Lectures in Quantum Field Theory
Lecture 1

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Summary of Lectures

- Part 1 – Relativistic Quantum Mechanics (Lecture 1)
  - The basic principles of Quantum Mechanics and Special Relativity
  - Klein-Gordon and Dirac equations
  - Gamma matrices and spinors
  - Covariance
  - Solutions for the free particle
  - Minimal coupling
  - Non relativistic limit and the Pauli equation
  - Charge conjugation and antiparticles
  - Massless spin 1/2 particles

Summary . . .

- Part 2 – Quantum Field Theory (Two Lectures)
  - QED as an example (Lecture 2)
    - QED as a gauge theory
    - Propagators and Green functions
    - Feynman rules for QED
    - Example 1: Compton scattering
    - Example 2: $e^- + e^+ \to \mu^- + \mu^+$ in QED
  - Non Abelian Gauge Theories (NAGT) (Lecture 3)
    - Radiative corrections and renormalization
    - Non Abelian gauge theories: Classical theory
    - Non Abelian gauge theories: Quantization
    - Feynman rules for a NAGT
    - Example: Vacuum polarization in QCD

Summary . . .

- Part 3 – The Standard Model (Lecture 4)
  - Gauge group and particle content of the massless SM
  - Spontaneous Symmetry Breaking and the Higgs mechanism
  - The SM with masses after Spontaneous Symmetry Breaking
  - Interactions dictated by the gauge symmetry group
  - Example 1: The decay $Z \to f\overline{f}$ in the SM
  - Example 2: $e^+e^- \to \mu^+\mu^-$ in the SM
  - Example 3: Muon decay
  - How the gauge symmetry corrects for the bad high energy behaviour in $\nu_e\overline{\nu}_e \to W_L^+W_L^-$
  - The need for the Higgs to correct the bad high energy behaviour in $W_L^+W_L^- \to W_L^+W_L^-$

Basic Ideas of Quantum Mechanics and Special Relativity

- In this lecture we will put together the ideas of Quantum Mechanics and Special Relativity
- This will lead to the substitution of Schrödinger equation by the Klein-Gordon (spin 0) and Dirac equations (spin 1/2)
- The idea of describing Quantum Physics by an equation will be abandoned in favour of a description in terms of a variable number of particles allowing for the creation and annihilation of particles
- However, the formalism that we will develop in this first lecture will be very useful for the Quantum Field Theory that we will address in the following lectures

Principles of Quantum Mechanics

We list the basic principles of Quantum Mechanics:

- For a given physical state there is a state function $|\Phi\rangle$ that contains all the possible information about the system
  - In many cases we will deal with the representation of state $|\Phi\rangle$ in terms of the coordinates, the so-called wave function $\Psi(q_i, s_i, t)$.
  - $|\Psi(q_i, s_i, t)|^2 \geq 0$ has the interpretation of a density of probability of finding the system in a state with coordinates q_i, internal quantum numbers s_i at time t
- Physical observables are represented by hermitian linear operators, as

$$p_i \to -i\hbar \frac{\partial}{\partial q_i} \;, \qquad E \to i\hbar \frac{\partial}{\partial t}$$

- A state $|\Phi_n\rangle$ is an eigenstate of the operator Ω if

$$\Omega\, |\Phi_n\rangle = \omega_n\, |\Phi_n\rangle$$

  where $|\Phi_n\rangle$ is the eigenstate that corresponds to the eigenvalue ω_n. If Ω is hermitian then ω_n are real numbers.

Principles of Quantum Mechanics ...

- For a complete set of commuting operators $\{\Omega_1, \Omega_2, \ldots\}$, there exists a complete set of orthonormal eigenfunctions, Ψ_n. An arbitrary state (wave function) can expanded in this set

$$\Psi = \sum_n a_n \Psi_n$$

- The result of the measurement of the observable Ω is any of the eigenvalues ω_n with probability $|a_n|^2$. The average value of the observable is

$$< \Omega >_\Psi = \sum_s \int dq_1 \ldots \Psi^*(q_i, s_i, t) \Omega \Psi(q_i, s_i, t) = \sum_n |a_n|^2 \omega_n$$

- The time evolution of a system is given by

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi$$

(The Hamiltonian H is a linear and hermitian operator)

- Linearity implies the superposition principle and hermiticity leads to the conservation of probability

$$\frac{d}{dt} \sum_s \int dq_1 \cdots \Psi^* \Psi = \frac{i}{\hbar} \sum_s \int dq_1 \cdots [(H\Psi)^* \Psi - \Psi^* (H\Psi)] = 0$$

Principles of Special Relativity

Special relativity is based on the principles of relativity and of the constancy of the speed of light in all reference frames. For our purposes it is enough to recall:

- The coordinates of two reference frames are related by

$$x'^{\mu} = a^{\mu}{}_{\nu}\, x^{\nu}, \qquad \mu, \nu = 0, 1, 2, 3$$

- The invariance of the interval

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = dx^{\mu} dx_{\mu}$$

where the metric is diagonal and given by $g_{\mu\nu} = \text{diag}(+ - - -)$, restricts the coefficients $a^{\mu}{}_{\nu}$ to obey

$$g_{\mu\nu} a^{\mu}{}_{\alpha} a^{\nu}{}_{\beta} dx^{\alpha} dx^{\beta} = g_{\alpha\beta} dx^{\alpha} dx^{\beta}$$

which implies

$$a^{\mu}{}_{\alpha} g_{\mu\nu} a^{\nu}{}_{\beta} = g_{\alpha\beta} \qquad \text{or in matrix form} \qquad \boxed{a^T g\, a = g}$$

Principles of Special Relativity ...

- The matrices that obey $a^T g\, a = g$ constitute the **Lorentz group**, designated by $O(3,1)$
- We can easily verify that

$$\det\ a = \pm 1$$

- Transformations with $\det\ a = +1$ constitute the **proper Lorentz group** a subgroup of the Lorentz group. They can be built from infinitesimal transformations. Examples are **rotations** and **Lorentz transformations**

$$a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad ; \quad \beta = \frac{V}{c}$$

and V is the relative velocity of reference frame S' with respect to S

- Examples of transformations with $\det\ a = -1$ are the space (**Parity**) and time (**Time Reversal**) inversions. For instance

$$a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad a = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Parity　　　　Time Reversal

- The transformations with $\det\ a = -1$ do not form a subgroup of the full Lorentz group (they do not contain the identity)

The Klein-Gordon Equation

We start with the free particle:

- In non-relativistic quantum mechanics we get Schrödinger equation from the fundamental equation

$$i\hbar \frac{\partial}{\partial t}\psi = H\psi$$

using the non-relativistic free particle Hamiltonian

$$H = \frac{p^2}{2m}$$

and making the substitution $\vec{p} \to -i\hbar\vec{\nabla}$. We get

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \psi$$

- The idea is to use the relativistic form of $H = E$, the energy of the free particle.

