, x_\beta = \frac{\partial x^\beta}{\partial x'^\mu}\, x_\beta\,. \tag{A.14}$$

A generic contravariant vector transforms similarly to (A.6) as

$$A'^\mu = \Lambda^\mu_{\cdot\alpha}\, A^\alpha\,, \tag{A.15}$$

while a generic covariant vector transforms similarly to (A.14) as

$$B'_\mu = \left(\Lambda^{-1}\right)^\beta_{\cdot\mu}\, B_\beta\,. \tag{A.16}$$

Thus,

$$A.B = A_\mu B^\mu \tag{A.17}$$

is Lorentz invariant. [**QE:** Find the expressions for $p.x$ and $p^2 = p.p$.]

Consider the distance between two spacetime "points"

$$\Delta x^2 = \Delta x.\Delta x = \Delta x^\mu \Delta x_\mu = c^2 \Delta t^2 - |\Delta\vec{r}|^2\,. \tag{A.18}$$

Since no signal can travel faster than the speed of light c, $\Delta x^2 > 0$ (which implies $c > |\Delta\vec{r}|/\Delta t$), implies that the two events can be causally connected. Likewise, $\Delta x^2 < 0$ (which implies $c < |\Delta\vec{r}|/\Delta t$), implies that there can be no causal connection between the two events. In the limiting case $\Delta x^2 = 0$, only signals (eg., massless particles) traveling at the speed of light can connect the two events. By analogy with the space-time 4-vector x^μ, we use for any 4-vector the nomenclature

$$\begin{array}{ll} A^2 > 0 & \text{time-like} \\ A^2 = 0 & \text{light-like} \\ A^2 < 0 & \text{space-like} \end{array} \quad . \tag{A.19}$$

Comparing the chain rule

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\beta}{\partial x'^\mu}\,\frac{\partial}{\partial x^\beta} \tag{A.20}$$

with (A.14), we conclude that the derivative with respect to a contravariant variable transforms as a covariant vector operator, and we write

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = (\partial_0, \vec{\nabla})\,. \tag{A.21}$$

This should be compared with (A.8). Likewise,

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = (\partial_0, -\vec{\nabla})\,, \tag{A.22}$$

which should be compared with the transformation for x^μ (A.6). Notice the difference in signs of the space components in (A.5) and (A.22). The D'Alembertian can be written in Lorentz invariant form as

$$\Box = \partial^2 = \partial_\mu\partial^\mu = \partial_0^2 - (\vec{\nabla})^2\,. \tag{A.23}$$

One can now define tensors with multiple contravariant and/or covariant indices, which transform as (A.15) and (A.16), respectively. We can now understand the placement of the lower index α in $\Lambda^\mu_{\ .\alpha}$ of (A.6).

Using exercise REFneeded, one can show that

$$\widetilde{dp} \equiv \frac{d^3p}{(2\pi)^3\,2E_p}\,, \tag{A.24}$$

and

$$\widetilde{\delta}(p-q) \equiv (2\pi)^3\,2E_p\,\delta^3(\vec{p}-\vec{q})\,, \tag{A.25}$$

where $E_p \equiv +\sqrt{|\vec{p}|^2+m^2}$, are Lorentz invariant. Consider a function $f(p)$ of the 4-vector p. Assuming that p^0 is always to be taken as the positive solution $p^0 = E_p \equiv +\sqrt{|\vec{p}|^2+m^2}$, then

$$\int \widetilde{dp}\,f(p)\,\widetilde{\delta}(p-q) = f(q)\,. \tag{A.26}$$

A.1.1 Rapidity

Defining the rapidity η as

$$\beta = \tanh \eta \,, \tag{A.27}$$

one finds

$$\gamma = \cosh \eta \,, \qquad \beta\gamma = \sinh \eta \,. \tag{A.28}$$

Thus, the LT may be written as

$$\begin{cases} ct' = \cosh(\eta)\ ct - \sinh(\eta)\ z \\ z' = -\sinh(\eta)\ ct + \cosh(\eta)\ z \end{cases} . \tag{A.29}$$

Thenceforth, we will use natural units $c = 1$ (and $\hbar = 1$) and define the contravariant 4-vectors $x^\mu = (x^0, x^1, x^2, x^3) = (t, \vec{x})$ and $p^\mu = (p^0, p^1, p^2, p^3) = (E, \vec{p})$. For a generic boost in the direction $\hat{\eta}$, define $\vec{\eta} = \eta\hat{\eta}$ where $\eta = \text{arctanh}(\beta)$. The special relativity relations

$$E = m\gamma \,, \quad \text{and} \quad \vec{p} = m\gamma\vec{\beta} \,, \tag{A.30}$$

can be written as

$$E = m\cosh\eta \,, \qquad \vec{p} = m\hat{\eta}\sinh\eta \,. \tag{A.31}$$

A.1.2 Warning: passive versus active transformations

Thus far, we have used the convention common in elementary Physics, in which we move the referential, as in figure A.1. This is called the passive convention. This will lead us to eqs. (??) and (??). Maggiore [78] and Peskin&Schroeder [79] use the opposite (active) convention, in which one keeps the referential and (actively) moves the point position. Thus, they get opposite signs for the angles and boosts. In particular, the (-) signs of the sin/sinh terms in eqs. (??) and (??) become (+) signs. For them, eqs. (??)-(??), which have the opposite signs, hold.

A.1.3 Electromagnetism in covariant notation

Consider the 4-vector $A^\mu = (\phi, \vec{A})$, where ϕ and $\vec{A}$ are the electromagnetic scalar and vector potentials, respectively, from which the electromagnetic fields are obtained as

$$\begin{aligned} \vec{B} &= \vec{\nabla} \times \vec{A} \,, \\ \vec{E} &= -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi \,. \end{aligned} \tag{A.32}$$

The Maxwell tensor is

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \,, \tag{A.33}$$

which is antisymmetric in $\mu \leftrightarrow \nu$. Looking at each component,

$$F^{0k} = -E^k\,, \qquad F^{ij} = -\epsilon^{ijk}B^k\,, \tag{A.34}$$

meaning that

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}. \tag{A.35}$$

Thus, we can obtain the inhomogeneous Maxwell equations

$$\begin{aligned} \vec{\nabla}.\vec{E} =& \rho\,, \\ \vec{\nabla}\times\vec{B} =& \vec{J} + \frac{\partial\vec{E}}{\partial t}\,, \end{aligned} \tag{A.36}$$

from

$$\partial_\mu F^{\mu\nu} = J^\nu\,, \tag{A.37}$$

which implies trivially the continuity equation ("local charge conservation"):

$$0 = \partial_\nu J^\nu = \frac{\partial\rho}{\partial t} + \vec{\nabla}.\vec{J}\,. \tag{A.38}$$

[**QE:** Show this.]

Consider the dual tensor

$$\mathcal{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\,F_{\alpha\beta}. \tag{A.39}$$

As a consequence of the antisymmetry of $F^{\mu\nu}$,

$$\partial_\mu\mathcal{F}^{\mu\nu} = 0\,, \tag{A.40}$$

which is equivalent to the Bianchi identity

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0\,. \tag{A.41}$$

Writing (A.40) in terms of fields, we recover the homogeneous Maxwell equations

$$\begin{aligned} \vec{\nabla}.\vec{B} =& 0\,, \\ \vec{\nabla}\times\vec{E} =& -\frac{\partial\vec{B}}{\partial t}\,. \end{aligned} \tag{A.42}$$

A gauge transformation is induced by any function of spacetime $\lambda(x)$ as

$$A^\mu \to A^\mu + \partial^\mu\lambda\,. \tag{A.43}$$

The Maxwell tensor in (A.33) is gauge invariant. [**QE:** Show this.] Given this gauge invariance, one can perform calculations in any gauge of choice. Two common choices are

$$\partial_\mu A^\mu = 0, \qquad \text{Lorentz gauge}\,, \tag{A.44}$$

$$A^0 = 0\,, \quad \vec{\nabla}.\vec{A} = 0 \qquad \text{Radiation gauge}\,. \tag{A.45}$$

A.1.4 Relativistic kinematics

We refer to energy conservation and (3-)momentum conservation collectively as "momentum" conservation.

Consider the kinematics of Compton scattering

$$e^-(p) \; + \; \gamma(k) \to e^-(p') \; + \; \gamma(k') \,, \tag{A.46}$$

in the Laboratory (Lab) frame, where the initial electron is at rest and the outgoing photon's direction makes an angle θ with the incoming photon's direction. The letters in parenthesis in (A.46) indicate our notation for the respective momenta. Without loss of generality, one can write

$$p = (m, \vec{0}) \,, \qquad\qquad p' = (E', \vec{p}') \,, \tag{A.47}$$

$$k = (k, 0, 0, k) \,, \qquad\qquad k' = (k', k' \sin\theta, 0, k' \cos\theta) \,, \tag{A.48}$$

where $\vec{k} = \vec{p}' + \vec{k}'$ and m is the mass of the electron. Notice the two different uses of k: as the 4-vector, and as the magnitude of the 3-vector (ditto for k'). The exact meaning is always clear from the context.

Using conservation of 4-momentum one can show that

$$k' = \frac{k}{1 + \frac{k}{m}(1 - \cos\theta)} \,, \tag{A.49}$$

which is easily transformed into the usual Compton formula for the shift in wavelength

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta) \,. \tag{A.50}$$

[**QE:** Show (A.49) and (A.50).] The quantity

$$\lambda_c = \frac{h}{mc} \,, \tag{A.51}$$

(or, sometimes, $\hbar/(mc)$) is the Compton wavelength. Up to a factor of two, this is the wavelength of a photon with enough energy to create an electron-positron pair, and can be taken as a rough estimate for the regime where the need for QFT arises.

Take $a, b, c > 0$ and the triangle function

$$\lambda(a,b,c) = a^2 + b^2 + c^2 - 2(ab + ac + bc) = \left[a - \left(\sqrt{b} + \sqrt{c}\right)^2\right] \left[a - \left(\sqrt{b} - \sqrt{c}\right)^2\right] \,, \tag{A.52}$$

where the equality obviously holds with any permutation of a, b, c. One can show that $\lambda \leq 0$ if and only if it is possible to form a triangle with sides of length $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$. [**QE:** Show this.] This function is very useful in $1 \to 2$ decays and $2 \to 2$ scattering processes.

Consider a particle of mass m_0 at rest decaying into particles of mass m_1 and m_2. In m_0's rest frame, the 3-momenta of the daughter particles have magnitude [**QE:** Show this.]

$$|\vec{p}_{\text{final}}| = \frac{\sqrt{\lambda(m_0^2\,,\,m_1^2\,,\,m_2^2)}}{2m_0} = \frac{1}{2m_0}\sqrt{\left[m_0^2-(m_1+m_2)^2\right]\left[m_0^2-(m_1-m_2)^2\right]}. \tag{A.53}$$

Consider the $2\to 2$ process $A\,B\ \to\ C\,D$ and the Mandelstam variables

$$\begin{aligned} s =&(p_A+p_B)^2,\\ t =&(p_A-p_C)^2,\\ u =&(p_A-p_D)^2. \end{aligned} \tag{A.54}$$

Clearly, $s+t+u=\sum_\alpha m_\alpha^2$. Also, $\sqrt{s}$ coincides with the system's energy (initial and final) in the center of momentum frame (CM), where

$$\vec{p}_A+\vec{p}_B=0=\vec{p}_C+\vec{p}_D. \tag{A.55}$$

Moreover,

$$\begin{aligned} \eta_i \equiv&4s\,|\vec{p}_i|^2=\lambda(s,m_A^2,m_B^2)\,,\\ \eta_f \equiv&4s\,|\vec{p}_f|^2=\lambda(s,m_C^2,m_D^2)\,, \end{aligned} \tag{A.56}$$

where $|\vec{p}_i|=|\vec{p}_A|=|\vec{p}_B|$ and $|\vec{p}_f|=|\vec{p}_C|=|\vec{p}_D|$. [**QE:** Show all results mentioned after (A.54).]

