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Advanced Quantum Field Theory
Chapter 2
Physical States. S Matrix. LSZ Reduction

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Fall 2013

Lecture 3

Physical states

- In the previous chapter we saw, for the case of free fields, how to construct the space of states, the so-called *Fock space* of the theory. When we consider the real physical case, with interactions, we are no longer able to solve the problem exactly. For instance, the interaction between electrons and photons is given by a set of nonlinear coupled equations,

$$(i\partial\!\!\!/ - m)\psi = eA\!\!\!/\,\psi$$
$$\partial_\mu F^{\mu\nu} = e\overline{\psi}\gamma^\nu\psi$$

  that do not have an exact solution.

- In practice we have to resort to approximation methods. In the following chapter we will learn how to develop a covariant perturbation theory. Here we are going just to study the general properties of the theory.

- Let us start by the physical states. As we do not know how to solve the problem exactly, we can not prove the assumptions we are going to make about these states. However, these are *reasonable* assumptions, based on Lorentz covariance. We choose our states to be eigenstates of energy and momentum, and of all the other observables that commute with P^μ.

Physical states

- Besides that, we will also assume the following properties:
  - The eigenvalues of p^2 are non-negative and $p^0 > 0$.
  - There exists one non-degenerate base state, with the minimum of energy, which is Lorentz invariant. This state is called the vacuum state $|0\rangle$ and by convention

$$p^\mu |0\rangle = 0$$

  - There exist one particle states $\left|p^{(i)}\right\rangle$, such that,

$$p_\mu^{(i)} p^{(i)\mu} = m_i^2$$

    for each stable particle with mass m_i.
  - The vacuum and the one-particle states constitute the discrete spectrum of p^ν.

In states

- As we are mainly interested in scattering problems, we should construct states that have a simple interpretation in the limit $t \to -\infty$. At that time, the particles that are going to participate in the scattering process have not interacted yet (we assume that the interactions are adiabatically switched off when $|t| \to \infty$ which is appropriate for scattering problems).

- We look for operators that create one particle states with the physical mass. To be explicit, we start by an hermitian scalar field given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2 - V(x)$$

where $V(x)$ is an operator made of more than two interacting fields φ at point x. For instance, those interactions can be self-interactions of the type

$$V(x) = \frac{\lambda}{4!}\varphi^4(x)$$

- The field φ satisfies the following equation of motion

$$(\Box + m^2)\varphi(x) = -\frac{\partial V}{\partial \varphi(x)} \equiv j(x)$$

In states

- The equal time canonical commutation relations are,

$$[\varphi(\vec{x},t)\varphi(\vec{y},t)] = [\pi(\vec{x},t)\pi(\vec{y},t)] = 0$$
$$[\pi(\vec{x},t),\varphi(\vec{y},t)] = -i\delta^3(\vec{x}-\vec{y}) , \quad \text{where} \quad \pi(x) = \dot{\varphi}(x)$$

if we assume that $V(x)$ has no derivatives.

- We designate by $\varphi_{in}(x)$ the operator that creates one-particle states. It will be a functional of the fields $\varphi(x)$. Its existence will be shown by explicit construction. We require that $\varphi_{in}(x)$ must satisfy the conditions:

  i) $\varphi_{in}(x)$ and $\varphi(x)$ transform in the same way for translations and Lorentz transformations. For translations we have then

$$i\,[P^\mu, \varphi_{in}(x)] = \partial^\mu \varphi_{in}(x)$$

  ii) The spacetime evolution of $\varphi_{in}(x)$ corresponds to that of a free particle of mass m, that is

$$(\Box + m^2)\varphi_{in}(x) = 0$$

In states

- From these definitions it follows that $\varphi_{in}(x)$ creates one-particle states from the vacuum. In fact, let us consider a state $|n\rangle$, such that,

$$P^\mu |n\rangle = p_n^\mu |n\rangle$$

- Then

$$\partial^\mu \langle n|\varphi_{in}(x)|0\rangle = i \langle n| [P^\mu, \varphi_{in}(x)] |0\rangle = ip_n^\mu \langle n|\varphi_{in}(x)|0\rangle$$

and therefore

$$\Box \langle n|\varphi_{in}(x)|0\rangle = -p_n^2 \langle n|\varphi_{in}(x)|0\rangle$$

- Then

$$(\Box + m^2) \langle n|\varphi_{in}(x)|0\rangle = (m^2 - p_n^2) \langle n|\varphi_{in}(x)|0\rangle = 0$$

where we have used the fact that $\varphi_{in}(x)$ is a free field

- Therefore the states created from the vacuum by φ_{in} are those for which $p_n^2 = m^2$, that is, the one-particle states of mass m

In states

- The Fourier decomposition of $\varphi_{in}(x)$ is then the same as for free fields,

$$\varphi_{in}(x) = \int \widetilde{dk}\ \left[a_{in}(k)e^{-ik\cdot x} + a^{\dagger}_{in}(k)e^{ik\cdot x}\right]$$

  where $a_{in}(k)$ and $a^{\dagger}_{in}(k)$ satisfy the usual algebra for creation and annihilation operators. In particular, by repeated use of $a^{\dagger}_{in}(k)$ we can create one state of n particles.

- To express $\varphi_{in}(x)$ in terms of $\varphi(x)$ we start by introducing the retarded Green's function of the Klein-Gordon operator,

$$(\Box_x + m^2)\Delta_{\rm ret}(x-y;m) = \delta^4(x-y)$$

  where

$$\Delta_{\rm ret}(x-y;m) = 0 \ \text{ if } \ x^0 < y^0$$

- We can then write

$$\sqrt{Z}\varphi_{in}(x) = \varphi(x) - \int d^4y \Delta_{\rm ret}(x-y;m) j(y)$$

In states

- The field $\varphi_{in}(x)$, satisfies the two initial conditions.
- The constant $\sqrt{Z}$ was introduced to normalize φ_{in} in such a way that it has amplitude 1 to create one-particle states from the vacuum. The fact that $\Delta_{\rm ret} = 0$ for $x_0 \to -\infty$, suggests that $\sqrt{Z}\varphi_{in}(x)$ is, in some way, the limit of $\varphi(x)$ when $x_0 \to -\infty$.
- In fact, as φ and φ_{in} are operators, the correct asymptotic condition must be set on the matrix elements of the operators. Let $|\alpha\rangle$ and $|\beta\rangle$ be two normalized states. We define the operators

$$\varphi^f(t) = i\int d^3x f^*(x) \overleftrightarrow{\partial}_0 \varphi(x)\,, \quad \varphi^f_{in} = i\int d^3x f^*(x) \overleftrightarrow{\partial}_0 \varphi_{in}(x)$$

where $f(x)$ is a normalized solution of the Klein-Gordon equation. φ^f_{in} does not depend on time (for plane waves $f = e^{-ik\cdot x}$ and $\varphi^f_{in} = a_{in}$).