The Klein-Gordon Equation ...

- In special relativity the energy and momentum are related through

$$p_\mu p^\mu = m^2 c^2, \quad p^\mu \equiv \left(\frac{E}{c}, \vec{p}\right)$$

giving

$$E^2 = p^2 c^2 + m^2 c^4 \to E = \pm\sqrt{p^2c^2 + m^2c^4}$$

- Classically we require energies to be positive, so we could choose

$$H = \sqrt{p^2c^2 + m^2c^4}$$

- We are lead to interpret the square root of an operator. To avoid this problem we can find an equation for H^2. Iterating the original equation and noticing that $\left[i\hbar\frac{\partial}{\partial t}, H\right] = 0$, we get

$$-\hbar^2 \frac{\partial^2}{\partial t^2}\psi = (-\hbar^2 c^2 \vec{\nabla}^2 + m^2 c^4)\psi \to \left[\Box + \left(\frac{mc}{\hbar}\right)^2\right]\psi = 0, \quad \Box = \partial_\mu \partial^\mu$$

The Klein-Gordon Equation ...

- Now we have no problem with the operators but we reintroduce the negative energy solutions. We will see that the negative energy solutions can not be avoided and its interpretation will be related to the antiparticles.

- It was not because of the negative energy solutions that the Klein-Gordon equation was abandoned, but because of the density of probability. Using the Klein-Gordon equation and its hermitian conjugate we get

$$\psi^* \left[\Box + \left(\frac{mc}{\hbar}\right)^2\right] \psi - \psi \left[\Box + \left(\frac{mc}{\hbar}\right)^2\right] \psi^* = 0$$

or

$$0 = \psi^* \Box \psi - \psi \Box \psi^* = \partial_\mu (\psi^* \overset{\leftrightarrow}{\partial}{}^\mu \psi) \to \partial_\mu J^\mu = 0 \quad ; \quad J^\mu = \psi^* \overset{\leftrightarrow}{\partial}{}^\mu \psi$$

- In the usual identification $J^\mu = (\rho c, \vec{J})$ gives

$$\rho = \frac{1}{c^2}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) \to \boxed{\text{Not positive defined}}$$

- The Klein-Gordon equation was abandoned by the wrong reasons: $\boxed{\text{Spin 0}}$

Dirac Equation

- Dirac approach was to treat time and space on the same footing, in the spirit of relativity. As in the fundamental equation the time derivative appears linearly, Dirac postulated the same behaviour for the space derivatives, writing,

$$\boxed{i\hbar\frac{\partial\psi}{\partial t} = \left(-i\hbar c\vec{\alpha}\cdot\vec{\nabla} + \beta mc^2\right)\psi \equiv H\psi}$$

- The dimensionless constants α^i and β can not be numbers as we will see in a moment. So, Dirac postulated that $\vec{\alpha}$ and β are $N\times N$ hermitian matrices (for H to be hermitian) acting on a column vector with N entries,

$$\psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_N \end{bmatrix}$$

- This matrix equation must obey the conditions:
  - Give the correct relation $E^2 = p^2c^2 + m^2c^4$ for the free particle
  - Give a probability that is positive defined
  - Must be covariant under Lorentz transformations

- For the energy-momentum relation, it is enough that each component satisfies the Klein-Gordon equation. We iterate the equation to get

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = \left(-i\hbar c \alpha^i \nabla_i + \beta m c^2\right) i\hbar \frac{\partial \psi}{\partial t}$$

$$= \left[-\hbar^2 c^2 \frac{\alpha^i \alpha^j + \alpha^j \alpha^i}{2} \nabla_i \nabla_j - i\hbar m c^2 (\alpha^i \beta + \beta \alpha^i) \nabla_i + \beta^2 m^2 c^4\right] \psi$$

- This gives the relations

$$\begin{cases} \alpha^i \alpha^j + \alpha^j \alpha^i = 2\delta^{ij} \\ \alpha^i \beta + \beta \alpha^i = 0 \\ (\alpha^i)^2 = \beta^2 = 1 \to \boxed{\text{Eigenvalues} = \pm 1} \end{cases}$$

- We have to construct 4 anti-commuting matrices with square equal to unity. This is not possible for $N = 2$ (only 3 Pauli matrices). It is not possible for N odd, because

$$\alpha^i = -\beta \alpha^i \beta \to \text{Tr}(\alpha^i) = \text{Tr}(-\beta \alpha^i \beta) = -\text{Tr}(\alpha^i) \to \boxed{\text{Tr}(\alpha^i) = 0}$$

Dirac Equation . . .

- The smallest is $N = 4$. An explicit representation (not unique) is

$$\alpha^i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} \quad ; \quad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \boxed{\text{Blocks } 2 \times 2}$$

where σ_i are the Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \; ; \; \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \; ; \; \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Let us now look at the probability current. As α^i and β are hermitian we get

$$-i\hbar \frac{\partial \psi^\dagger}{\partial t} = \psi^\dagger (i\hbar c \alpha^i \overleftarrow{\partial}_i + \beta m c^2)$$

- Doing the usual trick of multiplying and subtracting we get

$$i\hbar \frac{\partial}{\partial t}(\psi^\dagger \psi) = -i\hbar c \nabla_i (\psi^\dagger \alpha^i \psi) \rightarrow \boxed{\frac{\partial}{\partial t}(\psi^\dagger \psi) + \vec{\nabla} \cdot (\psi^\dagger c \vec{\alpha} \psi) = 0}$$

Dirac Equation ...