Using the notation in figure A.2, we can show that, except for the angle θ, all

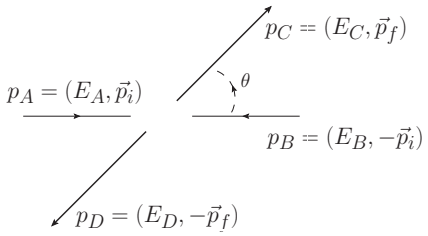


Figure A.2: Notation for $2\to 2$ scattering in the CM frame. θ is the angle between the directions of the incoming $\vec{p}_A$ and the outgoing $\vec{p}_C$.

momenta and energies are fixed by energy-momentum conservation. Namely [**QE:** Show this.]:

$$\begin{aligned} p_A =&\frac{1}{2\sqrt{s}}\left(s+m_A^2-m_B^2\,,\,0\,,\,0\,,\,\sqrt{\eta_i}\right)\,,\\ p_B =&\frac{1}{2\sqrt{s}}\left(s-m_A^2+m_B^2\,,\,0\,,\,0\,,\,-\sqrt{\eta_i}\right)\,, \end{aligned}$$

$$p_C = \frac{1}{2\sqrt{s}} \left(s + m_C^2 - m_D^2 \,,\, \sqrt{\eta_f} \sin\theta \,,\, 0 \,,\, \sqrt{\eta_f} \cos\theta \right) \,,$$
$$p_D = \frac{1}{2\sqrt{s}} \left(s - m_C^2 + m_D^2 \,,\, -\sqrt{\eta_f} \sin\theta \,,\, 0 \,,\, -\sqrt{\eta_f} \cos\theta \right) \,. \tag{A.57}$$

For future reference (when performing the integration over the available final state phase space), we note that

$$d\,t = 2\,|\vec{p}_f|\,|\vec{p}_i|\,d\cos\theta = |\vec{p}_f|\,|\vec{p}_i|\,\frac{d\Omega}{\pi}\,, \tag{A.58}$$

where

$$\int\int d\Omega = \int\int \sin\theta d\theta\, d\varphi = \int 2\pi\, d\cos\theta\,. \tag{A.59}$$

In the extreme relativistic limit (where energies are far larger than any masses), one has [**QE:** Show this. Do not use previous results; start from scratch.]

$$p_A = (p, \vec{p}_i)\,, \qquad p_B = (p, -\vec{p}_i)\,,$$
$$p_C = (p, \vec{p}_f)\,, \qquad p_D = (p, -\vec{p}_f)\,, \tag{A.60}$$

where $p = |\vec{p}_i| = |\vec{p}_f|$. Also

$$p = \frac{\sqrt{s}}{2},$$
$$t = -\frac{s}{2}(1 - \cos\theta),$$
$$u = -\frac{s}{2}(1 + \cos\theta), \tag{A.61}$$

where θ is the CM scattering angle and $p = |\vec{p}_i| = |\vec{p}_f|$ is the common magnitude of the 3-momenta of the initial and final particles in the CM frame. In this particular case, t and u are always negative or zero.

A.2 Quantum Mechanics

The Einstein $E = h\nu$ and de Broglie $p = h/\lambda$ equations relate the energy E with the frequency ν, and the momentum p with the wavelength λ. They may be written as

$$E = \hbar\omega \qquad \vec{p} = \hbar\vec{k}\,, \tag{A.62}$$

or, in covariant notation

$$p^\mu = \hbar k^\mu == k^\mu\,, \tag{A.63}$$

meaning that there is no distinction between energy-momentum and frequency-wavenumber when using natural units $\hbar = 1$.

Given the classical relation between energy and momentum, one can find a quantum mechanical equation (to be applied to a wave function) by the substitutions

$$E \to i\frac{\partial}{\partial t}\,, \qquad \vec{p} \to -i\vec{\nabla}\,, \tag{A.64}$$

which can be condensed in 4-vector notation as

$$p^\mu \to i\partial^\mu\,. \tag{A.65}$$

A.2.1 Schrodinger equation

Starting from the non-relativistic relation

$$E = \frac{p^2}{2m} + V\,, \tag{A.66}$$

and using (A.64), we find Schrodinger's equation

$$i\frac{\partial}{\partial t}\psi(\vec{x},t) = \hat{H}\psi(\vec{x},t) = -\frac{\vec{\nabla}^2}{2m}\psi(\vec{x},t) + V\psi(\vec{x},t)\,, \tag{A.67}$$

with the Hamiltonian operator defined as

$$\hat{H} = -\frac{\vec{\nabla}^2}{2m} + V\,. \tag{A.68}$$

Rarely, we will stress that some quantities are operators by including a hat over them. We look for fixed frequency solutions of the type

$$\psi(\vec{x},t) = e^{-i\omega t}\phi(\vec{x})\,. \tag{A.69}$$

Substituting into (A.67), we find

$$\hat{H}\,\phi(\vec{x}) = E\,\phi(\vec{x})\,, \tag{A.70}$$

with $E = \omega$, which is known as the time-independent Schrodinger equation. Eq. (A.70) shows the correspondence

$$\begin{aligned} e^{-i\omega t} &\longrightarrow \text{positive energy solutions}\,,\\ e^{+i\omega t} &\longrightarrow \text{negative energy solutions}\,. \end{aligned} \tag{A.71}$$

Consider the free ($V = 0$) Schrodinger equation. In this case $\hat{H}$ commutes with the momentum operator $\hat{P} = -i\vec{\nabla}$ because $\hat{H} = \hat{P}^2/(2m)$. The common eigenstates are

$$\phi_{\vec{k}}(\vec{x}) = e^{i\vec{k}.\vec{x}}\,. \tag{A.72}$$

Thus, the free solution is the plane wave

$$\psi = N e^{i(\vec{k}.\vec{x}-\omega t)} \,. \tag{A.73}$$

Since, for this plane wave

$$\omega\,\psi = i\frac{\partial}{\partial t}\,\psi = -\frac{\vec{\nabla}^2}{2m}\,\psi = -\frac{|\vec{k}|^2}{2m}\,\psi\,, \tag{A.74}$$

we conclude that the eigenvalue of the energy operator $E_{\vec{p}} = \omega$ and the eigenvalue of the momentum operator $\vec{p} = \vec{k}$ are related by

$$E_{\vec{p}} = -\frac{|\vec{p}|^2}{2m}\,, \tag{A.75}$$

showing that a complete set of states involves only *positive energy solutions*.

Rewriting Schrodinger's equation (A.67) and multiplying by the complex conjugated wavefunction (ψ^*), we obtain

$$0 = \psi^*\frac{\partial\psi}{\partial t} - i\psi^*\frac{\vec{\nabla}^2}{2m}\,\psi\,, \tag{A.76}$$

with complex conjugate

$$0 = \psi\frac{\partial\psi^*}{\partial t} + i\psi\frac{\vec{\nabla}^2}{2m}\,\psi^*\,. \tag{A.77}$$

The sum of the last two equations may be written as [**QE:** Show this.]

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}.\vec{J} = 0\,, \tag{A.78}$$

where

$$\rho = |\psi|^2\,, \quad \vec{J} = \frac{-i}{2m}\left[\psi^*\,\vec{\nabla}\psi - \psi\,\vec{\nabla}\psi^*\right]\,. \tag{A.79}$$

Since ρ is interpreted as a probability density, $\vec{J}$ is a probability current and (A.78) is the probability's continuity equation, which should be compared with the continuity equation in electromagnetism (A.38). More importantly, ρ is always positive and, thus, any probability of the type

$$\text{Prob} = \int |\psi|^2 > 0\,. \tag{A.80}$$

The probability is normalized such that, integrating over all space the (total) probability yields 1. If multiplied by the number of particles, then ρ can be interpreted as a number density and (A.80) gives the total number of particles inside the integrated region. Integrating (A.78) over some finite volume V, bounded by a close surface S, we find

$$-\frac{\partial}{\partial t}\int_V \rho\,dV = \int_V \vec{\nabla}.\vec{J}\,dV = \int_S \vec{J}.\vec{n}\,dS\,, \tag{A.81}$$

where $\vec{n}$ is the unit vector normal to the surface S. The interpretation of this result is that the rate of decrease of the number of particles within the volume V equals the total flux of particles leaving through the surface S. Thus, probability (number of particles) is conserved.

For a plane wave

$$\begin{aligned} \psi(\vec{x},t) =& Ne^{i(\vec{k}.\vec{x}-\omega t)}\,, \\ \rho =& |N|^2\,, \\ J =& |N|^2\frac{\vec{k}}{m}\,. \end{aligned} \tag{A.82}$$

Noting that $\vec{k}/m = \vec{v}$ is a velocity, the analogy with the electromagnetic relation between charge density and current is perfect.

A.2.2 Klein-Gordon equation

We retrace the steps that led to the Schrodinger equation, but start from the relativistic relation between momentum and energy:

$$E^2 - |\vec{p}|^2 = m^2 \tag{A.83}$$

Performing the substitutions (A.64), we obtain the Klein-Gordon (KG) equation

$$-\frac{\partial^2\varphi}{\partial t^2} + \vec{\nabla}^2\varphi = m^2\varphi\,. \tag{A.84}$$

It admits the plane wave solution

$$\varphi(\vec{x},t) = Ne^{i(\vec{k}.\vec{x}-\omega t)}\,. \tag{A.85}$$

Indeed, direct substitution of (A.85) in (A.84) yields $\omega^2 - |\vec{k}|^2 = m^2$. Thus, the energy and momentum eigenvalues are related by

$$E = \pm\sqrt{|\vec{p}|^2 + m^2}\,, \tag{A.86}$$

meaning that we *need both positive and negative energies* in order to get a complete set of states. But, the negative energies are unbounded from below, and the theory has no minimum. Moreover, such negative energies correspond to negative probability densities (and negative probabilities) as we will now show.

We start by multiplying (A.84) by $-i\varphi^*$. Then, we complex conjugate (A.84) and multiply it by $i\varphi$. Finally we add the two results, finding

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}.\vec{J} = 0\,, \tag{A.87}$$

where

$$\rho = i\left(\varphi^*\frac{\partial^2\varphi}{\partial t^2} - \varphi\frac{\partial^2\varphi^*}{\partial t^2}\right),$$

$$\vec{J} = -i\left(\varphi^*\,\vec{\nabla}\varphi - \varphi\,\vec{\nabla}\varphi^*\right). \tag{A.88}$$

For the plane wave in (A.85), we find

$$\rho = 2E|N|^2,$$

$$\vec{J} = 2\vec{p}|N|^2. \tag{A.89}$$

Thus, ρ cannot be interpreted here as a probability density, since it is negative for the negative values of the energy E. [**QE:** Show eqs. (A.87)-(A.89).]

Since (A.83) is relativistic, it is interesting to re-obtain the KG equation in full relativistic notation. In covariant notation, (A.83) reads

$$p^\mu p_\mu = m^2 \longrightarrow i\partial^\mu\, i\partial_\mu\,\varphi(x) = m^2\varphi(x), \tag{A.90}$$

where (A.65) was used. Recalling (A.23), one obtains

$$\left(\Box + m^2\right)\varphi(x) = 0. \tag{A.91}$$

The plane wave solution may be written in covariant notation as

$$\varphi(x) = Ne^{ik.x}, \tag{A.92}$$

where $k^2 = m^2$. This can be checked by direct substitution, noting that

$$\partial_\mu e^{-ik_\mu x^\mu} = -ik_\mu e^{-ik.x}. \tag{A.93}$$

The 4-current is

$$J^\mu = (\rho, \vec{J}) = i\left(\varphi^*\,\partial^\mu\varphi - \varphi\,\partial^\mu\varphi^*\right), \tag{A.94}$$

which, for plane waves, yields

$$J^\mu = 2p^\mu|N|^2. \tag{A.95}$$

Historically, the presence of negative energies and probabilities led Schrodinger to discard this equation as a good quantum mechanical wave equation. Later, Dirac developed an equation linear in time, which solved the problem of a positive ρ but still had to contend with negative energies. In a twist o genius, Dirac proposed that all negative energy states were filled by infinitely many electrons and, thus, only positive energy electrons were allowed. This is the so-called Dirac sea. Still, he had to contend with the possibility that a photon would hit a negative energy electron, leaving behind a "hole". This lead him to invent the antiparticle of the electron (positron), with opposite charge, later shown experimentally to exist. The (relativistic) Dirac equation, with its Dirac sea imagery, is not a formally consistent framework; it is a hack that gives some correct solutions. We now know that the only formalism consistent with relativity, quantum mechanics and the cluster decomposition principle (related to locality) is Quantum Field Theory (see Weinberg's book [33]).