- Then the asymptotic condition of Lehmann, Symanzik e Zimmermann (LSZ), is

$$\lim_{t\to-\infty} \langle\alpha|\,\varphi^f(t)\,|\beta\rangle = \sqrt{Z}\,\langle\alpha|\,\varphi^f_{in}\,|\beta\rangle$$

Spectral representation for scalar fields

- We saw that Z had a physical meaning as the square of the amplitude for the field $\varphi(x)$ to create one-particle states from the vacuum. Let us now find a formal expression for Z and show that $0 \leq Z \leq 1$.

- We start by calculating the expectation value in the vacuum of the commutator of two fields,

$$i\Delta'(x,y) \equiv \langle 0|\left[\varphi(x),\varphi(y)\right]|0\rangle$$

- As we do not know how to solve the equations for the interacting fields φ, we can not solve exactly the problem of finding the Δ', in contrast with the free field case. We can, however, determine its form using general arguments of Lorentz invariance and the assumed spectra for the physical states.

- We introduce a complete set of states between the two operators and we use the invariance under translations in order to obtain,

$$\begin{aligned}\langle n|\varphi(y)|m\rangle =& \langle n|\, e^{iP\cdot y}\varphi(0)e^{-iP\cdot y}\,|m\rangle \\ =& e^{i(p_n-p_m)\cdot y}\,\langle n|\varphi(0)|m\rangle\end{aligned}$$

Spectral representation for scalar fields

- Therefore we get

$$\Delta'(x,y) = -\,i \sum_n \langle 0|\varphi(0)|n\rangle \langle n|\varphi(0)|0\rangle \left(e^{-ip_n\cdot(x-y)} - e^{ip_n\cdot(x-y)}\right)$$
$$\equiv \Delta'(x-y)$$

that is, like in the free field case, Δ' is only a function of the difference $x-y$.

- Introducing now

$$1 = \int d^4q\ \delta^4(q-p_n)$$

we get

$$\Delta'(x-y) = -\,i \int \frac{d^4q}{(2\pi)^3} \left[(2\pi)^3 \sum_n \delta^4(p_n-q) |\langle 0|\varphi(0)|n\rangle|^2\right] \left(e^{-iq\cdot(x-y)} - e^{iq\cdot(x-y)}\right)$$
$$= -\,i \int \frac{d^4q}{(2\pi)^3} \rho(q) \left(e^{-iq\cdot(x-y)} - e^{iq\cdot(x-y)}\right)$$

Spectral representation for scalar fields

- We have defined the density $\rho(q)$ (spectral amplitude),

$$\rho(q) = (2\pi)^3 \sum_n \delta^4(p_n - q) |\, \langle 0|\varphi(0)|n\rangle \,|^2$$

- This spectral amplitude measures the contribution to Δ' of the states with 4-momentum q^μ. $\rho(q)$ is Lorentz invariant (as can be shown using the invariance of $\varphi(x)$ and the properties of the vacuum and of the states $|n\rangle$) and vanishes when q is not in future light cone, due the assumed properties of the physical states.
- Then we can write

$$\rho(q) = \overline{\rho}(q^2)\theta(q^0)$$

and we get

$$\begin{aligned}\Delta'(x-y) =& -i \int \frac{d^4q}{(2\pi)^3}\, \overline{\rho}(q^2)\theta(q^0)(e^{-iq\cdot(x-y)} - e^{iq\cdot(x-y)})\\ =& -i \int \frac{d^4q}{(2\pi)^3} \int d\sigma^2 \delta(q^2-\sigma^2)\overline{\rho}(\sigma^2)\theta(q^0)\left[e^{-iq\cdot(x-y)} - e^{iq\cdot(x-y)}\right]\\ =& \int_0^\infty d\sigma^2 \overline{\rho}(\sigma^2)\Delta(x-y;\sigma)\end{aligned}$$

Spectral representation for scalar fields

- Where

$$\Delta(x-y;\sigma) = -i \int \frac{d^4q}{(2\pi)^3} \delta(q^2-\sigma^2)\theta(q^0)(e^{-iq\cdot(x-y)} - e^{iq\cdot(x-y)})$$

  is the invariant function for the commutator of free fields with mass σ.

- The above relation is known as the spectral decomposition of the commutator of two fields. This expression will allow us to show that $0 \leq Z < 1$.

- To show that, we separate the states of one-particle from the sum. Let $|p\rangle$ be a one-particle state with momentum p. Then

$$\begin{aligned}\langle 0|\varphi(x)|p\rangle =& \sqrt{Z}\,\langle 0|\varphi_{in}(x)|p\rangle + \int d^4y \Delta_{\rm ret}(x-y;m)\,\langle 0|j(y)|p\rangle \\ =& \sqrt{Z}\,\langle 0|\varphi_{in}(x)|p\rangle\end{aligned}$$

  where we have used

$$\begin{aligned}\langle 0|j(y)|p\rangle =& \left\langle 0|(\Box+m^2)\varphi(y)|p\right\rangle = (\Box+m^2)e^{-ip\cdot y}\,\langle 0|\varphi(0)|p\rangle \\ =&(m^2-p^2)e^{-ip\cdot y}\,\langle 0|\varphi(0)|p\rangle = 0\end{aligned}$$

Spectral representation for scalar fields

- On the other hand

$$\langle 0|\varphi_{in}(x)|p\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik\cdot x} \langle 0|a_{in}(k)|p\rangle$$
$$= e^{-ip\cdot x}$$

and then

$$\rho(q) = (2\pi)^3 \int \widetilde{dp}\ \delta^4(p-q) Z + \text{ contributions from more than one particle}$$
$$= Z\delta(q^2-m^2)\theta(q^0) + \cdots$$

- Therefore

$$\Delta'(x-y) = Z\Delta(x-y;m) + \int_{m_1^2}^{\infty} d\sigma^2 \overline{\rho}(\sigma^2)\Delta(x-y;\sigma)$$

where m_1 is the mass of the lightest state of two or more particles.

Spectral representation for scalar fields

- Finally noticing that

$$\frac{\partial}{\partial x^0}\Delta'(x-y)|_{x^0=y^0} = \frac{\partial}{\partial x^0}\Delta(x-y;\sigma)|_{x^0=y^0} = -\delta^3(\vec{x}-\vec{y})$$

we get the relation

$$1 = Z + \int_{m_1^2}^{\infty} d\sigma^2 \overline{\rho}(\sigma^2)$$

- This means

$$0 \leq Z < 1$$

where this last step results from the assumed positivity of $\overline{\rho}(\sigma^2)$.