- Using

$$\rho = \psi^\dagger \psi, \quad \vec{j} = \psi^\dagger c \vec{\alpha} \psi \quad \rightarrow \quad \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0}$$

- Integrating in all the space we get

$$\frac{d}{dt} \int d^3x \; \psi^\dagger \psi = 0$$

which allow us to identify $\psi^\dagger \psi$ with a positive density of probability

- The notation anticipates the fact that $\vec{j}$ is a vector in 3-dimensional space. Actually, we will show in the following that $j^\mu = (c\rho, \vec{j})$ is a conserved 4-vector, $\partial_\mu j^\mu = 0$, and that Dirac equation is covariant, that is it, maintains its form in all inertial reference frames

Dirac gamma matrices

- We start by introduction a convenient notation due to Dirac. We multiply the Dirac equation on the left by $\frac{1}{c}\beta$ and introduce the matrices,

$$\gamma^0 \equiv \beta \qquad ; \qquad \gamma^i \equiv \beta\alpha^i \qquad i = 1, 2, 3$$

- The Dirac equation reads

$$\boxed{(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0 \quad \text{or} \quad (i\hbar\partial\!\!\!/ - mc)\psi = 0}$$

where we introduced Feynman slash notation, $\partial\!\!\!/ \equiv \gamma^\mu\partial_\mu$

- In the Dirac representation we have

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad ; \qquad \gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}$$

- The normalization and anti-commuting relations can be written as

$$\boxed{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}}$$

- We should note that, despite the suggestive form of the above equation, we have not yet proved its covariance

Different γ representations: Equivalence transformations

- Let us consider two representations, γ^μ and $\tilde{\gamma}^\mu$. Both satisfy Dirac equation,

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0, \quad (i\hbar\tilde{\gamma}^\mu\partial_\mu - mc)\tilde{\psi} = 0$$

- Both describe the same Physics, so must exist a relation between ψ and $\tilde{\psi}$. Let it be

$$\psi = U\tilde{\psi}$$

where U is a constant matrix that admits inverse. Substituting in the above equation and multiplying on the left by U^{-1} we get

$$\boxed{\tilde{\gamma}^\mu = U^{-1}\gamma^\mu U}$$

- This kind of transformations are called equivalence transformations and they should not change the observable physical quantities despite the fact that the wave function is changed. For future use, we also note that

$$\boxed{\text{Tr}\left[\tilde{\gamma}^\mu\tilde{\gamma}^\nu\cdots\tilde{\gamma}^\sigma\right] = \text{Tr}\left[\gamma^\mu\gamma^\nu\cdots\gamma^\sigma\right]}$$

Proof of the Covariance

- Let us consider the Dirac equation in two inertial frames O and O'

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi(x) = 0, \quad (i\hbar\gamma'^\mu\partial'_\mu - mc)\psi'(x') = 0$$

- γ'^μ satisfies the same anti-commuting relations as γ^μ, as well as the relations, $\gamma'^{0\dagger} = \gamma'^0$ and $\gamma'^{i\dagger} = -\gamma'^i$. We can then show that γ'^μ and γ^μ are related by an equivalence transformation,

$$\gamma'^\mu = U^{-1}\gamma^\mu U$$

where U is an unitary matrix.

- We can therefore transfer all the changes into the wave function and keep the same representation in all inertial frames. The wave functions $\psi'(x')$ and $\psi(x)$ must then be related through

$$\boxed{\psi'(x') = \psi'(ax) = S(a)\psi(x) = S(a)\psi(a^{-1}x'), \quad x'^\mu = a^\mu{}_\nu x^\nu}$$

and the matrix $S(a)$ must depend on the relative velocity and/or the rotation between the two frames (for proper Lorentz transformations)

Proof of the Covariance ...

- Substituting $\psi'(x') = S(a)\psi(x)$ we get

$$(i\hbar\gamma^\mu \frac{\partial}{\partial x'^\mu} - mc)S(a)\psi(x) = 0$$

- Now we use

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = (a^{-1})^\nu{}_\mu \partial_\nu$$

to get

$$\left[i\hbar S^{-1}(a)\gamma^\mu S(a)(a^{-1})^\nu{}_\mu\, \partial_\nu - mc\right]\psi(x) = 0$$

- Comparing with the equation in frame O, we have the same equation if

$$S^{-1}(a)\gamma^\mu S(a)(a^{-1})^\nu{}_\mu = \gamma^\nu \quad \to \quad \boxed{S(a)\gamma^\mu S^{-1}(a)a^\nu{}_\mu = \gamma^\nu}$$

These are the fundamental relations that will allow to find $S(a)$. If we succeed we have proved the covariance

Proof of the Covariance ...

- To obtain $S(a)$ we start by considering infinitesimal transformations,

$$a^{\nu}{}_{\mu} = g^{\nu}{}_{\mu} + \omega^{\nu}{}_{\mu} + \cdots \quad \text{with} \quad \omega^{\mu\nu} = -\omega^{\nu\mu}$$

- The last equation means that there are only six independent parameters. We will see that they will be identified with the three degrees of freedom of a rotation plus the three degrees of freedom of a Lorentz transformation (boost) in an arbitrary direction

- For infinitesimal transformations we define then

$$S = 1 - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu} + \cdots\,, \quad S^{-1} = 1 + \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu} + \cdots$$

where the matrices $\sigma_{\mu\nu} = -\sigma_{\nu\mu}$ are anti-symmetric.

- Substituting in the original equation and keeping terms only linear in $\omega^{\mu\nu}$ we get

$$[\gamma^{\mu}, \sigma_{\alpha\beta}] = 2i(g^{\mu}{}_{\alpha}\gamma_{\beta} - g^{\mu}{}_{\beta}\gamma_{\alpha})$$

Proof of the Covariance ...

- Using the anti-commutation relations for the γ's we can verify that the solution is

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$$

- This determines S and S^{-1} for infinitesimal transformations. However, for these continuous transformations (**Lie groups**) the solution exponentiates, giving the final result

$$S = e^{-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}}$$

- To find an explicit form we distinguish the case of Lorentz boosts from the rotations. For rotations we define the vectors,

$$(\theta^1, \theta^2, \theta^3) \equiv ({\omega^2}_3, {\omega^3}_1, {\omega^1}_2), \quad (\Sigma^1, \Sigma^2, \Sigma^3) \equiv (\sigma^{23}, \sigma^{31}, \sigma^{12})$$

- Then

$$S_R = e^{\frac{i}{2}\vec{\theta}\cdot\vec{\Sigma}}, \quad \vec{\Sigma} \equiv \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \quad \text{(Dirac representation)}$$

a generalization of the way 2-spinors transform under rotations.

Proof of the Covariance ...

- Using

$$(\vec{\theta}\cdot\vec{\Sigma})(\vec{\theta}\cdot\vec{\Sigma}) = \vec{\theta}\cdot\vec{\theta}$$

we can get another useful form

$$\boxed{S_R = \cos\frac{\theta}{2} + i\hat{\theta}\cdot\vec{\Sigma}\sin\frac{\theta}{2}}$$

where $\hat{\theta}$ is an unit vector in the direction of the rotation

- For the Lorentz boosts we define the 3-vector $\vec{\omega}$ such that $(\omega^i \equiv \omega^{0i})$

$$\begin{cases} \hat{\omega} \equiv \hat{V} \\ \tanh\omega = \frac{V}{c} \end{cases}$$

- Using now

$$\sigma^{0i} = \frac{i}{2}\left[\gamma^0, \gamma^i\right] = i\gamma^0\gamma^i = i\alpha^i$$

Proof of the Covariance . . .