Problems

A.1 We wish to prove the relation

$$\frac{d^3p'}{E'} = \frac{d^3p}{E}\,, \tag{A.96}$$

in two different ways.

a) Find the transformations for the energy-momentum components and prove Eq. (A.96).

b) Consider a function $g(x)$ with zeros at points a_j $(j = 1, \cdots n)$ where the derivatives are nonvanishing: $g'(x)|_{x=a_j} \neq 0$. It is known that, then,

$$\delta\left[g(x)\right] = \sum_{j=1}^{n} \left[\left|\frac{d\,g}{dx}\right|_{x=a_j}\right]^{-1} \delta(x - a_j)\,. \tag{A.97}$$

Take first $g(x) = x^2 - a^2$ and find an expression for $\delta(x^2 - a^2)$. Show that

$$\int dp^0\ \delta\left((p^0)^2 - |\vec{p}|^2 - m^2\right)\theta(p^0) = \frac{1}{2E}\,, \tag{A.98}$$

where $E = +\sqrt{|\vec{p}|^2 + m^2}$ and $\theta(x)$ is the Heaviside step function

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}\,, \tag{A.99}$$

Use the result to show that

$$\int d^4p\ \delta\left(p^2 - m^2\right)\theta(p^0) = \int \frac{d^3\vec{p}}{2E}\,, \tag{A.100}$$

and recover Eq. (A.96).

c) For completeness, use $g(k^0) = (k^0)^2 - |\vec{k}|^2 - m^2$ and $\omega = +\sqrt{|\vec{k}|^2 + m^2}$ to show that

$$\begin{aligned}\int dk^0 e^{-i(k^0t - \vec{k}.\vec{x})}\ f(k^0, \vec{k})\ \ \delta\left((k^0)^2 - |\vec{k}|^2 - m^2\right)& \\ = e^{-i(-\omega t - \vec{k}.\vec{x})}\ \frac{f(-\omega, \vec{k})}{2\omega} + e^{-i(\omega t - \vec{k}.\vec{x})}\ \frac{f(\omega, \vec{k})}{2\omega}\,,&\end{aligned} \tag{A.101}$$

where the first (second) term corresponds to the negative (positive) energy solutions.

We now wish to prove

$$2E'\ \delta^3(\vec{p}' - \vec{q}') = 2E\ \delta^3(\vec{p} - \vec{q})\,. \tag{A.102}$$

d) Prove (A.102) by setting the boost along the z axis and following the same strategy as in **a)**.

A.2 Consider the transformation of a boosted 4-vector $A^\mu = (A^0, \vec{A})$ given by

$$\begin{aligned}(A^0)' =& \cosh(\eta)\, A^0 - \sinh(\eta)\, \vec{A}_\parallel\,,\\ (\vec{A}_\parallel)' =& -\sinh(\eta)\, A^0\, \hat{\eta} + \cosh(\eta)\, \vec{A}_\parallel\,,\\ (\vec{A}_\perp)' =& \vec{A}_\perp\,,\end{aligned} \tag{A.103}$$

where

$$\begin{aligned}\vec{A}_\parallel =& (\vec{A}.\hat{\eta})\,\hat{\eta}\,,\\ \vec{A}_\perp =& \vec{A} - (\vec{A}.\hat{\eta})\,\hat{\eta}\,,\end{aligned} \tag{A.104}$$

are the components of $\vec{A}$ parallel and perpendicular to the boost $\vec{\eta}$, respectively. Obviously, $\vec{A}.\hat{\eta} = \vec{A}_\parallel.\eta$. Convince yourself that this contains (A.29). Show that the 3-vector part may be written as

$$(\vec{A})' = \vec{A} + (\gamma - 1)\,(\vec{A}.\hat{\eta})\,\hat{\eta} - \gamma\beta\hat{\eta}A^0\,. \tag{A.105}$$

A.3

a) Prove Eqs. (A.40), (A.41), and (A.42).

b) Write the invariants $F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu}\mathcal{F}^{\mu\nu}$, and $\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$ in terms of $\vec{E}$ and $\vec{B}$.

c) Obviously, the tensor $F^{\mu\nu}$ transforms under a Lorentz transformation as

$$F'^{\mu\nu} = \Lambda^\mu_{\cdot\alpha}\Lambda^\nu_{\cdot\beta}F^{\alpha\beta}\,. \tag{A.106}$$

Using as an example a boost along z, infer that the electromagnetic fields transform under a boost as

$$\begin{aligned}E'_\parallel =& E_\parallel\,,\\ E'_\perp =& \gamma\left(\vec{E} + \vec{\beta}\times\vec{B}\right)_\perp\,,\\ B'_\parallel =& B_\parallel\,,\\ B'_\perp =& \gamma\left(\vec{B} - \vec{\beta}\times\vec{E}\right)_\perp\,.\end{aligned} \tag{A.107}$$

d) Show that if $\vec{E}$ and $\vec{B}$ are orthogonal in one reference inertial frame, then they are orthogonal in all inertial frames.

A.4 Some integrals have tensor structure in Minkowski space. Looking at the Lorentz indices, one can relate them to scalar integrals. Some further tricks may allow the latter to be calculated in a simple fashion. Consider the six-dimensional tensor integral

$$I_{\alpha\beta} = \int \frac{d^3k_1}{2k_1^0}\frac{d^3k_2}{2k_2^0}\ \delta^4(\Delta - k_1 - k_2)\ k_{1\,\alpha}k_{2\,\beta}\,, \tag{A.108}$$

where $k_1^2 = 0 = k_2^2$ refer to massless particles (*e.g.* neutrinos), and Δ is some external 4-momentum.

a) Argue that $I_{\alpha\beta}$ must be of the form

$$I_{\alpha\beta} = A\Delta^2 g_{\alpha\beta} + B\Delta_\alpha\Delta_\beta\,, \tag{A.109}$$

where A and B may be functions of Δ^2.

b) Since the delta function forces $\Delta = k_1 + k_2$, show that

$$k_1.k_2 = k_1.\Delta = k_2.\Delta = \frac{\Delta^2}{2}\,. \tag{A.110}$$

c) Contract both sides of (A.109) with $g^{\alpha\beta}$, and find one relation between A and B. Repeat, contracting with $\Delta^\alpha\Delta^\beta$, and find a second relation. Show that

$$I_{\alpha\beta} = \frac{1}{12}\left(\Delta^2 g_{\alpha\beta} + 2\Delta_\alpha\Delta_\beta\right) I\,, \tag{A.111}$$

where

$$I = \int \frac{d^3k_1}{2k_1^0}\frac{d^3k_2}{2k_2^0}\ \delta^4(\Delta - k_1 - k_2)\,. \tag{A.112}$$

d) Use $\delta^3(\Delta - k_1 - k_2)$ to perform the d^3k_2 integral. Next, go to the CM frame of the two "neutrinos" where $k_1^0 = |\vec{k_1}| = |\vec{k_2}| = k_2^0$. Explain why $d^3k_1 = |\vec{k_1}|^2 d|\vec{k_1}|d\Omega = (k_1^0)^2 dk_1^0 d\Omega$, and conclude that $I = \pi/2$. Thus, show that

$$I_{\alpha\beta} = \frac{\pi}{24}\left(\Delta^2 g_{\alpha\beta} + 2\Delta_\alpha\Delta_\beta\right)\,. \tag{A.113}$$

A.5 Consider the process $A\,B \ \to\ C\,D$ in the s channel. Using crossing symmetry, show that

a) t is the square of the CM energy for the process $\overline{D}\,B \ \to\ C\,\overline{A}$.

b) u is the square of the CM energy for the process $A\,\overline{D} \;\to\; C\,\overline{B}$.

For a more complete analysis, show that

c) In going from $A\,B \;\to\; C\,D$ to $\overline{D}\,B \;\to\; C\,\overline{A}$, the Mandelstam variables change as $s \leftrightarrow t$, with u fixed.

d) In going from $A\,B \;\to\; C\,D$ to $A\,\overline{D} \;\to\; C\,\overline{B}$, the Mandelstam variables change as $s \leftrightarrow u$, with t fixed.

e) How do the Mandelstam variables change in going from $A\,B \;\to\; C\,D$ to $A\,B \;\to\; D\,C$?

A.6 In the extreme relativistic limit ($m_e, m_\mu << \sqrt{s}$), the differential cross section of the process $e^-e^+ \to \mu^-\mu^+$ is proportional to

$$d\sigma \sim \frac{u^2 + t^2}{s^2}. \tag{A.114}$$

Determine the equivalent form for the process $e^-\mu^- \to e^-\mu^-$.

A.7 **[HM 4.6 (Ref. [80])]** Take a fermion f, its conjugate $\bar{f}$ and the s-channel process to be $f\bar{f} \to f\bar{f}$ ($A\,B \;\to\; C\,D$).

a) Show that one can take

$$\begin{aligned} p_A &= (E, \vec{p}_i) & p_B &= (E, -\vec{p}_i), \\ p_C &= (E, \vec{p}_f) & p_D &= (E, -\vec{p}_f), \end{aligned} \tag{A.115}$$

where $|\vec{p}_i| = |\vec{p}_f|$, and explain the meaning of all variables.

b) Verify that

$$\begin{aligned} s =& 4(p^2 + m^2), \\ t =& -2p^2(1 - \cos\theta), \\ u =& -2p^2(1 + \cos\theta), \end{aligned} \tag{A.116}$$

where θ is the CM scattering angle and $p = |\vec{p}_i| = |\vec{p}_f|$ is the common magnitude of the 3-momenta of the initial and final particles in the CM frame.

c) Show that the process is physically allowed, provided $s \geq 4m^2$, $t \leq 0$, and $u \leq 0$. Note that $t = 0$ ($u = 0$) corresponds to forward (backward) scattering.

d) **[HM 4.7(Ref. [80])]** For the crossing reaction $ff \to ff$ ($A\,\overline{D} \;\to\; C\,\overline{B}$), show that u becomes the square of the CM energy and that this process would become physical in a different kinematic region: $u \geq 4m^2$, $t \leq 0$, and $s \leq 0$. [Note that, for example, $-p_D = (E, \vec{p})$, where E and $\vec{p}$ refer to the incoming $\overline{D}$.]

Appendix B

Wick's theorem

To evaluate the amplitudes that appear in the calculation of the S-matrix elements we have to move the annihilation operators to the right until they act on the vacuum. The final result from these manipulations can be stated in the form of a theorem, known as Wick's theorem, which relates the time ordered with the normal ordered product and can be stated in the following form,

Wick's Theorem:

$$
\begin{aligned}
&T(\varphi(x_1)\cdots\varphi(x_n)) = \\
&= :\varphi(x_1)\cdots\varphi(x_n): + [\langle 0|\, T(\varphi(x_1)\varphi(x_2))\,|0\rangle : \varphi(x_3)\cdots\varphi(x_n): + \text{perm.} \\
&+ \langle 0|\, T(\varphi(x_1)\varphi(x_2))\,|0\rangle \langle 0|\, T(\varphi(x_3)\varphi(x_4))\,|0\rangle : \varphi(x_5)\cdots\varphi(x_n): + \text{perm.} \\
&+ \cdots \\
&+ \begin{cases} \langle 0|\, T(\varphi(x_1)\varphi(x_2))\,|0\rangle \cdots \langle 0|\, T(\varphi(x_{n-1})\varphi(x_n))\,|0\rangle + \text{perm.} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{n even} \\ \langle 0|\, T(\varphi(x_1)\varphi(x_2)\,|0\rangle \cdots \langle 0|\, T(\varphi(x_{n-2})\varphi(x_{n-1}))\,|0\rangle\, \varphi(x_n) + \text{perm.} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{n odd} \end{cases}
\end{aligned}
\tag{B.1}
$$

In these expressions all the fields are in the interaction picture and obbey the free field commutation relations.

