Wess-Zumino Model : Interactions

1) the free Lagrangian

We saw how describe a massless multiplet with a complex spin-0 field ϕ, a massless chiral spinor χ and a non propagating field f. As each of these supermultiplets will describe the leptons and quarks plus their partners we generalize the free W-Z model to

$$\mathcal{L}_{\text{Free WZ}} = \partial_\mu \phi_i^\dagger \partial^\mu \phi_i + \bar{\chi}_i \bar{\sigma}^\mu \partial_\mu \chi_i + f_i^\dagger f_i$$

where i runs over all degrees of freedom. the action is invariant under

$$\begin{cases} \delta_\varepsilon \phi_i = \varepsilon \chi_i \, ; \; \delta_\varepsilon \chi_i = -i\sqrt{2}\,(\sigma^\mu \bar{\varepsilon})\,\partial_\mu \phi_i + \sqrt{2}\,\varepsilon f_i \\ \delta_\varepsilon f_i = -i\sqrt{2}\,\bar{\varepsilon}\,\bar{\sigma}^\mu \partial_\mu \chi_i \end{cases}$$

2) Interactions

Requirements : $\begin{cases} \cdot \text{ renormalizable (dimension } \leq 4) \\ \cdot \text{ supersymmetric} \end{cases}$

the most general interaction Lagrangian that is renormalizable is

$$\mathcal{L}_{int} = W_i(\phi,\phi^\dagger)F_i - \frac{1}{2}W_{ij}(\phi,\phi^\dagger)\chi_i\chi_j + h.c.$$

where, for the moment W_i, W_{ij} are arbitrary functions of $\phi_i, \phi_j^\dagger$. As $[W_i]$ must be ≤ 2, it can only depend on ϕ_i, $\phi_i^\dagger$ up to the second power. In the same way W_{ij} can only depend on the first power of ϕ_i, $\phi_i^\dagger$. furthermore since

$$\chi_i\chi_j = \chi_j\chi_i \Rightarrow \boxed{W_{ij} = W_{ji}}$$

<u>Now: terms with 4 spinors:</u>

$$\delta_\varepsilon\left(-\frac{1}{2}W_{ij}\,\chi_i\chi_j + h.c.\right)$$

$$= -\frac{1}{2}\frac{\partial W_{ij}}{\partial\phi_k}(\varepsilon\chi_k)(\chi_i\chi_j) - \frac{1}{2}\frac{\partial W_{ij}}{\partial\phi_k^\dagger}(\bar\varepsilon\bar\chi_k)(\chi_i\chi_j)$$

$$+ h.c.$$

these terms can not cancel against each other. they cannot cancel against the other terms in $\mathcal{L}_{int}$ ($\mathcal{L}_{free\,WZ}$ is invariant). the first term can vanish if

$$\frac{\partial W_{ij}}{\partial\phi_k} \text{ is symmetric in } i,j,k$$

because

$$(\varepsilon\chi_k)(\chi_i\chi_j) + (\varepsilon\chi_i)(\chi_j\chi_k) + (\varepsilon\chi_j)(\chi_k,\chi_i) = 0$$

the second term only vanishes iff W_{ij} does not depend on $\phi_i^\dagger$ only on ϕ_i. So this term will be supersymmetric iff

$$W_{ij} = M_{ij} + y_{ijk}\,\phi_k$$

with y_{ijk} symmetric in (i,j,k). If we write

$$\boxed{W \equiv \frac{1}{2} M_{ij}\,\phi_i\phi_j + \frac{1}{6} y_{ijk}\,\phi_i\phi_j\phi_k}$$

then

$$\boxed{W_{ij} \equiv \frac{\partial^2 W}{\partial\phi_i\partial\phi_j}}$$

<u>terms with ∂_μ:</u>

$$\delta_\varepsilon \mathcal{L}_{int} = \dots + W_i\,(-i\sqrt{2}\,\bar{\varepsilon}\bar{\sigma}^\mu\partial_\mu\chi_i)$$

$$- W_{ij}\,(\delta_\varepsilon\chi_i\,\delta_\varepsilon\chi_j) \quad + h.c$$

$$= \dots - i\sqrt{2}\,W_i\,\bar{\varepsilon}\bar{\sigma}^\mu\partial_\mu\chi_i$$

$$+ i\sqrt{2}\,W_{ij}\,(\chi_i\sigma^\mu\bar{\varepsilon})\,\partial_\mu\phi_j + h.c.$$