- We obtain for the case of the Lorentz transformations (boosts)

$$S_L = e^{-\frac{1}{2}\vec{\omega}\cdot\vec{\alpha}}, \qquad \boxed{S_L = \cosh\frac{\omega}{2} - \hat{\omega}\cdot\vec{\alpha}\sinh\frac{\omega}{2}}$$

  where we have used $(\vec{\omega}\cdot\vec{\alpha})^2 = \vec{\omega}\cdot\vec{\omega}$

- We can easily convince ourselves that while S_R is an unitary matrix but the same is not true for S_L. However, we can show (using $[\gamma^0, \vec{\Sigma}] = 0$ and $\{\gamma^0, \vec{\alpha}\} = 0$) that

$$S^{-1} = \gamma^0 S^\dagger \gamma^0$$

  both for S_R and S_L

- This is important to show that the current,

$$j^\mu(x) = c\psi^\dagger(x)\gamma^0\gamma^\mu\psi(x)$$

  is a 4-vector as we will prove now

Proof of the Covariance ...

- The current in frame O' is

$$\begin{aligned} j'^{\mu} =& c\psi'^{\dagger}(x')\gamma^0\gamma^{\mu}\psi'(x') \\ =& c\psi^{\dagger}(x)S^{\dagger}\gamma^0\gamma^{\mu}S\psi(x) \\ =& c\psi^{\dagger}(x)\gamma^0\gamma^0S^{\dagger}\gamma^0\gamma^{\mu}S\psi(x) \\ =& c\psi^{\dagger}(x)\gamma^0S^{-1}\gamma^{\mu}S\psi(x) \end{aligned}$$

  and using $S^{-1}\gamma^{\mu}S = a^{\mu}{}_{\nu}\gamma^{\nu}$ we get

$$j'^{\mu} = a^{\mu}{}_{\nu}j^{\nu}$$

  as it should for a 4-vector

- The combination $\psi^{\dagger}\gamma^0$ appears so frequently that it is convenient to define a symbol for it,

$$\boxed{\overline{\psi} \equiv \psi^{\dagger}\gamma^0}$$

  the so-called Dirac adjoint

Parity

Let us consider one case of discrete symmetries, the space inversion or parity, P.

- Parity corresponds to the Lorentz transformation with $\det\ a = -1$,

$$a^\mu{}_\nu \equiv \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

- We want to find a matrix S_P that satisfies

$$S_P^{-1} \gamma^\mu S_P = a^\mu{}_\nu \gamma^\nu \quad \to \quad \boxed{[\gamma^0, S_P] = 0,\ \{\gamma^i, S_P\} = 0}$$

- We can easily verify that the relation is satisfied by,

$$\boxed{\mathcal{P} \equiv S_P = e^{i\varphi}\gamma^0}$$

where $e^{i\varphi}$ is an arbitrary phase

Covariant Bilinears

- Any 4×4 matrix can be expanded in terms of a basis of 16 linearly independent, 4×4 matrices.

- To define this basis it is convenient to introduce the matrix

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \gamma_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$ (In the Dirac representation)

- From the definition we have the properties,

$$\{\gamma_5, \gamma^\mu\} = 0, \quad (\gamma_5)^2 = 1$$

- Now we define the 16 matrices of the basis

$$\Gamma^S = 1 \qquad \Gamma^P = \gamma_5$$

$$\Gamma^V_\mu = \gamma_\mu \qquad \Gamma^A_\mu \equiv \gamma_5\gamma_\mu$$

$$\Gamma^T_{\mu\nu} = \sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$$

Covariant Bilinears . . .

- The labels S, V, T, A and P have the meaning of **S**calar, **V**ector, **T**ensor, **A**xial-Vector and **P**seudo-Scalar and have to do with the transformations properties, under Lorentz transformations, of the bilinears

$$\overline{\psi}\,\Gamma^a\psi \qquad a = S, V, T, A \text{ and } P$$

- As an example

$$\begin{aligned}\overline{\psi'}(x')\,\Gamma^A\psi'(x') =& \overline{\psi'}(x')\gamma_5\gamma^\mu\psi'(x')\\ =& \overline{\psi}(x)S^{-1}\gamma_5\gamma^\mu S\psi(x)\\ =& \det a\ a^\mu{}_\nu\overline{\psi}(x)\gamma_5\gamma^\nu\psi(x)\end{aligned}$$

where we have used $[S,\gamma_5]=0$ for proper Lorentz transformations (rotations and boosts) and $\{\mathcal{P},\gamma_5\}=0$ for the space inversion (parity). This shows that $\overline{\psi}(x)\gamma_5\gamma_\mu\psi(x)$ is an axial-vector

Plane Wave Solutions

- From now on we take a system of units where $\hbar = c = 1$. The Dirac equation then reads

$$(i\partial\!\!\!/ - m)\psi(x) = 0$$

- This equation has plane wave solutions

$$\psi(x) = w(\vec{p})e^{-ip_\mu x^\mu}$$

if $p_\mu p^\mu = m^2$.

- This implies $(p^0)^2 = E^2 = \vec{p}\cdot\vec{p} + m^2$ showing that we have negative energy solutions

- We always take $p^0 = E = \sqrt{|\vec{p}|^2 + m^2} > 0$

$$\boxed{\psi^r(x) = w^r(\vec{p})e^{-i\varepsilon_r p_\mu x^\mu}}$$

where $\varepsilon_r = \pm 1$ for the positive and negative energy solutions, respectively

Plane Waves ...

- We start by finding $w^r(\vec{p})$ in the rest frame of the particle. Dirac equation reduces then to

$$\left(i\gamma^0 \frac{\partial}{\partial t} - m \right) \psi = 0$$

- Using

$$\psi^r = w^r(0) e^{-i\varepsilon_r m t}$$

we have

$$m\left(\varepsilon_r \gamma^0 - 1\right)\psi^r = 0, \quad \varepsilon_r = \begin{cases} +1 & r = 1,2 \\ -1 & r = 3,4 \end{cases}, \quad \gamma^0 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$$

- With the solutions ($N = \sqrt{2m}$)

$$w^{(1)}(0) = N \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, w^2(0) = N \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, w^3(0) = N \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, w^4(0) = N \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Plane Waves ...