Proof:

The proof of the theorem is done by induction. For $n = 1$ it is certainly true (and trivial). Also for $n = 2$ we can shown that

$$
T(\varphi(x_1)\varphi(x_2)) =: \varphi(x_1)\varphi(x_2) : + \text{c-number} \tag{B.2}
$$

where the *c-number* comes from the commutations that are needed to move the annihilation operators to the right. To find this constant, we do not have to do any calculation, just to notice that

$$\langle 0| : \cdots : |0\rangle = 0 \tag{B.3}$$

Then

$$T(\varphi(x_1)\varphi(x_2)) =: \varphi(x_1)\varphi(x_2) : + \langle 0| T(\varphi(x_1)\varphi(x_2)) |0\rangle \tag{B.4}$$

which is in agreement with Eq. (B.1).

Continuing with the induction, let us assume that Eq. (B.1) is valid for a given n. We have to show that it remains valid for $n+1$. Let us consider then $T(\varphi(x_1)\cdots\varphi(x_{n+1}))$ and let us assume that t_{n+1} is the earliest time. Then

$$\begin{aligned} T(\varphi(x_1)\cdots\varphi(x_{n+1})) = & \\ &= T(\varphi(x_1)\cdots\varphi(x_n))\varphi(x_{n+1}) \\ &= : \varphi(x_1)\cdots\varphi(x_n) : \varphi(x_{n+1}) \\ &\quad + \sum_{\text{perm}} \langle 0| T(\varphi(x_1)\varphi(x_2)) |0\rangle : \varphi(x_3)\cdots\varphi(x_n) : \varphi(x_{n+1}) \\ &\quad + \cdots \end{aligned} \tag{B.5}$$

To write Eq. (B.5) in the form of Eq. (B.1) it is necessary to find the rule showing how to introduce $\varphi(x_{n+1})$ inside the normal product. For that, we introduce the notation,

$$\varphi(x) = \varphi^{(+)}(x) + \varphi^{(-)}(x) \tag{B.6}$$

where $\varphi^{(+)}(x)$ contains the annihilation operator and $\varphi^{(-)}(x)$ the creation operator. Then we can write,

$$: \varphi(x_1)\cdots\varphi(x_n) := \sum_{A,B} \prod_{i\in A} \varphi^{(-)}(x_i) \prod_{j\in B} \varphi^{(+)}(x_j) \tag{B.7}$$

where the sum runs over all the sets A, B that constitute partitions of the n indices. Then

$$\begin{aligned} : \varphi(x_1)\cdots\varphi(x_n) : \varphi(x_{n+1}) = & \\ &= \sum_{A,B} \prod_{i\in A} \varphi^{(-)}(x_i) \prod_{j\in B} \varphi^{(+)}(x_j)[\varphi^{(+)}(x_{n+1}) + \varphi^{(-)}(x_{n+1})] \\ &= \sum_{A,B} \prod_{i\epsilon A} \varphi^{(-)}(x_i) \prod_{j\in B} \varphi^{(+)}(x_j)\varphi^{(+)}(x_{n+1}) \\ &\quad + \sum_{A,B} \prod_{i\in A} \varphi^{(-)}(x_i)\varphi^{(-)}(x_{n+1}) \prod_{j\in B} \varphi^{(+)}(x_j) \end{aligned}$$

$$+\sum_{A,B}\prod_{i\in A}\varphi^{(-)}(x_i)\sum_{k\in B}\prod_{j\in B j\neq k}\varphi^{(+)}(x_j)\,\langle 0|\,\varphi^{(+)}(x_k)\varphi^{(-)}(x_{n+1})\,|0\rangle \qquad \text{(B.8)}$$

we can now write,

$$\begin{aligned}\langle 0|\,\varphi^{(+)}(x_k)\varphi^{(-)}(x_{n+1})\,|0\rangle &= \langle 0|\,\varphi(x_k)\varphi(x_{n+1})\,|0\rangle \\ &= \langle 0|\,T(\varphi(x_k)\varphi(x_{n+1}))\,|0\rangle \qquad \text{(B.9)}\end{aligned}$$

where we have used the fact that t_{n+1} is the earliest time. We can then write Eq. (B.8) in the form,

$$\begin{aligned}&:\varphi(x_1)\cdots\varphi(x_n):\varphi(x_{n+1}) =: \varphi(x_1)\cdots\varphi(x_{n+1}) : \\ &\quad +\sum_k :\varphi(x_1)\cdots\varphi(x_{k-1})\varphi(x_{k+1})\cdots\varphi(x_n):\langle 0|\,T(\varphi(x_k)\varphi(x_{n+1}))\,|0\rangle\end{aligned} \qquad \text{(B.10)}$$

With this result, Eq. (B.5) takes the general form of Eq. (B.1) for the $n+1$ case, ending the proof of the theorem. To fully understand the theorem, it is important to do in detail the case $n = 4$, to see how things work. The importance of the Wick's theorem for the applications comes from the following two corollaries.

Corollary 1 : If n is odd, then $\langle 0|\,T(\varphi(x_1)\cdots\varphi(x_n))\,|0\rangle = 0$, as results trivially from Eqs. (B.1) e (B.3) and from,

$$\langle 0|\,\varphi(x)\,|0\rangle = 0 \qquad \text{(B.11)}$$

Corollary 2: If n is even

$$\begin{aligned}&\langle 0|\,T(\varphi(x_1)\cdots\varphi(x_n))\,|0\rangle = \\ &\quad = \sum_{\text{perm}}\delta_p\,\langle 0|\,T(\varphi(x_1)\varphi(x_2))\,|0\rangle\cdots\langle 0|\,T(\varphi(x_{n-1})\varphi(x_n))\,|0\rangle\end{aligned} \qquad \text{(B.12)}$$

where δ_p is the sign of the permutation that is necessary to introduce in case of fermion fields. This result, that in practice is the most important one, also results from Eqs. (B.1), (B.3) and (B.11).

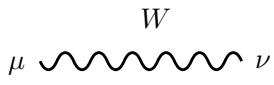
Appendix C

Feynman Rules for the Standard Model

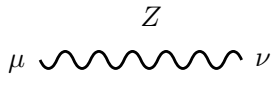
In this appendix we collect the Feynman rules for the Standard Model [1], necessary for the calculation in lowest order (tree level). Therefore we choose the *unitary gauge* where we only deal with the physical particles and we follow the conventions of Romão e Silva [59]. For other gauges see [12].

C.1 Propagators

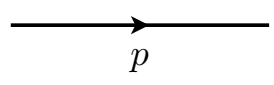
$$-i\frac{g_{\mu\nu}}{q^2} \tag{C.1}$$



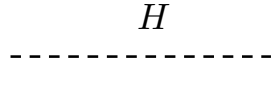
$$-i\frac{g_{\mu\nu}-\frac{q_\mu q_\nu}{M_W^2}}{q^2-M_W^2+iM_W\Gamma_W} \tag{C.2}$$



$$-i\frac{g_{\mu\nu}-\frac{q_\mu q_\nu}{M_Z^2}}{q^2-M_Z^2+iM_Z\Gamma_Z} \tag{C.3}$$



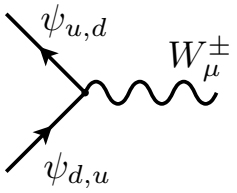
$$i\frac{\not{p}+m_f}{p^2-m_f^2} \tag{C.4}$$



$$\frac{i}{p^2-M_H^2} \tag{C.5}$$

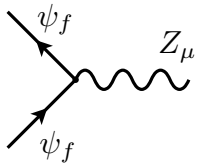
[1] More precisely to the electroweak pat of the SM.

C.2 Charged Current

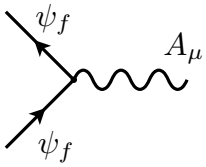


$$-i\frac{g}{\sqrt{2}}\gamma_\mu\frac{1-\gamma_5}{2} \tag{C.6}$$

C.3 Neutral Current

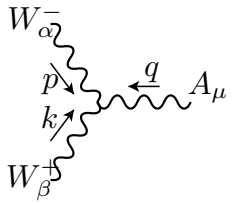


$$i\frac{g}{\cos\theta_W}\gamma_\mu\left(g_V^f-g_A^f\gamma_5\right)$$

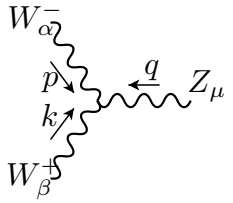


$$-ieQ_f\gamma_\mu$$

C.4 Interactions with three gauge bosons

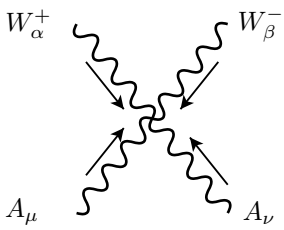


$$-ie\left[g_{\alpha\beta}(p-k)_\mu+g_{\beta\mu}(k-q)_\alpha+g_{\mu\alpha}(q-p)_\beta\right] \tag{C.7}$$

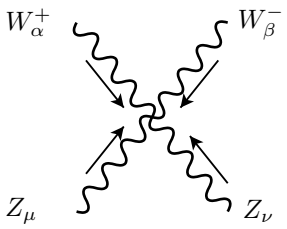


$$-ig\cos\theta_W\left[g_{\alpha\beta}(p-k)_\mu+g_{\beta\mu}(k-q)_\alpha+g_{\mu\alpha}(q-p)_\beta\right] \tag{C.8}$$

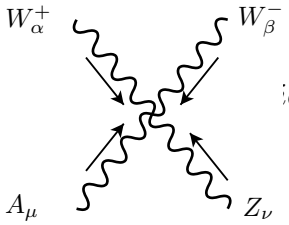
C.5 Interactions with four gauge bosons



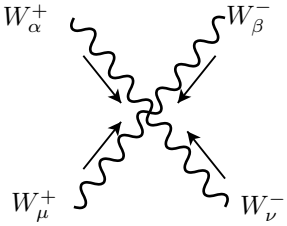
$$-ie^2\left[2g_{\alpha\beta}g_{\mu\nu}-g_{\alpha\mu}g_{\beta\nu}-g_{\alpha\nu}g_{\beta\mu}\right] \tag{C.9}$$



$$-ig^2 \cos\theta_W \left[2g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right] \quad \text{(C.10)}$$

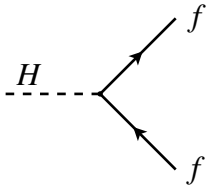


$$ieg \cos\theta_W \left[2g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right] \quad \text{(C.11)}$$

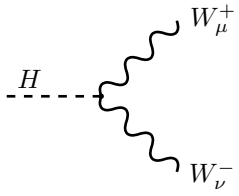


$$ig^2 \left[2g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right] \quad \text{(C.12)}$$

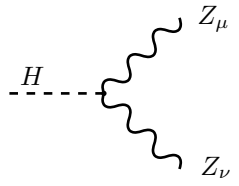
C.6 Cubic interactions with the Higgs boson



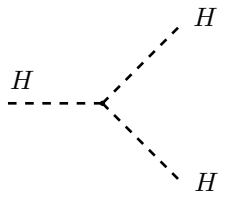
$$-i\frac{g}{2}\frac{m_f}{M_W} \quad \text{(C.13)}$$



$$ig\, M_W\, g_{\mu\nu} \quad \text{(C.14)}$$

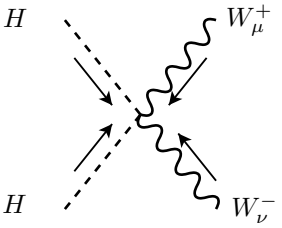


$$i\frac{g}{\cos\theta_W} M_Z\, g_{\mu\nu} \quad \text{(C.15)}$$

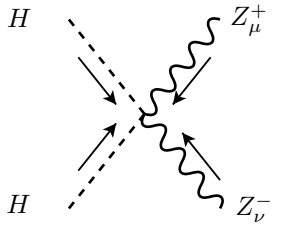


$$-\frac{3}{2}\,i\,g\frac{M_H^2}{M_W} \tag{C.16}$$

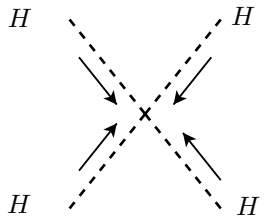
C.7 Quartic interactions with the Higgs boson



$$\frac{i}{2}g^2 g_{\mu\nu} \tag{C.17}$$



$$\frac{i}{2}\frac{g^2}{\cos^2\theta_W}g_{\mu\nu} \tag{C.18}$$



$$-\frac{3}{4}\,i\,g^2\frac{M_H^2}{M_W^2} \tag{C.19}$$

Appendix D

Useful techniques for renormalization

D.1 μ parameter

The reason for the μ parameter introduced in section 9.1.1 is the following. In dimension $d = 4-\epsilon$, the fields A_μ and ψ have dimensions given by the kinetic terms in the action,

$$\int d^d x \left[-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + i\,\overline{\psi}\gamma \cdot \partial\psi \right] \tag{D.1}$$