Out states

- In the same way as we reduced the dynamics of $t \to -\infty$ to the free fields φ_{in}, it is also possible to define in the limit $t \to +\infty$ the corresponding free fields, $\varphi_{out}(x)$.
- These free fields will be the final state of a scattering problem.
- The formalism is copied from the case of φ_{in}, and therefore we will present the results without going into the details of the derivations. $\varphi_{out}(x)$ obey the following relations:

$$i\,[P^\mu, \varphi_{out}] = \partial^\mu \varphi_{out}$$
$$(\Box + m^2)\varphi_{out} = 0$$

and has the expansion

$$\varphi_{out}(x) = \int \widetilde{dk}\ \left[a_{out}(k)e^{-ik\cdot x} + a^\dagger_{out}(k)e^{ik\cdot x}\right]$$

- The asymptotic condition is now

$$\lim_{t\to\infty} \langle\alpha|\,\varphi^f(t)\,|\beta\rangle = \sqrt{Z}\,\langle\alpha|\,\varphi^f_{out}\,|\beta\rangle$$

Out states

- Also

$$\sqrt{Z}\varphi_{out}(x) = \varphi(x) - \int d^4y \Delta_{\rm adv}(x-y;m) j(y)$$

where the Green's functions $\Delta_{\rm adv}$ satisfy

$$(\Box_x + m^2)\Delta_{\rm adv}(x-y;m) = \delta^4(x-y)$$
$$\Delta_{\rm adv}(x-y;m) = 0 \quad ; \quad x^0 > y^0 \ .$$

- For one-particle states we get

$$\begin{aligned}\langle 0|\varphi(x)|p\rangle =& \sqrt{Z}\,\langle 0|\varphi_{out}(x)|p\rangle\\ =& \sqrt{Z}\,\langle 0|\varphi_{in}(x)|p\rangle\\ =& \sqrt{Z} e^{-ip\cdot x}\end{aligned}$$

S matrix

- We have now all the formalism needed to study the transition amplitudes from one initial state to a given final state, the so-called S matrix elements. Let us start by an initial state with n non-interacting particles

$$|p_1 \cdots p_n \; ; in\rangle \equiv |\alpha \; ; in\rangle$$

where $p_1 \cdots p_n$ are the 4-momenta of the n particles. Other quantum numbers are assumed but not explicitly written.

- The final state will be, in general, a state with m particles

$$|p_1' \cdots p_m' \; ; out\rangle \equiv |\beta \; ; out\rangle$$

- The S matrix element $S_{\beta\alpha}$ is defined by the amplitude

$$S_{\beta\alpha} \equiv \langle\beta \; ; out|\alpha \; ; in\rangle$$

- The S matrix is an operator that induces an isomorphism between the in and out states, that by assumption are a complete set of states,

$$\langle\beta \; ; out| = \langle\beta \; ; in|\, S \, , \quad \langle\beta \; ; in| = \langle\beta \; ; out|\, S^{-1}$$
$$\langle\beta \; ; out|\alpha \; ; in\rangle = \langle\beta \; ; in|S|\alpha \; ; in\rangle = \langle\beta \; ; out|S|\alpha; out\rangle$$

S matrix

- From the assumed properties for the states we can show the following results for the S matrix.

  i) $\langle 0|S|0\rangle = \langle 0|0\rangle = 1$ (stability and unicity of the vacuum)

  ii) The stability of the one-particle states gives

$$\langle p\,; in|S|p\,; in\rangle = \langle p\,; out|p\,; in\rangle = \langle p\,; in|p\,; out\rangle = 1$$

  because $|p\,; in\rangle = |p\,; out\rangle$.

  iii) $\varphi_{in}(x) = S\varphi_{out}(x)S^{-1}$

  iv) The S matrix is unitary. To show this we have

$$\delta_{\beta\alpha} = \langle \beta\,; out|\alpha\,; out\rangle = \left\langle \beta\,; in|SS^\dagger|\alpha\,; in\right\rangle$$

  and therefore

$$SS^\dagger = 1$$

S matrix

v) The S matrix is Lorentz invariant. In fact we have

$$\begin{aligned}\varphi_{in}(ax+b) =& U(a,b)\varphi_{in}(x)U^{-1}(a,b) = US\varphi_{out}(x)S^{-1}U^{-1}\\ =& USU^{-1}\varphi_{out}(ax+b)US^{-1}U^{-1} \ .\end{aligned}$$

But

$$\varphi_{in}(ax+b) = S\varphi_{out}(ax+b)S^{-1} \ ,$$

and therefore we get finally

$$S = U(a,b)SU^{-1}(a,b) \ .$$

LSZ reduction formula for scalar fields

- The S matrix elements are the quantities that are directly connected to the experiment. In fact, $|S_{\beta\alpha}|^2$ represents the transition probability from the initial state $|\alpha\,;in\rangle$ to the final $|\beta\,;out\rangle$.

- We are going in this section to use the previous formalism to express these matrix elements in terms of the so-called Green functions for the interacting fields. In this way the problem of the calculation of these probabilities is transferred to the problem of calculating these Green functions. These, of course, can not be evaluated exactly, but we will learn in the next chapter how to develop a covariant perturbation theory for that purpose.

- Let us then proceed to the derivation of the relation between the S matrix elements and the the Green functions of the theory. This technique is known as the LSZ *reduction* from the names of Lehmann, Symanzik e Zimmermann that have introduced it. The starting point is, by definition

$$\langle p_1 \cdots ; out|q_1 \cdots ; in\rangle = \left\langle p_1, \cdots ; out|a^\dagger_{in}(q_1)|q_2, \cdots ; in\right\rangle$$

Using

$$a^\dagger_{in}(q_1) = -i\int_t d^3x e^{-iq_1\cdot x}\overset{\leftrightarrow}{\partial}_0\varphi_{in}(x)$$

LSZ reduction formula for scalar fields

- The last integral is time-independent, and therefore can be calculated for an arbitrary time t. If we take $t \to -\infty$ and use the asymptotic condition for the in fields, we get

$$\langle p_1 \cdots ; out | q_1 \cdots ; in\rangle = - \lim_{t\to -\infty} i Z^{-1/2} \int_t d^3x e^{-iq_1\cdot x} \overset{\leftrightarrow}{\partial}_0 \langle p_1 \cdots ; out | \varphi(x) | q_2 \cdots ; in\rangle$$

- In a similar way one can show that

$$\begin{aligned}\left\langle p_1 \cdots ; out | a^\dagger_{out}(q_1) | q_2 \cdots ; in\right\rangle &= \\ &= - \lim_{t\to \infty} i Z^{-1/2} \int_t d^3x e^{-iq_1\cdot x} \overset{\leftrightarrow}{\partial}_0 \langle p_1 \cdots ; out | \varphi(x) | q_2 \cdots ; in\rangle \ .\end{aligned}$$

- Then, using the result,

$$\begin{aligned}\left(\lim_{t\to\infty} - \lim_{t\to -\infty}\right) \int d^3x f(\vec{x},t) &= \lim_{t_f\to\infty, t_i\to -\infty} \int_{t_i}^{t_f} dt \frac{\partial}{\partial t} \int d^3x f(\vec{x},t) \\ &= \int d^4x \partial_0 f(\vec{x},t)\end{aligned}$$

LSZ reduction formula for scalar fields

- Subtracting the last two equations we get

$$\langle p_1 \cdots ; out | q_1 \cdots ; in \rangle = \left\langle p_1 \cdots ; out | a^\dagger_{out}(q_1) | q_2 \cdots ; in \right\rangle$$
$$+ \, i Z^{-1/2} \int d^4x \, \partial_0 \left[e^{-iq_1 \cdot x} \overleftrightarrow{\partial}_0 \langle p_1 \cdots ; out | \varphi(x) | q_2 \cdots ; in \rangle \right]$$

- The first term on the right-hand side of corresponds to a sum of disconnected terms, in which at least one of the particles is not affected by the interaction (it will vanish if none of the initial momenta coincides with one of the final momenta). Its form is

$$\left\langle p_1 \cdots ; out | a^\dagger_{out}(q_1) | q_2 \cdots ; in \right\rangle =$$
$$= \sum_{k=1}^{n} (2\pi)^3 2 p_k^0 \, \delta^3(\vec{p}_k - \vec{q}_1) \, \langle p_1, \cdots, \widehat{p}_k, \cdots ; out | q_2, \cdots ; in \rangle$$

where $\widehat{p}_k$ means that this momentum was taken out from that state.