- To get $w^r(\vec{p})$ we perform a Lorentz boost to the instantaneous frame that moves with velocity $-\vec{v}$, that is

$$\tanh\omega = |\vec{v}| = \beta \to \cosh\omega = \gamma,\ \sinh\omega = \gamma\beta$$

- We get then

$$\begin{aligned} w^r(\vec{p}) =& e^{-\frac{1}{2}\vec{\omega}\cdot\vec{\alpha}} w^r(0) \\ =& \left[\cosh\frac{\omega}{2}1 - \hat{\omega}\cdot\vec{\alpha}\sinh\frac{\omega}{2}\right] w^r(0) \\ =& \frac{1}{\sqrt{2m}\sqrt{E+m}}(\varepsilon_r \not{p} + m) w^r(0) \end{aligned}$$

and

$$\begin{aligned} w^{r\dagger}(\vec{p}) =& w^{r\dagger}(0)(\not{p}\gamma^0 + m)\frac{1}{\sqrt{2m}\sqrt{E+m}} \\ \overline{w}^r(\vec{p}) =& \overline{w}^r(0)(\varepsilon_r \not{p} + m)\frac{1}{\sqrt{2m}\sqrt{E+m}} \end{aligned}$$

Plane Waves . . .

Using these expressions we can derive the following relations:

$$(\not{p} - \varepsilon_r m) w^r(\vec{p}) = 0$$

$$\overline{w}^r(\vec{p})(\not{p} - \varepsilon_r m) = 0$$

$$\overline{w}^r(\vec{p}) w^{r'}(\vec{p}) = 2m\ \delta_{rr'} \varepsilon_r$$

$$\sum_{r=1}^{4} \varepsilon_r w^r_\alpha(\vec{p}) \overline{w}^r_\beta(\vec{p}) = 2m\ \delta_{\alpha\beta}$$

$$w^{r\dagger}(\varepsilon_r \vec{p}) w^{r'}(\varepsilon_{r'} \vec{p}) = 2E\ \delta_{rr'}$$

- $\overline{w}^r(\vec{p}) w^r(\vec{p})$ and $\overline{\psi}\psi$ are scalars
- $w^{r\dagger}(\varepsilon_r \vec{p}) w^r(\varepsilon_r \vec{p})$ and $\psi^\dagger \psi$ are the time component of a 4-vector.

Spin of the Dirac Equation

- The Poincaré group, translations, P^μ, plus Lorentz transformations, $J^{\mu\nu}$, has two invariants $P^2 = P_\mu P^\mu$ and $W^2 = W_\mu W^\mu$ where W_μ is the Pauli-Lubanski 4-vector,

$$\boxed{W_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma}$$

- One can show that their eigenvalues are

$$\boxed{P^2 = m^2, \quad W^2 = -m^2 s(s+1)}$$

where m is the mass and s the spin (integer or half-integer)

- For the Dirac equation and for infinitesimal transformations

$$\psi'(x) \equiv \left(1 - \frac{i}{2} J_{\mu\nu}\omega^{\mu\nu}\right)\psi(x), \quad \psi'(x') = \left(1 - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right)\psi(x)$$

leading to

$$\boxed{J_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu} + i(x_\mu\partial_\nu - x_\nu\partial_\mu)}$$

Spin of the Dirac Equation ...

- Using this $J_{\mu\nu}$ into W_μ we get

$$W_\mu = -\frac{i}{4}\varepsilon_{\mu\nu\rho\sigma}\sigma^{\nu\rho}\partial^\sigma$$

- We can easily show that for the Dirac equation we have,

$$W^2 = -\frac{3}{4}m^2$$

which confirms that the Dirac equation describes spin $s = \frac{1}{2}$.

- For the plane wave solutions we obtain

$$W_\mu = -\frac{1}{4}\varepsilon_r\ \varepsilon_{\mu\nu\rho\sigma}\sigma^{\nu\rho}p^\sigma = -\frac{1}{4}\gamma_5[\gamma_\mu, \not{p}]\ \varepsilon_r$$

- In the rest frame

$$\boxed{W^0 = 0\ , \quad \frac{\vec{W}}{m} = \frac{1}{2}\vec{\Sigma}\ \varepsilon_r \quad \text{with} \quad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}}$$

Spin of the Dirac Equation ...

- The spin operator in an arbitrary direction is

$$-\frac{W\cdot s}{m} = \frac{1}{2m}\gamma_5 \not{s} \not{p}\, \varepsilon_r , \qquad s^\mu s_\mu = -1,\ p_\mu s^\mu = 0$$

- Using this operator and choosing $s^\mu = (0,0,0,1)$ in the rest frame we get

$$\frac{1}{2m}\gamma_5 \not{s}\not{p}\varepsilon_r = -\frac{1}{2}\gamma_5\gamma^3 = -\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\sigma_3 & 0 \\ 0 & -\frac{1}{2}\sigma_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & & & \\ & -\frac{1}{2} & & \\ & & -\frac{1}{2} & \\ & & & \frac{1}{2} \end{bmatrix}$$

- This shows that $w^r(0)$ are the eigenstates with eigenvalues $\pm 1/2$ along the z axis: $+1/2$ for $r = 1,4$ and $-1/2$ for $r = 2,3$

Spinors u and v

- Instead of using the spinors $w^r(p)$ it is conventional to use different names for the positive energy solutions, $u(p,s)$ and for the negative energy, $v(p,s)$

- The u spinors satisfy

$$\boxed{(\not p - m)u(p,s) = 0} \quad \text{and in rest frame} \quad \vec{\Sigma} \cdot \vec{s}\, u(p,s) = u(p,s)$$

- The v spinors satisfy

$$\boxed{(\not p + m)v(p,s) = 0} \quad \text{and in rest frame} \quad \vec{\Sigma} \cdot \vec{s}\, v(p,s) = -v(p,s)$$

showing that it has spin $-\vec{s}$ in the rest frame.

- With these definitions we have the identification

$$\begin{aligned} w^1(\vec{p}) &= u(p, s_z) \\ w^2(\vec{p}) &= u(p, -s_z) \\ w^3(\vec{p}) &= v(p, -s_z) \\ w^4(\vec{p}) &= v(p, s_z) \end{aligned}$$

Spinors u and v ...

- The explicit expressions for u and v are (Dirac representation)

$$\boxed{u(p,s) = \sqrt{E+m}\begin{bmatrix} \chi(s) \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi(s) \end{bmatrix}} \qquad \boxed{v(p,s) = \sqrt{E+m}\begin{bmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi(-s) \\ \chi(-s) \end{bmatrix}}$$

where $\chi(s)$ is a Pauli two component spinor.