We have therefore

$$\begin{aligned} 0 &= -d + 2 + 2[A_\mu] \quad \Rightarrow [A_\mu] &= \tfrac{1}{2}(d-2) = 1 - \tfrac{\epsilon}{2} \\ 0 &= -d + 1 + 2[\psi] \quad\;\; \Rightarrow [\psi] &= \tfrac{1}{2}(d-1) = \tfrac{3}{2} - \tfrac{\epsilon}{2} \end{aligned} \tag{D.2}$$

Using these dimensions in the interaction term

$$S_I = \int d^d x \; e\overline{\psi}\gamma_\mu \psi A^\mu \tag{D.3}$$

we get

$$\begin{aligned} [S_I] &= -d + [e] + 2[\psi] + [A] \\ &= -4 + \epsilon + [e] + 3 - \epsilon + 1 - \frac{\epsilon}{2} \\ &= [e] - \frac{\epsilon}{2} \end{aligned} \tag{D.4}$$

Therefore, if we want the action to be dimensionless (remember that we use the system where $\hbar = c = 1$), we have to set

$$[e] = \frac{\epsilon}{2} \tag{D.5}$$

We see then that in dimensions $d \neq 4$ the coupling constant has dimensions. As it is more convenient to work with a dimensionless coupling constant we introduce a parameter μ with dimensions of a mass and in $d \neq 4$ we will make the substitution

$$e \to e\mu^{\frac{\epsilon}{2}} \qquad (\epsilon = 4 - d) \tag{D.6}$$

while keeping e dimensionless.

D.2 $\Gamma(z)$ function and other useful relations

The Γ function is defined by the integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{D.7}$$

or equivalently

$$\int_0^\infty t^{z-1} e^{-\mu t} dt = \mu^{-z} \Gamma(z) \tag{D.8}$$

The function $\Gamma(z)$ has the following important properties

$$\begin{aligned} \Gamma(z+1) &= z\Gamma(z) \\ \Gamma(n+1) &= n! \end{aligned} \tag{D.9}$$

Another related function is the logarithmic derivative of the Γ function, with the properties,

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) \tag{D.10}$$

$$\psi(1) = -\gamma \tag{D.11}$$

$$\psi(z+1) = \psi(z) + \frac{1}{z} \tag{D.12}$$

where γ is the Euler constant. The function $\Gamma(z)$ has poles for $z = 0, -1, -2, \cdots$. Near the pole $z = -m$ we have ($\epsilon \to 0$)

$$\Gamma(-m+\epsilon) = \frac{(-1)^m}{m!} \frac{1}{\epsilon} + \frac{(-1)^m}{m!} \psi(m+1) + O(\epsilon) \tag{D.13}$$

From this we conclude that when $\epsilon \to 0$

$$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} + \psi(1) + O(\epsilon) \qquad \Gamma(-n+\frac{\epsilon}{2}) = \frac{(-1)^n}{n!}\left[\frac{2}{\epsilon} + \psi(n+1)\right] \tag{D.14}$$

For positive integers the function $\Gamma(z)$ has no poles. But as we have to expand everything up to order ϵ, before making $\epsilon \to 0$, we need the expansion near the positive integers. Using the definition in Eq. (D.10) we get for a general n, up to order ϵ

$$\Gamma(n+\epsilon) = \Gamma(n) + \Gamma(n)\psi(n)\,\epsilon \tag{D.15}$$

giving, in particular,

$$\Gamma(1+\frac{\epsilon}{2}) = 1 - \gamma\,\frac{\epsilon}{2} + O(\epsilon^2) \tag{D.16}$$

D.3 Feynman parameterization

The most general form for a *1–loop* is [1]

$$\hat{T}_n^{\mu_1\cdots\mu_p} \equiv \int \frac{d^dk}{(2\pi)^d}\,\frac{k^{\mu_1}\cdots k^{\mu_p}}{D_0 D_1\cdots D_{n-1}} \tag{D.17}$$

where

$$D_i = (k+r_i)^2 - m_i^2 + i\epsilon \tag{D.18}$$

and the momenta r_i are related with the external momenta (all taken to be incoming) through the relations,

$$\begin{aligned} r_j &= \sum_{i=1}^{j} p_i \quad ; \quad j = 1,\ldots,n-1 \\ r_0 &= \sum_{i=1}^{n} p_i = 0 \end{aligned} \tag{D.19}$$

as indicated in Fig. (D.1). In these expressions there appear in the denominators products of the denominators of the propagators of the particles in the loop. It is convenient to combine these products in just one common denominator. This is achieved by a technique due to Feynman. Let us exemplify with two denominators.

$$\frac{1}{ab} = \int_0^1 \frac{dx}{\left[ax + b(1-x)\right]^2} \tag{D.20}$$

The proof is trivial. In fact

$$\int dx\,\frac{1}{\left[ax+b(1-x)\right]^2} = \frac{x}{b\left[(a-b)x+b\right]} \tag{D.21}$$

[1] We introduce here the notation $\hat{T}$ to distinguish from a more standard notation used for the Passarino-Veltman integrals [81] as explained in Refs. [12, 82].

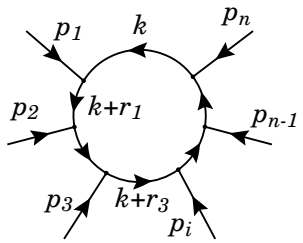


Figure D.1: Conventions for the momenta in the loop.

and therefore Eq. (D.20) immediately follows. Taking successive derivatives with respect to a and b we get

$$\frac{1}{a^p\, b^q} = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^1 dx\ \frac{x^{p-1}(1-x)^{q-1}}{[ax+b(1-x)]^{p+q}} \tag{D.22}$$

and using induction we obtain a general formula

$$\begin{aligned}\frac{1}{a_1 a_2 \cdots a_n} =& \Gamma(n) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \\ & \int_0^{1-x_1-\cdots-x_{n-2}} \frac{dx_{n-1}}{[a_1 x_1 + a_2 x_2 + \cdots + a_n(1-x_1-\cdots-x_{n-1})]^n}\end{aligned} \tag{D.23}$$

Before closing the section let us give an example that will be useful in the self-energy case. Consider the situation with the kinematics described in Fig. (D.2).

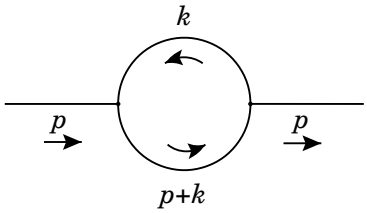


Figure D.2: Kinematics for the self-energy in ϕ^3.

We get

$$\begin{aligned} I &= \int \frac{d^d k}{(2\pi)^d}\ \frac{1}{[(k+p)^2 - m_1^2 + i\epsilon]\,[k^2 - m_2^2 + i\epsilon]} \\ &= \int_0^1 dx \int \frac{d^d k}{(2\pi)^d}\ \frac{1}{\left[k^2 + 2p\cdot k\, x + p^2\, x - m_1^2\, x - m_2^2\,(1-x) + i\epsilon\right]^2} \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[k^2 + 2P\cdot k - M^2 + i\epsilon\right]^2} \\
&= \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[(k+P)^2 - P^2 - M^2 + i\epsilon\right]^2}
\end{aligned} \tag{D.24}
$$

where in the last line we have completed the square in the term with the loop momenta k. The quantities P and M^2 are, in this case, defined by

$$P = xp \tag{D.25}$$

and

$$M^2 = -x\,p^2 + m_1^2\,x + m_2^2\,(1-x) \tag{D.26}$$

They depend on the masses, external momenta and Feynman parameters, but not in the loop momenta. Now changing variables $k \to k - P$ we get rid of the linear terms in k and finally obtain

$$I = \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[k^2 - C + i\epsilon\right]^2} \tag{D.27}$$

where C is independent of the loop momenta k and it is given by

$$C = P^2 + M^2 \tag{D.28}$$

Notice that the $i\epsilon$ factors will add correctly and can all be put as in Eq. (D.27).

D.4 Tensor integrals in dimensional regularization

We are frequently faced with the task of evaluating the tensor integrals of the form of Eq. (D.17),

$$\hat{T}_n^{\mu_1\cdots\mu_p} \equiv \int \frac{d^dk}{(2\pi)^d} \frac{k^{\mu_1}\cdots k^{\mu_p}}{D_0 D_1 \cdots D_{n-1}} \tag{D.29}$$

The first step is to reduce to one common denominator using the Feynman parameterization technique. The result is,

$$
\begin{aligned}
\hat{T}_n^{\mu_1\cdots\mu_p} &= \Gamma(n) \int_0^1 dx_1 \cdots \int_0^{1-x_1-\cdots-x_{n-2}} dx_{n-1} \int \frac{d^dk}{(2\pi)^d} \frac{k^{\mu_1}\cdots k^{\mu_p}}{\left[k^2 + 2k\cdot P - M^2 + i\epsilon\right]^n} \\
&= \Gamma(n) \int_0^1 dx_1 \cdots \int_0^{1-x_1-\cdots-x_{n-2}} dx_{n-1}\, I_n^{\mu_1\cdots\mu_p}
\end{aligned} \tag{D.30}
$$

where we have defined

$$I_n^{\mu_1\cdots\mu_p} \equiv \int \frac{d^dk}{(2\pi)^d}\ \frac{k^{\mu_1}\cdots k^{\mu_p}}{\left[k^2+2k\cdot P-M^2+i\epsilon\right]^n} \tag{D.31}$$

that we call, from now on, the tensor integral. In principle all these integrals can be written in terms of scalar integrals. It is however convenient to have a general formula for them. We start with the result,

$$\begin{aligned} I_{0,n} =& \int \frac{d^dk}{(2\pi)^d}\ \frac{1}{\left[k^2+2k\cdot P-M^2+i\epsilon\right]^n} \\ =& \frac{i}{(4\pi)^{d/2}}(-1)^n\frac{\Gamma(n-d/2)}{\Gamma(n)}\left(\frac{1}{C}\right)^{n-d/2} \end{aligned} \tag{D.32}$$

where we used the result in Eq. (9.22) and use the definition of the Γ function,

$$\left(\frac{1}{C}\right)^z = \frac{1}{\Gamma(z)}\int_0^\infty dt\, t^{z-1}e^{-tC} \tag{D.33}$$

to write

$$\int \frac{d^dk}{(2\pi)^d}\ \frac{1}{\left[k^2+2k\cdot P-M^2+i\epsilon\right]^n} = \frac{i}{(4\pi)^{d/2}}(-1)^n\frac{1}{\Gamma(n)}\int_0^\infty dt\, t^{n-1-d/2}e^{-tC} \tag{D.34}$$

Now we use

$$\frac{\partial}{\partial P^\mu}\ \frac{1}{\left[k^2+2k\cdot P-M^2+i\epsilon\right]^n} = -n\ \frac{2k_\mu}{\left[k^2+2k\cdot P-M^2+i\epsilon\right]^{n+1}} \tag{D.35}$$

to show that

$$\frac{k^{\mu_1}\cdots k^{\mu_p}}{\left[k^2+2k\cdot P-M^2+i\epsilon\right]^n} = \frac{(-1)^p}{2^p}\frac{\Gamma(n-p)}{\Gamma(n)}\frac{\partial}{\partial P_{\mu_1}}\cdots\frac{\partial}{\partial P_{\mu_p}}\ \frac{1}{\left[k^2+2k\cdot P-M^2+i\epsilon\right]^{n-p}} \tag{D.36}$$

We then use Eq. (D.34) to write

$$\begin{aligned} \int \frac{d^dk}{(2\pi)^d}\frac{1}{\left[k^2+2k\cdot P-M^2+i\epsilon\right]^{n-p}} =& \frac{i}{(4\pi)^{d/2}}(-1)^{n-p}\frac{1}{\Gamma(n-p)}\int_0^\infty dt\, t^{n-p-1-d/2}e^{-tC} \\ =& \frac{i}{16\pi^2}(-1)^{n-p}\frac{(4\pi)^{\epsilon/2}}{\Gamma(n-p)}\int_0^\infty dt\, t^{n-p-3+\epsilon/2}e^{-tC} \end{aligned} \tag{D.37}$$

Inserting Eq. (D.36) and Eq. (D.37) into Eq. (D.31) we finally get the result

$$I_n^{\mu_1\cdots\mu_p} = \frac{i}{16\pi^2}\,\frac{(4\pi)^{\epsilon/2}}{\Gamma(n)}\,(-1)^n\int_0^\infty \frac{dt}{(2t)^p}\ t^{n-3+\epsilon/2}\ \frac{\partial}{\partial P_{\mu_1}}\cdots\frac{\partial}{\partial P_{\mu_p}}\,e^{-t\,C} \tag{D.38}$$

where $C = P^2 + M^2$. After doing the derivatives the remaining integrals can be done using the properties of the Γ function (see section D.2). Notice that P, M^2 and therefore also C depend not only in the Feynman parameters but also in the exterior momenta. The advantage of having a general formula is that it can be programmed [83] and all the integrals can then be obtained automatically.