LSZ reduction formula for scalar fields

- For the second term we write,

$$\int d^4x\ \partial_0 \left[e^{-iq_i x} \overleftrightarrow{\partial}_0 \langle p_1 \cdots ; out | \varphi(x) | q_2 \cdots ; in \rangle \right]$$

$$= \int d^4x \left[-\partial_0^2 e^{-iq_1 \cdot x} \langle \cdots \rangle + e^{-iq_1 \cdot x} \partial_0^2 \langle \cdots \rangle \right]$$

$$= \int d^4x \left[(-\Delta^2 + m^2) e^{-iq_1 \cdot x} \langle \cdots \rangle + e^{-iq_1 \cdot x} \partial_0^2 \langle \cdots \rangle \right]$$

$$= \int d^4x e^{-iq_1 \cdot x} (\Box + m^2) \langle p_1 \cdots ; out | \varphi(x) | q_2 \cdots ; in \rangle$$

where we have used $(\Box + m^2) e^{-iq_1 \cdot x} = 0$, and have performed an integration by parts (whose justification would imply the substitution of plane waves by wave packets).

LSZ reduction formula for scalar fields

- Therefore, after this first step in the reduction we get,

$$\langle p_1, \cdots p_n; out | q_1 \cdots q_\ell; in \rangle =$$

$$= \sum_{k=1}^{n} 2p_k^0 (2\pi)^3 \delta^3(\vec{p}_k - \vec{q}_1) \langle p_1, \cdots \widehat{p}_k; \cdots p_n; out | q_2 \cdots q_2 \cdots q_\ell; in \rangle$$

$$+ iZ^{-1/2} \int d^4x e^{-iq_1 x} (\Box + m^2) \langle p_1 \cdots p_n; out | \varphi(x) | q_2 \cdots q_\ell; in \rangle$$

- We will proceed with the process until all the momenta in the initial and final state are exchanged by the field operators. To be specific, let us now remove one momentum in the final state.

- From now on we will no longer consider the disconnected terms, because in practice we are only interested in the cases where *all* the particles interact. Once we know the cases where all the particles interact, we can always calculate situations where some of the particles do not participate in the scattering

LSZ reduction formula for scalar fields

- We have then

$$\langle p_1 \cdots p_n; out|\varphi(x_1)|q_2 \cdots q_\ell; in\rangle = \langle p_2 \cdots p_n; out|a_{out}(p_1)\varphi(x_1)|q_2 \cdots q_\ell; in\rangle$$

$$= \lim_{y_1^0 \to \infty} iZ^{-1/2} \int d^3y_1\, e^{ip_1\cdot y_1} \overleftrightarrow{\partial}_{y_1^0} \langle p_2 \cdots p_n; out|\varphi(y_1)\varphi(x_1)|q_2 \cdots q_\ell; in\rangle$$

$$= \langle p_2 \cdots p_n; out|\varphi(x_1)a_{in}(p_1)|q_2 \cdots q_\ell; in\rangle$$

$$+ \lim_{y_1^0 \to \infty} iZ^{-1/2} \int d^3y_1\, e^{ip_1\cdot y_1} \overleftrightarrow{\partial}_{y_1^0} \langle p_2 \cdots p_n; out|\varphi(y_1)\varphi(x_1)|q_2 \cdots q_\ell; in\rangle$$

$$- \lim_{y_1^0 \to -\infty} iZ^{-1/2} \int d^3y_1\, e^{ip_1\cdot y_1} \overleftrightarrow{\partial}_{y_1^0} \langle p_2 \cdots p_n; out|\varphi(x_1)\varphi(y_1)|q_2 \cdots q_\ell; in\rangle$$

$$= \langle p_2 \cdots p_n; out|\varphi(x_1)a_{in}(p_1)|q_2 \cdots q_\ell; in\rangle$$

$$+ iZ^{-1/2}\left(\lim_{y_1^0 \to \infty} - \lim_{y_1^0 \to -\infty}\right)\int d^3y_1 e^{ip_1\cdot y_1} \overleftrightarrow{\partial}_{y_1^0} \langle p_2 \cdots p_n; out|T\varphi(y_1)\varphi(x_1)|q_2 \cdots q_\ell; in\rangle$$

LSZ reduction formula for scalar fields

- Applying the same procedure as before we obtain,

$$\langle p_1 \cdots p_n, ; out|\varphi(x_1)|q_2 \cdots q_\ell; in\rangle = \text{disconnected terms}$$

$$+ iZ^{-1/2} \int d^4y_1 e^{ip_1\cdot y_1} (\Box_{y_1} + m^2) \langle p_2 \cdots p_n; out|T\varphi(y_1)\varphi(x_1)|q_2 \cdots q_\ell; in\rangle$$

- It is not very difficult to generalize this method to obtain the final reduction formula for scalar fields,

$$\langle p_1 \cdots p_n; out|q_1 \cdots q_\ell; in\rangle = \text{disconnected terms}$$

$$+ \left(\frac{i}{\sqrt{Z}}\right)^{n+\ell} \int d^4y_1 \cdots d^4y_n d^4x_1 \cdots d^4x_\ell e^{\left[i\sum_1^n p_k\cdot y_k - i\sum_1^\ell q_r\cdot x_r\right]}$$

$$(\Box_{y_1} + m^2) \cdots (\Box_{x_\ell} + m^2) \langle 0|T\varphi(y_1) \cdots \varphi(y_n)\varphi(x_1) \cdots \varphi(x_\ell)|0\rangle$$

- This last equation is the fundamental equation in quantum field theory. It allows us to relate the transition amplitudes to the Green functions of the theory.

LSZ reduction formula for scalar fields

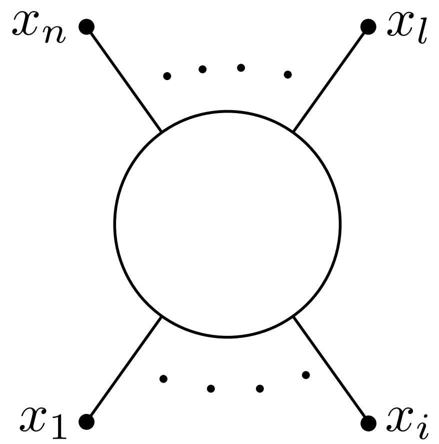
- The quantity

$$\langle 0|T\varphi(x_1)\cdots\varphi(x_n)|0\rangle \equiv G(x_1\cdots x_n)$$

is known as the complete green function for $n = m + \ell$ particles

- We will introduce the following diagrammatic representation for it

$$G(x_1\cdots x_n) =$$



- The factors $(\Box + m^2)$ in force the external particles to be on-shell. In fact, in momentum space $(\Box + m^2) \to (-p^2 + m^2)$. Therefore the amplitude will vanish unless the propagators of the external legs are on-shell, as in that case they will have a pole, $\frac{1}{p^2-m^2}$. We conclude that for the transition amplitudes only the truncated Green functions will contribute, that is the ones with the external legs removed. In the next chapter we will learn how to evaluate these Green functions in perturbation theory.