- As an example

$$v(p,\uparrow) = \sqrt{E+m}\begin{bmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi(\downarrow) \\ \chi(\downarrow) \end{bmatrix} = \sqrt{E+m}\begin{bmatrix} \frac{p_-}{E+m} \\ -\frac{p_z}{E+m} \\ 0 \\ 1 \end{bmatrix} = w^4(\vec{p})$$

where $p_- = p_x - ip_y$.

Wave Packets

- Dirac equation is linear so we can form superpositions of plane wave solutions, the so-called wave-packets

- Consider one formed only by positive energy solutions

$$\psi^{(+)}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E} \sum_{\pm s} b(p,s) u(p,s) e^{-ip\cdot x}$$

- We can show that

$$\int d^3x \psi^\dagger(x)\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E} \sum_{s} |b(p,s)|^2 = 1$$

and

$$\vec{J}^{(+)} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E} \frac{\vec{p}}{E} \sum_{s} |b(p,s)|^2 = < \frac{\vec{p}}{E} >$$

- $< \frac{\vec{p}}{E} >$ is the group velocity, so we recover a familiar result from wave packets in non-relativistic quantum mechanics

Wave Packets ...

- However the complete set of solutions must include also the negative energy solutions. Therefore

$$\psi(x)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E}\sum_s\left[b(p,s)u(p,s)e^{-ip\cdot x}+d^*(p,s)v(p,s)e^{ipx}\right]$$

- The probability gives

$$\int d^3x\psi^\dagger\psi=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E}\sum_s\left[|b(p,s)|^2+|d(p,s)|^2\right]=1$$

and the current, $(\tilde{p}\equiv(p^0,-\vec{p}))$

$$\begin{aligned}J^k=\int d^3x\overline{\psi}\gamma^k\psi=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E}\Bigg\{&\sum_s\left[|b(p,s)|^2+|d(p,s)|^2\right]\frac{p^k}{E}\\&+i\sum_{s,s'}b^*(\tilde{p},s')d^*(p,s)e^{2iEt}\overline{u}(\tilde{p},s')\sigma^{k0}v(p,s)\\&-i\sum_{s,s'}b(\tilde{p},s')d(p,s)e^{-2iEt}\overline{v}(p,s)\sigma^{k0}u(\tilde{p},s)\Bigg\}\end{aligned}$$

The cross terms oscillate very rapidly with frequencies $\omega=2E>2m\simeq 1\ \mathrm{MeV}>1.5\times10^{21}\ \mathrm{s}^{-1}$. They become important if we try to localize the electron at distances of order of its $\lambda_c=\frac{1}{m}\simeq4\times10^{-11}\ cm$

Minimal Coupling

- The interaction of charged particles with the electromagnetic field is given by the minimal coupling prescription,

$$\boxed{p^\mu \longrightarrow p^\mu - eA^\mu, \quad \text{or} \quad \partial_\mu \longrightarrow \partial_\mu + ieA_\mu, \quad e = -|e|}$$

- Therefore we get for the Dirac equation

$$(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m)\,\psi(x) = 0$$

- In the original form

$$i\frac{\partial \psi}{\partial t} = \left[-i\vec{\alpha}\cdot(\vec{\nabla} - ie\vec{A}) + \beta m + eA^0\right]\psi \equiv (H_0 + H')\psi$$

where

$$H_0 = -i\vec{\alpha}\cdot\vec{\nabla} + \beta m, \quad H' = -e\vec{\alpha}\cdot\vec{A} + eA^0$$

- Notice the analogy

$$H'_{\text{classic}} = -e\frac{\vec{v}}{c}\cdot\vec{A} + eA^0 \quad \rightarrow \quad \vec{v}_{op} = c\vec{\alpha}$$

Non-relativistic Limit of Dirac Equation: Free Particle

- We start by the free particle, defining

$$\psi = \begin{pmatrix} \hat{\varphi} \\ \hat{\chi} \end{pmatrix}$$

where $\hat{\chi}$ and $\hat{\varphi}$ are two component Pauli spinors.

- Using $\vec{\alpha}$ and β in the Dirac representation we get

$$\begin{cases} i\dfrac{\partial\hat{\varphi}}{\partial t} = -i\vec{\sigma}\cdot\vec{\nabla}\hat{\chi} + m\hat{\varphi} \\[2ex] i\dfrac{\partial\hat{\chi}}{\partial t} = -i\vec{\sigma}\cdot\vec{\nabla}\hat{\varphi} - m\hat{\chi} \end{cases}$$

- In the non-relativistic limit $E - m \ll m$ and therefore we make the redefinition

$$\begin{pmatrix} \hat{\varphi} \\ \hat{\chi} \end{pmatrix} = e^{-imt} \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

Non-relativistic Limit of Dirac Equation: Free Particle ...

- Substituting back we get for the redefined spinors

$$\begin{cases} i\dfrac{\partial\varphi}{\partial t} = -i\vec{\sigma}\cdot\vec{\nabla}\chi \\ i\dfrac{\partial\chi}{\partial t} = -i\vec{\sigma}\cdot\vec{\nabla}\varphi - 2m\chi \end{cases}$$

- As χ changes slowly we solve approximately the second equation, getting

$$\chi \simeq -i\frac{\vec{\sigma}\cdot\vec{\nabla}}{2\ m}\varphi = \frac{\vec{\sigma}\cdot\vec{p}}{2\ m}\varphi \ll \varphi$$

- Substituting in the first

$$i\frac{\partial\varphi}{\partial t} = -\frac{\nabla^2}{2\ m}\varphi$$

which is the Schrödinger equation for the free particle.

Therefore in the non-relativistic limit the *big* components, φ, obey the Schrödinger equation and the *small* components, χ, are neglected. So the negative energy solutions, the *small* components, disappear in the non-relativistic limit.

Non-relativistic Limit of Dirac Equation: Pauli Equation

- We start now from

$$i\frac{\partial\psi}{\partial t}=\left[-i\vec{\alpha}\cdot\vec{\pi}+\beta m+eA^0\right]\psi,\qquad \vec{\pi}\equiv\vec{\nabla}-ie\vec{A}$$

- With the previous decomposition we get

$$\begin{cases} i\dfrac{\partial\varphi}{\partial t}=\vec{\sigma}\cdot\vec{\pi}\chi+eA^0\varphi \\ i\dfrac{\partial\chi}{\partial t}=\vec{\sigma}\cdot\vec{\pi}\varphi+eA^0\chi-2\,m\chi \end{cases}$$

- Assuming $eA^0\ll 2\,m$, and slow time variation

$$\chi=\frac{\vec{\sigma}\cdot\vec{\pi}}{2\,m}\varphi$$

- And for the *big* components

$$\boxed{i\frac{\partial\varphi}{\partial t}=\left[\frac{(\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{\pi})}{2\,m}+eA^0\right]\varphi}$$

Non-relativistic Limit of Dirac Equation: Pauli Equation . . .