D.5 Explicit formulæ for the *1–loop* integrals

Although we have presented in the previous sections the general formulæ for all the integrals that appear in *1–loop*, Eqs. (9.22) and (D.38), in practice it is convenient to have expressions for the most important cases with the expansion on the ϵ already done. The results presented below were generated with the `Mathematica` package `OneLoop` [83] from the general expressions. In these results the integration on the Feynman parameters has still to be done (see Eq. (D.30)). This is in general a difficult problem. In Ref. [82] is explained an alternative way of expressing these integrals in a more convenient way for a numerical evaluation.

D.5.1 *Tadpole* integrals

With the definitions of Eqs. (9.22) and (D.38) we get

$$\begin{aligned} I_{0,1} &= \frac{i}{16\pi^2}\, C(1+\Delta_\epsilon - \ln C) \\ I_1^\mu &= 0 \\ I_1^{\mu\nu} &= \frac{i}{16\pi^2}\,\frac{1}{8}\, C^2\, g^{\mu\nu}(3+2\Delta_\epsilon - 2\ln C) \end{aligned} \tag{D.39}$$

where for the *tadpole* integrals

$$P = 0 \quad ; \quad C = m^2 \tag{D.40}$$

because there are no Feynman parameters and there is only one mass. In this case the above results are final.

D.5.2 *Self–Energy* integrals

For the integrals with two denominators we get,

$$\begin{aligned} I_{0,2} &= \frac{i}{16\pi^2}\,(\Delta_\epsilon - \ln C) \\ I_2^\mu &= \frac{i}{16\pi^2}\,(-\Delta_\epsilon + \ln C) P^\mu \\ I_2^{\mu\nu} &= \frac{i}{16\pi^2}\,\frac{1}{2}\,\Big[C g^{\mu\nu}(1+\Delta_\epsilon - \ln C) + 2(\Delta_\epsilon - \ln C) P^\mu P^\nu \Big] \\ I_2^{\mu\nu\alpha} &= \frac{i}{16\pi^2}\,\frac{1}{2}\,\Big[- C g^{\mu\nu}(1+\Delta_\epsilon - \ln C) P^\alpha - C g^{\nu\alpha}(1+\Delta_\epsilon - \ln C) P^\mu \\ &\quad - C g^{\mu\alpha}(1+\Delta_\epsilon - \ln C) P^\nu - 2(\Delta_\epsilon - \ln C) P^\alpha P^\mu P^\nu \Big] \end{aligned} \tag{D.41}$$

where, with the notation and conventions of Fig. (D.1), we have

$$P^\mu = x\, r_1^\mu \quad ; \quad C = x^2\, r_1^2 + (1-x)\, m_2^2 + x\, m_1^2 - x\, r_1^2 \tag{D.42}$$

D.5.3 *Triangle* integrals

For the integrals with three denominators we get,

$$\begin{aligned}
I_{0,3} &= \frac{i}{16\pi^2}\,\frac{-1}{2C} \\
I_3^\mu &= \frac{i}{16\pi^2}\,\frac{1}{2C}\; P^\mu \\
I_3^{\mu\nu} &= \frac{i}{16\pi^2}\,\frac{1}{4C}\,\Big[C g^{\mu\nu}(\Delta_\epsilon - \ln C) - 2P^\mu P^\nu\Big] \\
I_3^{\mu\nu\alpha} &= \frac{i}{16\pi^2}\,\frac{1}{4C}\,\Big[C g^{\mu\nu}(-\Delta_\epsilon + \ln C)P^\alpha + C g^{\nu\alpha}(-\Delta_\epsilon + \ln C)P^\mu \\
&\qquad + C g^{\mu\alpha}(-\Delta_\epsilon + \ln C)P^\nu + 2P^\alpha P^\mu P^\nu\Big] \\
I_3^{\mu\nu\alpha\beta} &= \frac{i}{16\pi^2}\,\frac{1}{8C}\,\Big[C^2\,(1 + \Delta_\epsilon - \ln C)\,\big(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} + g^{\alpha\beta}g^{\mu\nu}\big) \\
&\quad + 2C\,(\Delta_\epsilon - \ln C)\,\big(g^{\mu\nu}P^\alpha P^\beta + g^{\nu\beta}P^\alpha P^\mu + g^{\nu\alpha}P^\beta P^\mu + g^{\mu\alpha}P^\beta P^\nu \\
&\quad + g^{\mu\beta}P^\alpha P^\nu + g^{\alpha\beta}P^\mu P^\nu\big) - 4P^\alpha P^\beta P^\mu P^\nu\Big]
\end{aligned} \tag{D.43}$$

where

$$\begin{aligned}
P^\mu &= x_1\, r_1^\mu + x_2\, r_2^\mu \\
C &= x_1^2\, r_1^2 + x_2^2\, r_2^2 + 2x_1\, x_2\, r_1 \cdot r_2 + x_1\, m_1^2 + x_2\, m_2^2 \\
&\quad + (1 - x_1 - x_2)\, m_3^2 - x_1\, r_1^2 - x_2\, r_2^2
\end{aligned} \tag{D.44}$$

D.5.4 *Box* integrals

$$\begin{aligned}
I_{0,4} &= \frac{i}{16\pi^2}\,\frac{1}{6C^2} \\
I_4^\mu &= \frac{i}{16\pi^2}\,\frac{-1}{6C^2}\; P^\mu \\
I_4^{\mu\nu} &= \frac{i}{16\pi^2}\,\frac{-1}{12C^2}\,\Big[C g^{\mu\nu} - 2P^\mu P^\nu\Big]
\end{aligned}$$

$$
\begin{aligned}
I_4^{\mu\nu\alpha} &= \frac{i}{16\pi^2}\,\frac{1}{12C^2}\,\Big[C\,(g^{\mu\nu}P^\alpha + g^{\nu\alpha}P^\mu + g^{\mu\alpha}P^\nu) - 2P^\alpha P^\mu P^\nu\Big] \\
I_4^{\mu\nu\alpha\beta} &= \frac{i}{16\pi^2}\,\frac{1}{24C^2}\,\Big[C^2\,(\Delta_\epsilon - \ln C)\,\big(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} + g^{\alpha\beta}g^{\mu\nu}\big) \\
&\qquad - 2C\,\big(g^{\mu\nu}P^\alpha P^\beta + g^{\nu\beta}P^\alpha P^\mu + g^{\nu\alpha}P^\beta P^\mu + g^{\mu\alpha}P^\beta P^\nu \\
&\qquad + g^{\mu\beta}P^\alpha P^\nu + g^{\alpha\beta}P^\mu P^\nu\big) + 4P^\alpha P^\beta P^\mu P^\nu\Big]
\end{aligned}
\tag{D.45}
$$

where

$$
\begin{aligned}
P^\mu &= x_1\,r_1^\mu + x_2\,r_2^\mu + x_3\,r_3^\mu \\
C &= x_1^2\,r_1^2 + x_2^2\,r_2^2 + x_3^2\,r_3^2 + 2x_1\,x_2\,r_1\cdot r_2 + 2x_1\,x_3\,r_1\cdot r_3 + 2x_2\,x_3\,r_2\cdot r_3 \\
&\quad + x_1\,m_1^2 + x_2\,m_2^2 + x_3\,m_3^2 + (1 - x_1 - x_2 - x_3)\,m_4^2 \\
&\quad - x_1\,r_1^2 - x_2\,r_2^2 - x_3\,r_3^2
\end{aligned}
\tag{D.46}
$$

D.6 Divergent part of *1–loop* integrals

When we want to study the renormalization of a given theory it is often convenient to have expressions for the divergent part of the one-loop integrals, with the integration on the Feynman parameters already done. We present here the results for the most important cases. These divergent parts were calculated with the help of the package `OneLoop` [83]. The results are for the functions $\hat{T}_n^{\mu,\mu_2,\cdots\mu_n}$ defined in Eq. (D.29).

D.6.1 *Tadpole* integrals

$$
\begin{aligned}
\mathrm{Div}\left[\hat{T}_1\right] &= \frac{i}{16\pi^2}\,\Delta_\epsilon\,m^2 \\
\mathrm{Div}\left[\hat{T}_1^\mu\right] &= 0 \\
\mathrm{Div}\left[\hat{T}_1^{\mu\nu}\right] &= \frac{i}{16\pi^2}\,\frac{1}{4}\,\Delta_\epsilon\,m^4\,g^{\mu\nu}
\end{aligned}
\tag{D.47}
$$

D.6.2 *Self–Energy* integrals

$$
\begin{aligned}
\mathrm{Div}\left[\hat{T}_2\right] &= \frac{i}{16\pi^2}\,\Delta_\epsilon \\
\mathrm{Div}\left[\hat{T}_2^\mu\right] &= \frac{i}{16\pi^2}\,\left(-\frac{1}{2}\right)\,\Delta_\epsilon\,r_1^\mu
\end{aligned}
$$

$$\begin{aligned}
\mathrm{Div}\left[\hat{T}_2^{\mu\nu}\right] &= \frac{i}{16\pi^2}\frac{1}{12}\Delta_\epsilon\left[(3m_1^2+3m_2^2-r_1^2)g^{\mu\nu}+4r_1^\mu r_1^\nu\right] \\
\mathrm{Div}\left[\hat{T}_2^{\mu\nu\alpha}\right] &= \frac{i}{16\pi^2}\left(-\frac{1}{24}\right)\Delta_\epsilon\left[(4m_1^2+2m_2^2-r_1^2)\left(g^{\mu\nu}r_1^\alpha+g^{\nu\alpha}r_1^\mu+g^{\mu\alpha}r_1^\nu\right)\right. \\
&\qquad \left. +6\,r_1^\alpha r_1^\mu r_1^\nu\right] \qquad \text{(D.48)}
\end{aligned}$$