Now

$$W_{ij}\,(\chi_i \sigma^\mu \bar{\varepsilon})\,\partial_\mu \phi_j = -W_{ij}\,(\bar{\varepsilon}\,\bar{\sigma}^\mu \chi_i)\,\partial_\mu \phi_j$$

$$= -(\bar{\varepsilon}\,\bar{\sigma}^\mu \chi_i)\,\partial_\mu \frac{\partial W}{\partial \phi_i}$$

because

$$\partial_\mu \frac{\partial W}{\partial \phi_i} = \frac{\partial^2 W}{\partial \phi_i \partial \phi_j}\,\partial_\mu \phi_j$$

therefore the terms with ∂_μ give

$$\delta_\varepsilon \mathcal{L}_{int} = \ldots -i\sqrt{2}\,W_i\,\bar{\varepsilon}\,\bar{\sigma}^\mu \partial_\mu \chi_i$$

$$-i\sqrt{2}\,\bar{\varepsilon}\,\bar{\sigma}^\mu \chi_i\,\partial_\mu\left(\frac{\partial W}{\partial \phi_i}\right) + h.c.$$

They can not cancel against anything else so they must vanish or be a total derivative. They do not vanish but they are a total derivative if

$$\boxed{W_i \equiv \frac{\partial W}{\partial \phi_i}}$$

because then

$$\delta_\varepsilon \mathcal{L}_{int} = \ldots + \partial_\mu\left[-i\sqrt{2}\,\frac{\partial W}{\partial \phi_i}\,\bar{\varepsilon}\,\bar{\sigma}^\mu \chi_i\right]$$

So we determine

$$W_i = M_{ij}\phi_j + \frac{1}{2} y_{ijk}\phi_j\phi_k$$

we can finally show that the remaining terms cancel, that is

$$\boxed{\delta_\varepsilon \mathcal{L}_{int} = \partial_\mu X^\mu}$$

So the interactions are completely determined by W the superpotential. We have there free for the part containing F_i

$$\mathcal{L} = \ldots + F_i^+ F_i + W_i F_i + W_i^+ F_i^+$$

the Euler Lagrange equations give

$$F_i = -W_i^+ \quad ; \quad F_i^+ = -W_i$$

Substituting back into the Lagrangian

$$\boxed{\mathcal{L} = \mathcal{L}_{free\,WZ} - |W_i|^2 - \frac{1}{2}\left(W_{ij}\chi_i\chi_j + h.c.\right)}$$

<u>Example</u>: just one field that is

$$\boxed{W = \frac{1}{2}M\phi^2 + \frac{1}{6}y\phi^3}$$

one can easily show that

1) ϕ satisfies the Klein Gordon equation

$$\Box \phi + |M|^2 \phi = 0$$

2) χ satisfies the equations

$$\begin{cases} i\bar{\sigma}^\mu \partial_\mu \chi = M^* i\sigma_2 \chi^{+T} \\ i\sigma^\mu \partial_\mu (i\sigma_2 \chi^{+T}) = M\chi \end{cases}$$

and therefore

$$\Box \chi + |M|^2 \chi = 0 \qquad \Rightarrow \phi, \chi \text{ have the same mass.}$$

3) the interaction terms are

$$-\left|M\phi + \tfrac{1}{2} y \phi^2\right|^2 - \tfrac{1}{2}\left[(M + y\phi)\chi\chi + h.c.\right]$$

which give the mass term

$$\mathcal{L}_{mass} = -|M|^2 \phi^+ \phi - \tfrac{1}{2}(M\chi\chi + h.c.)$$

and the following interactions

i) A cubic interaction

$$\mathcal{L}_{cubic} = -\frac{1}{2}\left(M y^* \phi \phi^{+2} + M^* y \phi^+ \phi^2\right)$$

(ii) <u>A quartic interaction</u>

$$\mathcal{L}_{\text{quartic}} = -\frac{1}{4}|y|^2 \phi^2 \phi^{\dagger 2}$$

(iii) A Yukawa interaction between ϕ and χ

$$\mathcal{L}_{\text{Yukawa}} = -\frac{1}{2}\left(y \phi \chi\chi + h.c.\right)$$

Notice that the quartic coupling is the square of the Yukawa interaction, which is needed to cancel the quadratic divergences!