Reduction Formula for fermions: States *in* and *out*

- The definition of the *in* and *out* follows exactly the same steps as in the case of the scalar fields. We will therefore, for simplicity, just state the results with the details.

- The states $\psi_{in}(x)$ satisfy the conditions,

$$(i\partial\!\!\!/ - m)\psi_{in}(x) = 0$$

$$[P_\mu, \psi_{in}(x)] = -i\partial_\mu \psi_{in}(x) \ .$$

- The states $\psi_{in}(x)$ will create one-particle states and they are related with the fields $\psi(x)$ by,

$$\sqrt{Z_2}\psi_{in}(x) = \psi(x) - \int d^4y S_{\text{ret}}(x-y,m) j(y)$$

where $\psi(x)$ satisfies the Dirac equation,

$$(i\partial\!\!\!/ - m)\psi(x) = j(x)$$

and S_{ret} is the retarded Green function for the Dirac equation,

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Reduction Formula for fermions: States *in* and *out*

- Where the retarded Green function is

$$(i\partial\!\!\!/_x - m) S_{\text{ret}}(x-y,m) = \delta^4(x-y)$$

$$S_{\text{ret}}(x-y) = 0 \;;\; x^0 < y^0$$

- The fields $\psi_{in}(x)$, as free fields, have the Fourier expansion,

$$\psi_{in}(x) = \int \widetilde{dp} \sum_s \left[b_{in}(p,s) u(p,s) e^{-ip\cdot x} + d^\dagger_{in}(p,s) v(p,s) e^{ip\cdot x} \right]$$

where the operators b_{in}, d_{in} satisfy exactly the same algebra as in the free field case. The asymptotic condition is now,

$$\lim_{t\to -\infty} \langle\alpha|\, \psi^f(t)\, |\beta\rangle = \sqrt{Z_2}\, \langle\alpha|\, \psi^f_{in}\, |\beta\rangle$$

where $\psi^f(t)$ and ψ^f_{in} have a meaning similar to the scalar case

- For the ψ_{out} fields we get essentially the same expressions with ψ_{in} substituted by ψ_{out}.

Reduction Formula for fermions: States *in* and *out*

- The main difference with respect to the *in* States is in the asymptotic condition that now reads,

$$\lim_{t\to\infty} \langle\alpha|\, \psi^f(t)\, |\beta\rangle = \sqrt{Z_2}\, \langle\alpha|\, \psi^f_{out}\, |\beta\rangle$$

implying the following relation between the fields ψ_{out} and ψ,

$$\sqrt{Z_2}\psi_{out} = \psi(x) - \int d^4y S_{\text{adv}}(x-y;m) j(y)$$

- The advanced Green function is defined by

$$(i\partial\!\!\!/_x - m) S_{adv}(x-y;m) = \delta^4(x-y)$$

$$S_{\text{adv}}(x-y;m) = 0 \qquad x^0 > y^0 \ .$$

Spectral representation fermions

- Let us consider the vacuum expectation value of the anti-commutator of two Dirac fields,

$$\begin{aligned} S'_{\alpha\beta}(x,y) \equiv& i \langle 0| \{\psi_\alpha(x), \overline{\psi}_\beta(y)\} |0\rangle \\ =& i \sum_n \Big[\langle 0| \psi_\alpha(0) |n\rangle \langle n| \overline{\psi}_\beta(0) |0\rangle \, e^{-ip_n (x-y)} \\ & + \langle 0| \overline{\psi}_\beta(0) |n\rangle \langle n| \psi_\alpha(0) |0\rangle \, e^{ip_n \cdot (x-y)} \Big] \\ \equiv& S'_{\alpha\beta}(x-y) \end{aligned}$$

  where we have introduced a complete set of eigen-states of the 4-momentum.

- As before we introduce the spectral amplitude $\rho_{\alpha\beta}(q)$,

$$\rho_{\alpha\beta}(q) \equiv (2\pi)^3 \sum_n \delta^4(p_n - q) \langle 0| \psi_\alpha(0) |n\rangle \langle n| \overline{\psi}_\beta(0) |0\rangle$$

Spectral representation fermions

- We will now find the most general form for $\rho_{\alpha\beta}(q)$ using Lorentz invariance arguments. $\rho_{\alpha\beta}(q)$ is a 4×4 matrix in Dirac space, and can be written as

$$\rho_{\alpha\beta}(q) = \overline{\rho}(q)\delta_{\alpha\beta} + \rho_\mu(q)\gamma^\mu_{\alpha\beta} + \rho_{\mu\nu}(q)\sigma^{\mu\nu}_{\alpha\beta} + \tilde{\rho}(q)\gamma^5_{\alpha\beta} + \tilde{\rho}_\mu(q)(\gamma^\mu\gamma^5)_{\alpha\beta}$$

- Lorentz invariance arguments restrict the form of the coefficients $\overline{\rho}(q)$, $\rho_\mu(q)$, $\rho_{\mu\nu}(q)$, $\tilde{\rho}(q)$ and $\tilde{\rho}_\mu(q)$. Under Lorentz transformations the fields transform as

$$U(a)\psi_\alpha(0)U^{-1}(a) = S^{-1}_{\alpha\lambda}(a)\overline{\psi}_\lambda(0)$$
$$U(a)\overline{\psi}_\alpha(0)U^{-1}(a) = \overline{\psi}_\lambda(0)S_{\lambda\alpha}(a)$$
$$S^{-1}\gamma^\mu S = a^\mu{}_\nu\gamma^\nu$$

- Then we can show that the matrix (in Dirac space), $\rho_{\alpha\beta}$ must obey the relation,

$$\rho(q) = S^{-1}(a)\rho(qa^{-1})S(a)$$

where we have used a matrix notation. This relation gives the properties of the different coefficients

Spectral representation fermions

- For instance,

$$\rho^\mu(q) = a^\mu{}_\nu \rho^\nu(qa^{-1})$$

which means that ρ^μ transform as a $4-$vector.

- Using the fact that $\rho_{\alpha\beta}$ is a function of q and vanishes outside the future light cone, we can finally write

$$\rho_{\alpha\beta}(q) = \rho_1(q^2)\not{q}_{\alpha\beta} + \rho_2(q^2)\delta_{\alpha\beta} + \tilde{\rho}_1(q^2)(\not{q}\gamma^5)_{\alpha\beta} + \tilde{\rho}_2(q^2)\gamma^5_{\alpha\beta}$$

that is, $\rho_{\alpha\beta}(q)$ is determined up to four scalar functions of q^2.