D.6.3 *Triangle* integrals

$$\begin{aligned}
\mathrm{Div}\left[\hat{T}_3\right] &= 0 \\
\mathrm{Div}\left[\hat{T}_3^{\mu}\right] &= 0 \\
\mathrm{Div}\left[\hat{T}_3^{\mu\nu}\right] &= \frac{i}{16\pi^2}\frac{1}{4}\Delta_\epsilon\, g^{\mu\nu} \\
\mathrm{Div}\left[\hat{T}_3^{\mu\nu\alpha}\right] &= \frac{i}{16\pi^2}\left(-\frac{1}{12}\right)\Delta_\epsilon\left[g^{\mu\nu}(r_1^\alpha+r_2^\alpha)+g^{\nu\alpha}(r_1^\mu+r_2^\mu)+g^{\mu\alpha}(r_1^\nu+r_2^\nu)\right] \\
\mathrm{Div}\left[\hat{T}_3^{\mu\nu\alpha\beta}\right] &= \frac{i}{16\pi^2}\frac{1}{48}\Delta_\epsilon\left[(2m_1^2+2m_2^2+2m_3^2)\left(g^{\mu\alpha}g^{\nu\beta}+g^{\alpha\beta}g^{\mu\nu}+g^{\mu\beta}g^{\nu\alpha}\right)\right. \\
&\quad +g^{\alpha\beta}\left[2r_1^\mu r_1^\nu+r_1^\mu r_2^\nu+(r_1\leftrightarrow r_2)\right]+g^{\mu\beta}\left[2r_1^\alpha r_1^\nu+r_1^\alpha r_2^\nu+(r_1\leftrightarrow r_2)\right] \\
&\quad +g^{\nu\beta}\left[2r_1^\alpha r_1^\mu+r_1^\alpha r_2^\mu+(r_1\leftrightarrow r_2)\right]+g^{\mu\nu}\left[2r_1^\alpha r_1^\beta+r_1^\alpha r_2^\beta+(r_1\leftrightarrow r_2)\right] \\
&\quad +g^{\mu\alpha}\left[2r_1^\beta r_1^\nu+r_1^\beta r_2^\nu+(r_1\leftrightarrow r_2)\right]+g^{\nu\alpha}\left[2r_1^\beta r_1^\mu+r_1^\beta r_2^\mu+(r_1\leftrightarrow r_2)\right] \\
&\quad \left.+\left(-r_1^2+r_1\cdot r_2-r_2^2\right)\left(g^{\mu\alpha}g^{\nu\beta}+g^{\alpha\beta}g^{\mu\nu}+g^{\mu\beta}g^{\nu\alpha}\right)\right] \qquad \text{(D.49)}
\end{aligned}$$

D.6.4 *Box* integrals

$$\begin{aligned}
\mathrm{Div}\left[\hat{T}_4\right] &= \mathrm{Div}\left[\hat{T}_4^{\mu}\right]=\mathrm{Div}\left[\hat{T}_4^{\mu\nu}\right]=\mathrm{Div}\left[\hat{T}_4^{\mu\nu\alpha}\right]=0 \\
\mathrm{Div}\left[\hat{T}_4^{\mu\nu\alpha\beta}\right] &= \frac{i}{16\pi^2}\frac{1}{24}\Delta_\epsilon\left[g^{\mu\nu}g^{\alpha\beta}+g^{\mu\beta}g^{\alpha\nu}+g^{\mu\alpha}g^{\nu\beta}\right] \qquad \text{(D.50)}
\end{aligned}$$

Appendix E

Solutions to Problems

We give here the solutions to all problems where the solution is not given in the text of the problem. In some cases we just give one of the results that will be enough to check if you got the correct answer.

Solutions to Problems Chapter 1

Problem 1.1

a) See section 1.4.5)

b) Weight (72 kg) = 4.04×10^{31} MeV, Height (1.7m) = 8.61×10^{12} MeV^{-1}, Age (71 y) = 3.40×10^{30} MeV^{-1}.

Problem 1.2

a)

b)

c)

Problem 1.3

c)

Problem 1.4

a)

b)

c)

Problem 1.6

a)

b)

Problem 1.7

Problem 1.8

Problem 1.9

Problem 1.10

Problem 1.11

Problem 1.12

Problem 1.13

c)

Problem 1.14

d)

Problem 1.22

a)

b)

c)

d)

Problem 1.23

b)

c)

Problem 1.33

a)

b)

Problem 1.34

a)

Problem 1.35

Problem 1.42

Solutions to Problems Chapter 3

Problem 3.4

- First Order

$$S^{(1)} = (-i\lambda)\frac{1}{4!}\int d^4x_1 : \phi_1\phi_1\phi_1\phi_1 : ,$$

- Second Order

$$S^{(2)} = \sum_{i=A}^{E} S_i^{(2)} ,$$

where

$$S_A^{(2)} = \frac{(-i\lambda)^2}{2!}\left(\frac{1}{4!}\right)^2 \int d^4x_1 d^4x_2 : \phi_1\phi_1\phi_1\phi_1\phi_2\phi_2\phi_2\phi_2 : ,$$

$$S_B^{(2)} = -\frac{(-i\lambda)^2}{2!}\left(\frac{1}{4!}\right)^2 4\times 4 \int d^4x_1 d^4x_2 : \phi_1\phi_1\phi_1\underbrace{\phi_1\phi_2}\phi_2\phi_2\phi_2 : ,$$

$$S_C^{(2)} = \frac{(-i\lambda)^2}{2!}\left(\frac{1}{4!}\right)^2 3!\times 3!\times 2 \int d^4x_1 d^4x_2 : \phi_1\phi_1\underbrace{\phi_1\underbrace{\phi_1\phi_2}\phi_2}\phi_2\phi_2 : ,$$

$$S_D^{(2)} = -\frac{(-i\lambda)^2}{2!}\left(\frac{1}{4!}\right)^2 4\times 4\times 3! \int d^4x_1 d^4x_2 : \underbrace{\phi_1\phi_1\phi_1\underbrace{\phi_1\phi_2}\phi_2\phi_2}\phi_2 : ,$$

$$S_E^{(2)} = \frac{(-i\lambda)^2}{2!}\left(\frac{1}{4!}\right)^2 4! \int d^4x_1 d^4x_2 : \underbrace{\phi_1\phi_1\phi_1\underbrace{\phi_1\phi_2}\phi_2\phi_2\phi_2} : ,$$

Problem 3.5

$$i\,\mathcal{M}^a = \overline{u}(p_3)(ie\gamma_\mu)u(p_1)D_F^{\mu\nu}(p_1-p_3)\overline{u}(p_4)(ie\gamma_\nu)u(p_2) ,$$

$$i\,\mathcal{M}^b = (-1)\overline{u}(p_4)(ie\gamma_\mu)u(p_1)D_F^{\mu\nu}(p_1-p_4)\overline{u}(p_3)(ie\gamma_\nu)u(p_2) .$$

Problem 3.6

$$i\,\mathcal{M}^a = \overline{u}(p_3)(ie\gamma_\mu)S_F(p_3-k_1)(ie\gamma_\nu)v(p_4)\epsilon^\mu(k_1)\epsilon^\nu(k_2) ,$$

$$i\,\mathcal{M}^b = \overline{u}(p_3)(ie\gamma_\nu)S_F(p_3-k_2)(ie\gamma_\mu)v(p_4)\epsilon^\mu(k_1)\epsilon^\nu(k_2) .$$

Problem 3.7

$$i\,\mathcal{M}^a = \overline{v}(p_2)(ie\gamma_\nu)S_F(p_1-k_1)(ie\gamma_\mu)u(p_1)\epsilon^\mu(k_1)\epsilon^\nu(k_2) ,$$

$$i\,\mathcal{M}^b = \overline{v}(p_2)(ie\gamma_\mu)S_F(p_1-k_2)(ie\gamma_\nu)u(p_1)\epsilon^\mu(k_1)\epsilon^\nu(k_2) .$$

Solutions to Problems Chapter 4

Problem 4.4

a) $\overline{1} = 1$, $\overline{\gamma^\mu} = \gamma^\mu$, $\overline{\sigma^{\mu\nu}} = \sigma^{\mu\nu}$, $\overline{\gamma^\mu\gamma_5} = \gamma^\mu\gamma_5$, $\overline{\gamma_5} = -\gamma_5$.

c) $\overline{\gamma^\mu P_L} = \gamma^\mu P_L$, $\overline{\gamma^\mu P_R} = \gamma^\mu P_R$.

Solutions to Problems Chapter 5

Problem 5.1

a) $\mathcal{M} = (\mathcal{M}_1^{\mu\nu} + \mathcal{M}_2^{\mu\nu})\,\epsilon_\mu(k_1)\epsilon_\nu(k_2)$, where

$$\mathcal{M}_1^{\mu\nu} = -\,e^2\,\overline{u}(p_3)\gamma^\mu\frac{\not{p}_3 - \not{k}_1 + m}{(p_3-k_1)^2 - m^2}\gamma^\nu v(p_4)$$
$$\mathcal{M}_2^{\mu\nu} = -\,e^2\,\overline{u}(p_3)\gamma^\nu\frac{\not{k}_1 - \not{p}_4 + m}{(k_1-p_4)^2 - m^2}\gamma^\mu v(p_4)$$

b) We have for k_1,

$$k_{1\mu}\mathcal{M}_1^{\mu\nu} = e^2\,\overline{u}(p_3)\gamma^\nu v(p_4), \quad k_{1\mu}\mathcal{M}_2^{\mu\nu} = -e^2\,\overline{u}(p_3)\gamma^\nu v(p_4), \quad k_{1\mu}\mathcal{M}^{\mu\nu} = 0.$$

c)

$$\sigma_{\text{tot}} = \frac{4\pi\alpha}{s^3}\left[\left(s + 4m_e^2 s - 8m_e^2\right)\right)\ln\left(\frac{\sqrt{s}+\sqrt{s-4m_e^2}}{\sqrt{s}+\sqrt{s-4m_e^2}}\right) - (s+4m_e^2)\sqrt{s(s-4m_e^2)}\Bigg]$$

Problem 5.2

b) $\mathcal{M} = (\mathcal{M}_1^{\mu\nu} + \mathcal{M}_2^{\mu\nu} + \mathcal{M}_3^{\mu\nu})\,\epsilon_\mu(k)\epsilon_\nu(k')$, where

$$\mathcal{M}_1^{\mu\nu} = -\,e^2\frac{1}{(p+k)^2 - m^2}(2p+k)^\mu(p+k+p')^\nu$$
$$\mathcal{M}_2^{\mu\nu} = -\,e^2\frac{1}{(p'-k)^2 - m^2}(2p-k')^\nu(2p'-k)^\mu$$
$$\mathcal{M}_3^{\mu\nu} = 2e^2 g^{\mu\nu}$$

c) We get

$$\begin{aligned} k_\mu \mathcal{M}_1^{\mu\nu} &= -\, e^2 (p + k + p')^\nu \\ k_\mu \mathcal{M}_2^{\mu\nu} &= e^2 (2p - k')^\nu \\ k_\mu \mathcal{M}_3^{\mu\nu} &= 2e^2 k^\nu \\ k_\mu \mathcal{M}^{\mu\nu} &= e^2 \left[p + k - p' - k'\right]^\nu = 0 \end{aligned}$$

Problem 5.3

c) From Eq. (5.13) and Eq. (5.146) we get

$$\lim_{\beta\to 1, \theta\to 0} \frac{d\sigma}{d\Omega}_{\text{Mott}} = \frac{\alpha^2}{s} \frac{1}{\sin^4(\theta/2)} = \lim_{\theta\to 0} \frac{d\sigma}{d\Omega}_{\text{Møller}}$$

Solutions to Problems Chapter 8

Problem 8.4

b) We have

$$\mathcal{M} = \frac{g^2}{2c_W^2} \frac{1}{s - m_Z^2 + i M_Z \Gamma_Z} \, \overline{v}(p_2) \gamma^\mu (g_V^e - g_A^e \gamma_5) u(p_1) \; \overline{u}(p_3) \gamma_\mu P_L v(p_4)$$

c) At $\sqrt{s} = M_Z$ we have,

$$\frac{\sigma(e^+e^- \to \nu_e \overline{\nu}_e)}{\sigma(e^+e^- \to e^+e^-)} = \frac{(g_V^\nu)^2 + (g_A^\nu)^2}{(g_V^e)^2 + (g_A^e)^2} \simeq 2. \tag{E.1}$$

Problem 8.5

a) $\beta_e = \dfrac{M_W^2 - m_e^2}{M_W^2 + m_e^2}$

c) $\Gamma(W^- \to e^- \nu_e) = \dfrac{G_F M_W^3}{6\pi\sqrt{2}}$

Problem 8.9

We just give the results for $t \to b + W^+ + \gamma$, the others being similar.

b) We have for, $t(p_1) \to b(p_2) + W^+(p_3) + \gamma(k)$,

$$k_\mu \mathcal{M}_1^\mu = \frac{eg}{\sqrt{2}} \frac{2}{3} \, \overline{u}(p_2) \gamma^\alpha P_L u(p_1) \; \epsilon_\alpha(k)$$

$$k_\mu \mathcal{M}_2^\mu = \frac{eg}{\sqrt{2}} \frac{1}{3} \overline{u}(p_2)\gamma^\alpha P_L u(p_1)\ \epsilon_\alpha(k)$$
$$k_\mu \mathcal{M}_3^\mu = -\frac{eg}{\sqrt{2}} \overline{u}(p_2)\gamma^\alpha P_L u(p_1)\ \epsilon_\alpha(k)$$
$$k_\mu \left(\mathcal{M}_1^\mu + \mathcal{M}_2^\mu + \mathcal{M}_3^\mu\right) = 0.$$

Problem 8.10

b) $R = \dfrac{m_e^2(m_\pi^2 - m_e^2)^2}{m_\mu^2(m_\pi^2 - m_\mu^2)^2}$.

c) $\Gamma(\pi^+ \to \nu_\mu + \mu^+) = \dfrac{G_F^2 V_{ud}^2 f_\pi^2}{8\pi m_\pi^3} m_\mu^2 \left(m_\pi^2 - m_\mu^2\right)^2$, therefore $f_\pi \simeq 131.7$ MeV.

d) $\lim\limits_{m_e \to 0} R = 0$. Helicity suppression.