- To understand the meaning of this equation we notice that

$$(\vec{\sigma}\cdot\vec{\pi})(\vec{\sigma}\cdot\vec{\pi}) = \vec{\pi}\cdot\vec{\pi} - e\vec{\sigma}\cdot\vec{B}$$

- Then we get Pauli equation for the electron

$$\boxed{i\frac{\partial\varphi}{\partial t} = \left[\frac{(\vec{p}-e\vec{A})^2}{2\ m} - \frac{e}{2\ m}\vec{\sigma}\cdot\vec{B} + eA^0\right]\varphi}$$

- Putting back $\hbar$ and c we get

$$H_{\text{mag}} = -\frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B} \equiv -\vec{\mu}\cdot\vec{B}, \quad \text{with} \quad \vec{\mu} = \frac{e\hbar}{2mc}\vec{\sigma} = 2\left(\frac{e}{2mc}\right)\frac{\hbar\vec{\sigma}}{2}$$

  which shows that gyromagnetic ratio is $g=2$

- Being able to predict the correct value for g was one of the biggest successes of the Dirac equation

- Higher order corrections (in QED and in the SM) deviate g from 2.

- We define

$$a \equiv \frac{g-2}{2}$$

- The present situation for the electron is very precise (QED only)

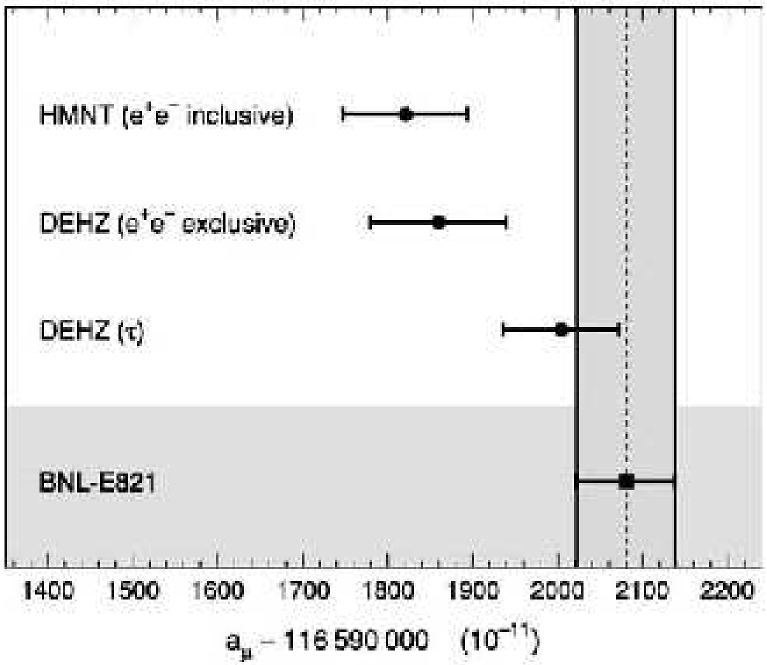
$$a_e^{th} = a_e^{exp} = (115965218073 \pm 28) \times 10^{-14}$$

and we use it to determine the fine structure constant.

- For the muon there is a 2σ difference

$$a_\mu^{th} = (116591841 \pm 81) \times 10^{-11}$$

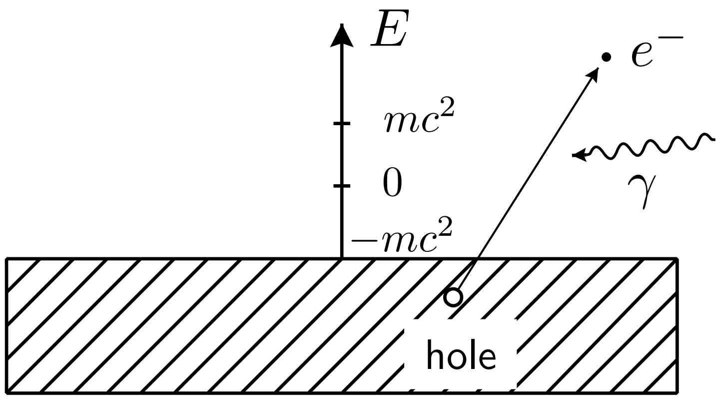
$$a_\mu^{exp} = (116592080 \pm 58) \times 10^{-11}$$

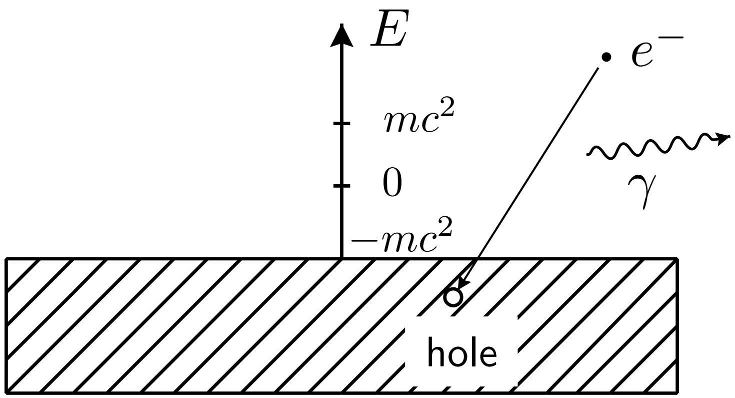


Dirac Hole Theory

- Despite all the successes of the Dirac equation the interpretation of the negative energy solutions still remains to be addressed
- This is not an academic problem. We have to explain why the electrons do not make transitions to the negative energy states. A simple calculation gives a rate of $10^8 \ \mathrm{s}^{-1}$ to decay into the energy interval $[-mc^2, -2mc^2]$
- Dirac hole theory makes use of the Pauli exclusion principle and defines the vacuum as the state with all the negative energy states filled. So, no transitions are allowed from states with $E > 0$ and atoms are stable
- Of course the vacuum has infinite energy and charge but as we only measure differences with respect to the vacuum this is not a problem
- The second quantization allows for a better understanding of this question and allows for the extension also to bosons
- The main consequence is the prediction of anti-particles. Consider that the vacuum has a *hole*. This means the absence of an electron with negative energy $-E$ and charge $-|e|$. But this can be interpreted as the presence of a particle with positive energy $+E$ and charge $+|e|$, the positron

Dirac Hole Theory: pair creation and annihilation





- Pair creation: An electron is excited from a negative energy state into a positive energy state. It leaves behind a *hole* interpreted as the positron. Therefore it was created a pair e^+e^-
- Pair annihilation: An electron with $E > 0$ makes a transition to a non-occupied $E < 0$ state, the hole corresponding to the positron. In the end both electron and positron disappear with the emission of radiation
- With Dirac hole theory we abandon the wave function interpretation as we can vary the number of particles. Only the second quantized creation and annihilation operators can provide a consistent description
- However, Dirac interpretation was a great success confirmed by the discovery of the anti-particles in the following years

Charge Conjugation

- From Dirac hole theory emerges a new symmetry of Nature: for each particle there exists an anti-particle. We call this symmetry *Charge Conjugation*. We now will define it more precisely.