- Requiring invariance under parity transformations we get

$$\rho_{\alpha\beta}(\vec{q}, q_0) = \gamma^0_{\alpha\lambda}\rho_{\lambda\delta}(-\vec{q}, q^0)\gamma^0_{\delta\beta}$$

and we obtain,

$$\widetilde{\rho}_1 = \widetilde{\rho}_2 = 0$$

Spectral representation fermions

- Therefore for the Dirac theory, that preserves parity, we get,

$$\rho_{\alpha\beta}(q) = \rho_1(q^2)\not{q}_{\alpha\beta} + \rho_2(q^2)\delta_{\alpha\beta}$$

- Repeating the steps of the scalar case we write,

$$S'_{\alpha\beta}(x-y) = \int_0^\infty d\sigma^2 \left\{\rho_1(\sigma^2) S_{\alpha\beta}(x-y;\sigma) + \right.$$
$$\left. + \left[\sigma\rho_1(\sigma^2) - \rho_2(\sigma^2)\right] \delta_{\alpha\beta}\Delta(x-y;\sigma)\right\}$$

where Δ and $S_{\alpha\beta}$ are the functions defined for free fields.

- We can then show that

i) ρ_1 e ρ_2 are real
ii) $\rho_1(\sigma^2) \geq 0$
iii) $\sigma\rho_1(\sigma^2) - \rho_2(\sigma^2) \geq 0$

- Using the previous relations we can extract of the one-particle states

Spectral representation fermions

- We get,

$$S'_{\alpha\beta}(x-y) = Z_2 S_{\alpha\beta}(x-y;m) + \int_{m_1^2}^{\infty} d\sigma^2 \left\{ \rho_1(\sigma^2) S_{\alpha\beta}(x-y;\sigma) + \left[\sigma\rho_1(\sigma^2) - \rho_2(\sigma^2)\right] \delta_{\alpha\beta} \Delta(x-y;\sigma) \right\}$$

where m_1 is the threshold for the production of two or more particles.

- Evaluating at equal times we can obtain

$$1 = Z_2 + \int_{m_1^2} d\sigma^2 \rho_1(\sigma^2)$$

that is

$$0 \leq Z_2 < 1$$

Reduction formula for fermions

- To get the reduction formula for fermions we will proceed as in the scalar case.

- The only difficulty has to do with the spinor indices. The creation and annihilation operators can be expressed in terms of the fields ψ_{in} by the relations,

$$b_{in}(p,s) = \int d^3x \overline{u}(p,s) e^{ip\cdot x} \gamma^0 \psi_{in}(x)$$

$$d^\dagger_{in}(p,s) = \int d^3x \overline{v}(p,s) e^{-ip\cdot x} \gamma^0 \psi_{in}(x)$$

$$b^\dagger_{in}(p,s) = \int d^3x \overline{\psi}_{in}(x) \gamma^0 e^{-ip\cdot x} u(p,s)$$

$$d_{in}(p,s) = \int d^3x \overline{\psi}_{in}(x) \gamma^0 e^{ip\cdot x} v(p,s)$$

with the integrals being time independent.

- In fact, to be more rigorous we should substitute the plane wave solutions by wave packets, but as in the scalar case, to simplify matter we will not do it.

Reduction formula for fermions

- To establish the reduction formula we start by extracting one electron from the initial state,

$$\langle \beta; out|(ps)\alpha; in\rangle = \left\langle \beta; out|b^{\dagger}_{in}(p,s)|\alpha, in\right\rangle$$

$$= \langle \beta - (p,s); out|\alpha; in\rangle + \left\langle \beta; out|b^{\dagger}_{in}(p,s) - b^{\dagger}_{out}(p,s)|\alpha; in\right\rangle$$

$$= \text{disconnected terms}$$

$$+ \int d^3x \, \langle \beta; out|\overline{\psi}_{in}(x) - \overline{\psi}_{out}(x)|\alpha; in\rangle \, \gamma^0 e^{-ip\cdot x} u(p,s)$$

$$= \text{disconnected terms}$$

$$- \left(\lim_{t\to+\infty} - \lim_{t\to-\infty}\right) \frac{1}{\sqrt{Z_2}} \int d^3x \, \langle \beta; out|\overline{\psi}(x)|\alpha; in\rangle \, \gamma^0 e^{-ip\cdot x} u(p,s)$$

$$= \text{disconnected terms}$$

$$- Z_2^{-1/2} \int d^4x \, \left[\langle \beta; out|\partial_0\overline{\psi}(x)|\alpha; in\rangle \, \gamma^0 e^{-ip\cdot x} u(p,s)\right.$$

$$\left. + \langle \beta; out|\overline{\psi}(x)|\alpha; in\rangle \, \gamma^0 \partial_0 (e^{-ip\cdot x} u(p,s))\right]$$

Reduction formula for fermions

- Using now

$$(i\gamma^0\partial_0 + i\gamma^i\partial_i - m)(e^{-ip\cdot x}u(p,s)) = 0$$

we get, after an integration by parts,

$$\left\langle \beta; out | b^\dagger_{in}(\rho, s) | \alpha; in \right\rangle = \text{disconnected terms}$$

$$- iZ_2^{-1/2} \int d^4x \, \langle \beta; out | \overline{\psi}(x) | \alpha; in \rangle \, (-i\overleftarrow{\not\partial}_x - m) e^{-ip\cdot x} u(p,s)$$

- In a similar way the reduction of an anti-particle from the initial state gives,

$$\left\langle \beta; out | d^\dagger_{in}(p, s) | \alpha; in \right\rangle = \text{disconnected terms}$$

$$+ iZ_2^{-1/2} \int d^4x e^{-ip\cdot x} \overline{v}(p,s)(i\not\partial_x - m) \, \langle \beta; out | \psi(x) | \alpha; in \rangle$$

Reduction formula for fermions

- The reduction of a particle or of an anti-particle from the final state give, respectively,

$$\langle \beta; out | b_{out}(p,s) | \alpha; in \rangle = \text{disconnected terms}$$

$$- iZ_2^{-1/2} \int d^4x e^{ip\cdot x} \overline{u}(p,s)(i\partial\!\!\!/_x - m) \langle \beta; out | \psi(x) | \alpha; in \rangle$$

and

$$\langle \beta; out | d_{out}(p,s) | \alpha; in \rangle = \text{disconnected terms}$$

$$+ iZ_2^{-1/2} \int d^4x \langle \beta; out | \overline{\psi}(x) | \alpha; in \rangle (-i\overleftarrow{\partial\!\!\!/}_x - m) v(p,s) e^{ip\cdot x}$$

- Notice the formal relation between one electron in the initial state and a positron in the final state. To go from one to the other one just has to do,

$$u(p,s)e^{-ip\cdot x} \to -v(p,s)e^{ip\cdot x}$$

Reduction formula for fermions

- To write the final expression we denote the momenta in the state $\langle in|$ by p_i or $\overline{p_i}$, respectively for particles or anti-particles, and those in the state $\langle out|$ by $p'_i, \overline{p}'_i$.