Problem 8.11

b) $\Gamma = \dfrac{G_F^2 V_{ud}^2 f_\pi^2}{16\pi} \dfrac{(m_\tau^2 - m_\pi^2)^2}{m_\tau}$

c) $\dfrac{\Gamma(\tau \to \nu_\tau + \pi)}{\Gamma(\tau \to \text{All})} = 10.8\%$

Problem 8.12

c) $3.16 \times 10^{-6}\ \text{MeV}^{-1}$.

Problem 8.16

c) $\dfrac{d\sigma}{d\Omega} = \dfrac{g_F^2 s}{4\pi^2}(1 - 2\epsilon)$

d) $\sigma = \dfrac{G_F^2 s}{\pi}(1 - 2\epsilon)$. $\dfrac{\delta\sigma}{\sigma} = 2\dfrac{\delta\epsilon}{\epsilon} = 10\%$.

Problem 8.18

a) See Eq. (8.55).

Problem 8.20

c) We have

$$\sigma = \frac{g^4}{4\pi c_W^4} \frac{g_V^4 + g_A^4 + 6 g_V^2 g_A^2}{s^2} \left[\frac{s^2 + 4M_Z^4)}{2(s - 2M_Z^2)} \ln\left(\frac{s - M_Z^2 + \sqrt{s(s - 4M_Z^2)}}{s - M_Z^2 - \sqrt{s(s - 4M_Z^2)}} \right) - \sqrt{s(s - 4M_Z^2)} \right]$$

Problem 8.22

b) $\Gamma = \dfrac{G_F}{24\pi\sqrt{2}} \dfrac{(M_Z^2 - M_H^2)^3}{M_Z^3}$

c) $\Gamma = \dfrac{G_F M_Z^3}{24\pi\sqrt{2}}$

Problem 8.23

c) We give the separate contributions in an obvious notation,

$$\sigma_\gamma = \frac{e^4}{192\pi M_W^4} \frac{\sqrt{1-\frac{4M_W^2}{s}}\,(s-4M_W^2)}{s^2} \left(20M_W^2 s + 12M_W^4 + s^2\right)$$

$$\sigma_Z = \frac{e^4\left(-2c_W^2 s_W^2 + c_W^4 + 5s_W^4\right)}{1536\pi M_W^4 s_W^4} \frac{\sqrt{1-\frac{4M_W^2}{s}}\,(s-4M_W^2)}{\left(M_Z^2 - s\right)^2} \left(20M_W^2 s + 12M_W^4 + s^2\right)$$

$$\sigma_\nu = \frac{e^4}{1536\pi M_W^4 s_W^4} \frac{\sqrt{1-\frac{4M_W^2}{s}}}{s^2\,(s-4M_W^2)} \Bigg[s\left(16M_W^2 s^2 - 128M_W^4 s + 192M_W^6 + s^3\right) + 24M_W^4\left(2M_W^2 - s\right)\sqrt{s\,(s-4M_W^2)}\,\ln\left(\frac{2M_W^2 - s + \sqrt{s(s-4M_W^2)}}{2M_W^2 - s - \sqrt{s(s-4M_W^2)}}\right)\Bigg]$$

$$\sigma_{\gamma Z} = \frac{e^4\left(c_W^2 - 3s_W^2\right)}{384\pi M_W^4 s_W^2} \frac{\left(1-\frac{4M_W^2}{s}\right)^{3/2}}{\left(s - M_Z^2\right)} \left(20M_W^2 s + 12M_W^4 + s^2\right)$$

$$\sigma_{\gamma\nu} = \frac{e^4}{384\pi M_W^4 s_W^2} \frac{\sqrt{1-\frac{4M_W^2}{s}}}{s^3\,(4M_W^2 - s)} \Bigg[s\left(-100M_W^4 s^2 + 14M_W^2 s^3 + 88M_W^6 s + 96M_W^8 + s^4\right) + 24M_W^6\sqrt{s\,(s-4M_W^2)}\left(M_W^2 + 2s\right)\ln\left(\frac{2M_W^2 - s + \sqrt{s(s-4M_W^2)}}{2M_W^2 - s - \sqrt{s(s-4M_W^2)}}\right)\Bigg]$$

$$\sigma_{Z\nu} = \frac{e^4\left(c_W^2 - s_W^2\right)}{768\pi M_W^4 s_W^4} \frac{\sqrt{1-\frac{4M_W^2}{s}}}{s^2\,(4M_W^2 - s)\,(s - M_Z^2)} \Bigg[s\left(-100M_W^4 s^2 + 14M_W^2 s^3 + 88M_W^6 s + 96M_W^8 + s^4\right) + 24M_W^6\sqrt{s\,(s-4M_W^2)}\left(M_W^2 + 2s\right) \times \ln\left(\frac{2M_W^2 - s + \sqrt{s(s-4M_W^2)}}{2M_W^2 - s - \sqrt{s(s-4M_W^2)}}\right)\Bigg]$$

Problem 8.24

c) $\dfrac{d\sigma}{d\Omega} = \dfrac{G_F^2 M_W^4}{\pi^2} \dfrac{s}{(s(1-\cos\theta) + M_W^2)^2}$

d) $\sigma_0 = \dfrac{G_F^2 M_W^2}{\pi}$, $\alpha = 0$.

Problem 8.25

c) $\dfrac{d\sigma}{d\Omega} = \dfrac{g^2}{64\pi^2} \dfrac{s(1-\cos^2\theta) + 4m_f^2}{[s(1-\cos\theta) + 2m_f^2]^2}$

d) $\sigma \simeq \dfrac{g^2}{16\pi^2} \dfrac{1}{s} \ln \dfrac{s}{m_f^2} + \mathcal{O}(1/s)$

Problem 8.26

c) We have for, $e^-(p_1) + e^+(p_2) \to \phi(p_3) + \gamma(k)$,

$$k_\mu \mathcal{M}_1^\mu = e\beta\, \overline{v}(p_2)\gamma_5 u(p_1), \quad k_\mu \mathcal{M}_2^\mu = -e\beta\, \overline{v}(p_2)\gamma_5 u(p_1), \quad k_\mu \left(\mathcal{M}_1^\mu + \mathcal{M}_2^\mu\right) = 0.$$

Problem 8.27

b) $\Gamma = \dfrac{\beta^2}{8\pi}\, m_\phi \sqrt{1 - \dfrac{4m_e^2}{m_\phi^2}}$

Problem 8.29

b) $\Gamma = \dfrac{1}{16\pi}\, \dfrac{\mu^2}{m_\chi} \sqrt{1 - \dfrac{4m_\phi^2}{m_\chi^2}}$

Problem 8.30

c) In the limit $m_\phi = m_\chi = 0$ we get
$\dfrac{d\sigma}{d\Omega} = \dfrac{\mu^4}{64\pi^2 s^3} \dfrac{(1-\cos\theta)^2}{(1+\cos\theta)^2}$

d) $\sigma = \dfrac{\mu^4}{16\pi s^3} \left[\cos\theta^{\min} - 4\dfrac{\cos\theta^{\min}}{\sin^2\theta^{\min}} + 2\ln\left(\tan^2\dfrac{\theta^{\min}}{2}\right)\right]$

Problem 8.31

c) We have for, $e^-(p_1) + e^+(p_2) \to \phi(p_3) + \phi(p_4) + \gamma(k)$,

$$k_\mu \mathcal{M}_1^\mu = \frac{e\mu\lambda}{s - m_\chi^2} \overline{v}(p_2) u(p_1), \quad k_\mu \mathcal{M}_2^\mu = -\frac{e\mu\lambda}{s - m_\chi^2} \overline{v}(p_2) u(p_1),$$

$$k_\mu \left(\mathcal{M}_1^\mu + \mathcal{M}_2^\mu\right) = 0.$$

Problem 8.32

b) $\Gamma = \dfrac{\lambda^2}{8\pi} \dfrac{\left(m_\chi^2 - 4m_e^2\right)^{3/2}}{m_\chi^2}$

Problem 8.33

c) $\dfrac{d\sigma}{d\Omega} = \dfrac{\lambda^2}{128\pi^2 s}\dfrac{(3+\cos\theta)^2}{1+\cos\theta}$

Problem 8.35

c) We have for, $e^-(p_1) + \phi(p_2) \to e^-(p_3) + \gamma(k)$,

$$k_\mu \mathcal{M}_1^\mu = e\beta\ \overline{v}(p_3)\gamma_5 u(p_1), \quad k_\mu \mathcal{M}_2^\mu = -e\beta\ \overline{v}(p_2)\gamma_5 u(p_1),$$
$$k_\mu \left(\mathcal{M}_1^\mu + \mathcal{M}_2^\mu\right) = 0.$$

Problem 8.36

b) Denoting by $C_Z = \dfrac{g^2}{2c_W^2}\dfrac{1}{t-m_Z^2}$, $C_W = -\dfrac{g^2}{2}\dfrac{1}{u-m_Z^2}$, $c_L = g_V + g_A$, $c_R = g_V - g_A$, we get,

$$\mathcal{M}(--;--) = 2C_Z c_L\ s_{34}^* s_{12} + 2C_W\ s_{43}^* s_{12},\ \mathcal{M}(-+;-+) = 2C_Z c_R\ s_{23}^* s_{41}$$

and $\overline{|\mathcal{M}|^2} = 2\left[s^2\left(C_Z^2 c_L^2 + C_W^2 - 2C_Z c_L C_W\right) + C_Z^2 c_R^2 u^2\right]$

Problem 8.38

c) We have for, $e^-(p_1) + \chi(p_2) \to e^-(p_3) + \gamma(k)$,

$$k_\mu \mathcal{M}_1^\mu = eg\ \overline{v}(p_3)u(p_1), \quad k_\mu \mathcal{M}_2^\mu = -eg\ \overline{v}(p_2)u(p_1),$$
$$k_\mu \left(\mathcal{M}_1^\mu + \mathcal{M}_2^\mu\right) = 0.$$

Problem 8.39

c) $\dfrac{d\sigma}{d\Omega} = \dfrac{(g\mu)^2}{128\pi^2}\dfrac{1}{s^2}$

d) $\sigma = \dfrac{(g\mu)^2}{32\pi}\dfrac{1}{s^2}$, so, $g\mu_2 = 10.16$ GeV.

Problem 8.40

c) $\Gamma_{\text{Tot}} = \dfrac{g^2}{64\pi m_t^3 M_W^2}\left(m_t^2 - M_W^2\right)^2\left(m_t^2 + 2M_W^2\right)$, $\Gamma_{\text{L}} = \dfrac{g^2}{64\pi m_t M_W^2}\left(m_t^2 - M_W^2\right)^2$

Problem 8.41

c) $\dfrac{d\sigma}{d\Omega} = \dfrac{G_F^2 s}{4\pi^2}$, $\quad \sigma = \dfrac{G_F^2 s}{\pi}$

Problem 8.43

a) We get,

$$\sigma = \frac{G_F^2 M_Z^4\left(g_A^2 + g_V^2\right)}{6\pi s^2}\frac{\sqrt{-2M_H^2\left(M_Z^2 + s\right) + M_H^4 + \left(M_Z^2 - s\right)^2}}{\left(M_Z^2 - s\right)^2}$$
$$\times\left[-2M_H^2\left(M_Z^2 + s\right) + M_H^4 + 10M_Z^2 s + M_Z^4 + s^2\right]$$

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