- We should have a correspondence between the negative energy solutions of Dirac equation for electrons,

$$(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi = 0$$

and the positive energy solutions of Dirac equation for positrons

$$(i\partial\!\!\!/ + eA\!\!\!/ - m)\psi_c = 0$$

where ψ_c is the wave function for the positron.

- We take the complex conjugate of the first equation

$$(-i\gamma^{\mu^*}\partial_\mu - e\gamma^{\mu^*}A_\mu - m)\psi^* = 0$$

Charge Conjugation . . .

- Using the relations $\gamma^{0T}\psi^* = \overline{\psi}^T$ and $\gamma^{0T}\gamma^{\mu^*}\gamma^{0T} = \gamma^{\mu T}$ we get

$$\left[-\gamma^{\mu T}(+i\partial_\mu + eA_\mu) - m\right]\overline{\psi}^T = 0$$

- If we find a matrix C such that

$$\boxed{C\gamma^{\mu T}C^{-1} = -\gamma^\mu}$$

we can identify

$$\boxed{\psi_c \equiv C\overline{\psi}^T}$$

- This can be shown by explicit construction. In the Dirac representation

$$C = i\gamma^2\gamma^0 = -C^{-1} = -C^\dagger = -C^T$$

or

$$C = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Charge Conjugation ...

- As an example, let us consider a negative energy electron with spin down

$$\psi = N \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{imt}$$

- Then ψ_c corresponds to a positive energy positron with spin up

$$\psi_c = C\overline{\psi}^T = C\gamma^0\psi^* = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} N \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{-imt} = N \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{-imt}$$

- For an arbitrary wave function we can show

$$\psi = \left(\frac{\varepsilon \not{p} + m}{2m}\right)\left(\frac{1+\gamma_5 \not{s}}{2}\right)\psi \;\to\; \psi_c = \left(\frac{-\varepsilon \not{p} + m}{2m}\right)\left(\frac{1+\gamma_5 \not{s}}{2}\right)\psi_c$$

that is, inverts the sign of energy and the direction of the spin

Massless Spin 1/2 Particles

- We always considered the Dirac equation for a massive electron. However, we could put the question, is there in Nature any massless spin $1/2$ particle?
- For many years the Standard Model of particle physics assumed that this was the case for neutrinos. Today we know that they have a very small mass (less than an eV), but they are not massless
- Despite this, massless spin $1/2$ particles can be very useful. For instance, in many situations we can neglect the electron mass compared with the energies in the center of mass of the process. For more reason the same is true for neutrinos
- Also, gauge theories are formulated in terms of massless particles, before the spontaneous breaking of the gauge theory
- All these are good reasons to look at massless spin $1/2$ particles and we will see that some surprises appear

Description in terms of 2-spinors: Weyl Equation

- For the massless case Dirac equation reads

$$i\frac{\partial\psi}{\partial t} = -i\vec{\alpha}\cdot\vec{\nabla}\psi$$

- We see that the β matrix disappears. This has the consequence that the relations

$$\alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij}$$

can be verified for 2×2 matrices, for instance the Pauli matrices. There are two possible choices

$$\vec{\alpha} = \pm\vec{\sigma}$$

- Let us consider plane wave solutions

$$\psi = \chi(p,s)e^{-ip\cdot x} \ \rightarrow\ \pm\vec{\sigma}\cdot\vec{p}\chi(p,s) = E\chi(p,s)$$

Description in terms of 2-spinors: Weyl Equation ...

- Consider first the case $\alpha = +\vec{\sigma}$. We take $\vec{p}$ along the positive z direction.

$$\chi(p,+) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \rightarrow \quad \frac{\vec{\sigma}.\vec{p}}{|\,\vec{p}\,|}\chi(p,+) = +\chi(p,+)$$

where we used $(|\,\vec{p}\,| = E)$. This solution corresponds to positive helicity

- If we consider the case $\vec{\alpha} = -\vec{\sigma}$ we get

$$\chi(p,-) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \rightarrow \quad \frac{\vec{\sigma}.\vec{p}}{|\,\vec{p}\,|}\chi(p,-) = -\chi(p,-)$$

corresponding to negative helicity

Description in terms of 4-Spinors

- Although it is enough to use 2-component spinors for massless spin $1/2$ particles in many applications it is convenient to use 4-component spinors
- For this it is convenient to choose the Weyl representation for the γ matrices

$$\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix} \quad ; \quad \beta = \gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad ; \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- If we write

$$\psi = \begin{bmatrix} \chi(+) \\ \chi(-) \end{bmatrix}$$

we get

$$i\frac{\partial}{\partial t}\chi(+) = -\, i\vec{\sigma}\cdot\vec{\nabla}\chi(+) - m\chi(-)$$

$$i\frac{\partial}{\partial t}\chi(-) = i\vec{\sigma}\cdot\vec{\nabla}\chi(-) - m\chi(+)$$

showing that the equations are coupled by the mass term

Description in terms of 4-Spinors

- In the limit $m \to 0$ the two equations become independent and they are just the two Weyl equations for the cases $\vec{\alpha} = \pm\vec{\sigma}$
- We notice also that

$$\gamma_5 \psi(\pm) = \pm\psi(\pm) \qquad \psi(+) = \begin{bmatrix} \chi(+) \\ 0 \end{bmatrix} \quad ; \quad \psi(-) = \begin{bmatrix} 0 \\ \chi(-) \end{bmatrix}$$

and therefore

$$\frac{1+\gamma_5}{2}\psi = \begin{bmatrix} \chi(+) \\ 0 \end{bmatrix}, \quad \frac{1-\gamma_5}{2}\psi = \begin{bmatrix} 0 \\ \chi(-) \end{bmatrix}$$

- This shows that chirality and helicity are the same thing in the limit of massless particles