- We also make the following conventions (needed to define the global sign),

$$|(p_1, s_1), \cdots, (\overline{p}_1, \overline{s}_1); \cdots; in\rangle = b^\dagger_{in}(p_1, s_1)\cdots d^\dagger_{in}(\overline{p}_1, \overline{s}_1)\cdots|0\rangle$$
$$\langle out; (p'_1, s'_1)\cdots, (\overline{p}'_1, \overline{s}'_1)\cdots| = \langle 0|\cdots d_{out}(\overline{p}'_1, \overline{s}'_1), \cdots b_{out}(p'_1, s'_1)$$

- Then, if $n(n')$ denotes the total number of particles (anti-particles), we get

$$\begin{aligned}
\langle out; (p'_1, s'_1)\cdots, (\overline{p}'_1, \overline{s}'_1)\cdots|(p_1, s_1), \cdots (\overline{p}_1, \overline{s}_1), \cdots; in\rangle &= \text{disc. terms} \\
+ (-iZ_2^{-1/2})^n (iZ_2^{-1/2})^{n'} \int d^4x_1\cdots d^4y_1\cdots d^4x'_1\cdots d^4y'_1\cdots & \\
e^{-i\sum(p_i\cdot x_i) - i\sum(\overline{p}_i\cdot y_i)} e^{+i\sum(p'_i\cdot x'_i) + i\sum(\overline{p}'_i\cdot y'_i)} & \\
\overline{u}(p'_1, s'_1)(i\partial\!\!\!/_{x'_1} - m)\cdots\overline{v}(\overline{p}_1, \overline{s}_1)(i\partial\!\!\!/_{y_1} - m) & \\
\langle 0|\, T(\cdots\overline{\psi}(y'_1)\cdots\psi(x'_1)\overline{\psi}(x_1)\cdots\psi(y_1)\cdots|0\rangle & \\
(-i\overleftarrow{\partial\!\!\!/}_{x_1} - m)u(p_1, s_1)\cdots(-i\overleftarrow{\partial\!\!\!/}_{y'_1} - m)v(\overline{p}'_1, \overline{s}'_1) &
\end{aligned}$$

Reduction formula for fermions

- Last equation is the fundamental expression that allows to relate the elements of the S matrix with the Green functions of the theory.
- The sign and ordering shown correspond to the previous conventions
- In terms of diagrams, we represent the Green function,

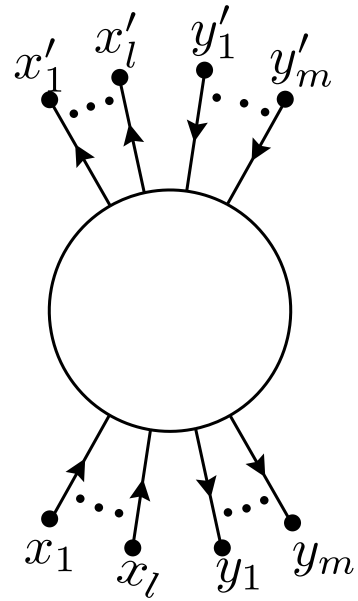
$$\langle 0|\, T\left[\overline{\psi}(y'_{m'})\cdots\overline{\psi}(y'_1)\psi(x'_{\ell'})\cdots\psi(x'_1)\overline{\psi}(x_1)\cdots\overline{\psi}(x_\ell)\psi(y_1)\cdots\psi(y_m)\right]|0\rangle$$

by the diagram of the figure

- With lepton number conservation, the number of particles minus anti-particles is conserved, that is

$$\ell - m = \ell' - m'$$

- The operators $(i\partial\!\!\!/ - m)$ e $(-i\overleftarrow{\partial\!\!\!/} - m)$ force the particles to be on-shell and remove the propagators from the external lines (truncated Green functions). In the next chapter we will learn how to determine these functions in perturbation theory.



Reduction formula for photons

- The LSZ formalism for photons, has some difficulties connected with the problems in quantizing the electromagnetic field. When one adopts a formalism (radiation gauge) where the only components of the field A^μ are transverse, the problems arise in showing the Lorentz and gauge invariance of the S matrix. In the formalism of the undefined metric, the difficulties are connected with the states of negative norm, besides the gauge invariance.

- We will see later a more satisfactory procedure to quantize all gauge theories, including Maxwell theory of the electromagnetic field and the resulting perturbation theory coincides with the one we get here. This is our justification to assume that we can define the in fields by the relation,

$$\sqrt{Z_3} A^\mu_{in}(x) = A^\mu(x) - \int d^4y D^{\mu\nu}_{\rm ret}(x-y) j_\nu(y)$$

and in the same way for the out fields,

$$\sqrt{Z_3} A^\mu_{out}(x) = A^\mu(x) - \int d^4y D^{\mu\nu}_{\rm adv}(x-y) j_\nu(y)$$

where

$$\Box A^\mu_{in} = \Box A^\mu_{out} = 0\,, \quad \Box A^\mu = j^\mu \ \text{ and } \Box D^{\mu\nu}_{\rm adv,\ ret} = \delta^{\mu\nu}\delta^\mu(x-y)$$

Reduction formula for photons

- The fields *in* and *out* are free fields, and therefore they have a Fourier expansion in plane waves and creation and annihilation operators of the form

$$A^{\mu}_{in}(x) = \int \widetilde{dk} \sum_{\lambda=0}^{3} \left[a_{in}(k,\lambda) \varepsilon^{\mu}(k,\lambda) e^{-ik\cdot x} + a^{\dagger}_{in}(k,\lambda) \varepsilon^{\mu *}(k,\lambda) e^{ik\cdot x} \right]$$

- Where

$$a_{in}(k,\lambda) = - i \int d^3x e^{ik\cdot x} \overleftrightarrow{\partial}_0 \varepsilon^{\mu}(k,\lambda) A^{in}_{\mu}(x)$$

$$a^{\dagger}_{in}(k,\lambda) = i \int d^3x e^{-ik\cdot x} \overleftrightarrow{\partial}_0 \varepsilon^{\mu *}(k,\lambda) A^{in}_{\mu}(x)$$

and, as usual, $a_{in}(k,\lambda)$ and $a^{\dagger}_{in}(k,\lambda)$ are time independent.

- Above, all the polarizations appear, but as the elements of the S matrix are between physical states, we are sure that the longitudinal and scalar polarizations do not contribute.

- In this formalism what is difficult to show is the spectral decomposition. We are not going to enter those details, just state that we can show that Z_3 is gauge independent and satisfies $0 \leq Z_3 < 1$.

Reduction formula for photons

- The reduction formula is easily obtained. We get

$$\langle \beta; out | (k\lambda) \alpha; in \rangle = \langle \beta - (k,\lambda); out | \alpha; in \rangle + \left\langle \beta; out | a^\dagger_{in}(k,\lambda) - a^\dagger_{out}(k,\lambda) | \alpha; in \right\rangle$$

$$= \text{disconnected terms}$$

$$+ i \int d^3x e^{-ik\cdot x} \overleftrightarrow{\partial}_0 \varepsilon^*_\mu(k,\lambda) \langle \beta; out | A^\mu_{in}(x) - A^\mu_{out}(x) | \alpha; in \rangle$$

$$= \text{disconnected terms}$$

$$- i(\lim_{t\to+\infty} - \lim_{t\to-\infty}) Z_3^{-1/2} \int d^3x e^{-ik\cdot x} \overleftrightarrow{\partial}_0 \langle \beta; out | A^\mu(x) | \alpha; in \rangle \varepsilon^*_\mu(k,\lambda)$$

$$= \text{disconnected terms}$$

$$- i Z_3^{-1/2} \int d^4x e^{-ik\cdot x} \overleftrightarrow{\partial}_0 \langle \beta; out | A^\mu(x) | \alpha; in \rangle \varepsilon^*_\mu(k,\lambda)$$

$$= \text{disconnected terms}$$

$$- i Z_3^{-1/2} \int d^4x e^{-ik\cdot x} \vec{\Box}_x \langle \beta; out | A^\mu(x) | \alpha; in \rangle \varepsilon^*_\mu(k,\lambda)$$

Reduction formula for photons

- The final formula for photons is then

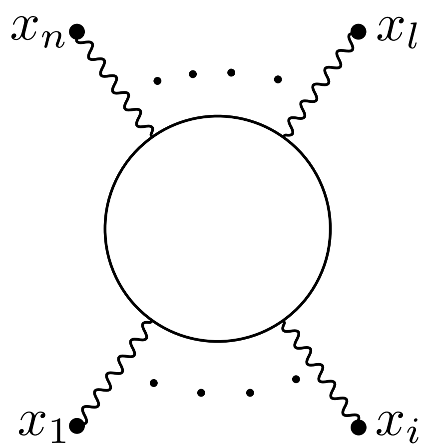
$$\langle k'_1 \cdots k'_n; out | k_1 \cdots k_\ell; in\rangle = \text{disconnected terms}$$

$$+ \left(\frac{-i}{\sqrt{Z_3}}\right)^{n+\ell} \int d^4y_1 \cdots d^4y_n d^4x_1 \cdots d^4x_\ell \; e^{\left[i\sum^n k'_i\cdot y_i - i\sum^\ell k_i\cdot x_i\right]}$$

$$\varepsilon^{\mu_1}(k_1,\lambda_1)\cdots\varepsilon^{\mu_\ell}(k_\ell,\lambda_\ell)\varepsilon^{*\mu'_1}(k'_1,\lambda'_1)\cdots\varepsilon^{*\mu'_n}(k'_n,\lambda'_n)$$

$$\Box_{y_1}\cdots\Box_{x_\ell}\,\langle 0|\,T(A_{\mu'_1}(y_1)\cdots A_{\mu'_n}(y_n)A_{\mu_1}(x_1)\cdots A_{\mu_\ell}(x_\ell)\,|0\rangle$$

- This corresponds to the diagram



Cross sections

- The reduction formulas are the fundamental results of this chapter. They relate the transition amplitudes from the initial to the final state with the Green functions of the theory. In the next chapter we will show how to evaluate these Green functions setting up the so-called covariant perturbation theory. Before we close this chapter, let us indicate how these transition amplitudes

$$S_{fi} \equiv \langle f; out | i; in \rangle$$

are related with the quantities that are experimentally accessible, the cross sections. Then the path between experiment (cross sections) and theory (Green functions) will be established.

- As we have seen in the reduction formulas there is always a trivial contribution to the S matrix, that corresponds to the so-called *disconnected terms*, when the system goes from the initial to the final state without interaction. The subtraction of this trivial contribution leads to the T matrix

$$S_{fi} = 1_{fi} - i(2\pi)^4 \delta^4(P_f - P_i) T_{fi}$$

where we have factorized explicitly the delta function expressing the 4-momentum conservation.

Cross sections

- If we neglect the trivial contribution, the transition probability from the initial to the final state will be given by

$$W_{f\leftarrow i} = \left|(2\pi)^4\delta^4(P_f - P_i)T_{fi}\right|^2$$

- To proceed we have to deal with the meaning of a square of a delta function. This appears because we are using plane waves. To solve this problem we can normalize in a box of volume V and consider that the interaction has a duration of T. Then

$$(2\pi)^4\delta^4(P_f - P_i) = \lim_{\substack{V\to\infty \\ T\to\infty}} \int_V d^3x \int_{-T/2}^{T/2} dx^0 e^{i(P_f-P_i)\cdot x} \; .$$

- However

$$F \equiv \int_V d^3x \int_{-T/2}^{T/2} dx^0 e^{i(P_f-P_i)\cdot x} = V\delta_{\vec{P}_f\vec{P}_j}\,\frac{2}{|E_f - E_i|}\sin\left|\frac{T}{2}(E_f - E_i)\right|$$

and the square of the last expression can be done, giving,

$$|F|^2 = V^2\delta_{\vec{P}_f,\vec{P}_i}\,\frac{4}{|E_f - E_i|^2}\sin^2\left|\frac{T}{2}(E_f - E_i)\right|$$

Cross sections

- If we want the transition rate by unit of volume (and unit of time) we divide by VT. Then

$$\Gamma_{fi} = \lim_{\substack{V\to\infty \\ T\to\infty}} V\delta_{\vec{P}_f,\vec{P}_i}\, 2\, \frac{\sin^2 \frac{T}{2}(E_f-E_i)}{\frac{T}{2}(E_f-E_i)^2} |T_{fi}|^2$$

- Using now the results

$$\lim_{V\to\infty} V\delta_{\vec{P}_f \vec{P}_j} = (2\pi)^3\delta^3(\vec{P}_f - \vec{P}_i)$$

$$\lim_{T\to\infty} 2\, \frac{\sin^2 \frac{T}{2}(E_f-E_i)}{\frac{T}{2}(E_f-E_i)^2} = (2\pi)\,\delta(E_f-E_i)$$

we get for the transition rate by unit volume and unit time,

$$\Gamma_{fi} \equiv (2\pi)^4\delta^4(P_f-P_i)|T_{fi}|^2$$

- To get the cross section we have to further divide by the incident flux, and normalize the particle densities to one particle per unit volume. Finally, we sum (integrate) over all final states in a certain energy-momentum range.

Cross sections

- We get,

$$d\sigma = \frac{1}{\rho_1 \rho_2} \frac{1}{|\vec{v}_{12}|} \Gamma_{fi} \prod_{j=3}^{n} \frac{d^3 p_j}{2 p_j^0 (2\pi)^3}$$

where $\rho_1 = 2E_1$ and $\rho_2 = 2E_2$

- An equivalent way of writing this equation is

$$d\sigma = \frac{1}{4\left[(p_i \cdot p_2)^2 - m_1^2 m_2^2\right]^{1/2}} (2\pi)^4 \delta^4(P_f - P_i) |T_{fi}|^2 \prod_{j=3}^{n} dp_j$$

that exhibits well the Lorentz invariance of each part that enters the cross section along the direction of collision. The incident flux and phase space factors are purely kinematics. The physics, with its interactions, is in the matrix element T_{fi}.

- We note that with our conventions, fermion and boson fields have the same normalization, that is, the one-particle states obey

$$\langle p | p' \rangle = 2p^0 (2\pi)^3 \delta^3(\vec{p} - \vec{p}')$$