Gauge hierarchy problem

Topic A

1) The scalar self-energy

Consider the following Lagrangian

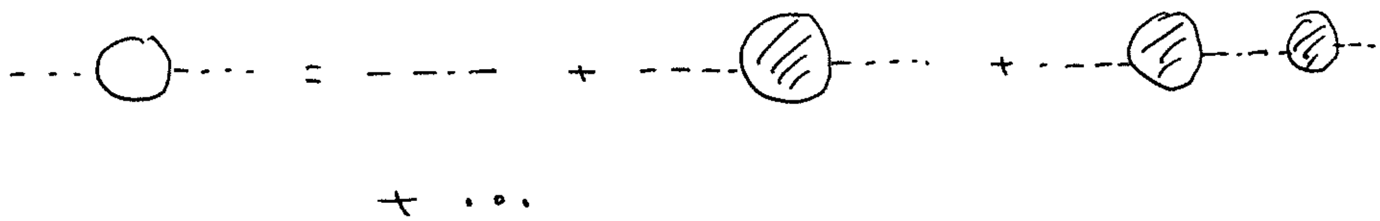
$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4$$

The free propagator is

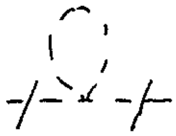
$$\Delta_F = \frac{i}{p^2 - m^2}$$

and the vertex

The full propagator is then given in perturbation theory by



where  is the 1-PI contribution of all orders in perturbation theory. In lowest order

 $=$  $+$ higher order

Now let us give names to these objects:

 $\equiv \Delta_F'$

 $= -i\,\Pi(p)$

Then we get

$$\Delta_F' = \Delta_F + \Delta_F(-i\Pi)\Delta_F + \Delta_F(-i\Pi)\Delta_F(-i\Pi)\Delta_F + \dots$$

$$= \Delta_F\big(1 - i\Pi\left(\Delta_F + \Delta_F(-i\Pi)\Delta_F + \dots\right)\big)$$

$$= \Delta_F\left(1 - i\,\Pi(p)\,\Delta_F'\right)$$

Multiplying on the left with Δ_F^{-1} and on the right with $\Delta_F'^{-1}$ we get

$$\Delta_F^{-1} = \Delta_F'^{-1} - i\,\Pi(p)$$

or

$$\Delta'^{-1}_F = \Delta_F^{-1} + i\,\Pi(p)$$

using $\Delta_F^{-1} = -i\left(p^2 - m^2\right)$

we get

$$\Delta_F'^{-1} = -i\left[p^2 - \left(m^2 + \Pi(p)\right)\right]$$

Therefore $\Pi(p)$ gives a contribution to the mass (also to the normalization, but we do not care about this here), hence the name of self-energy.

The scale at which $\Pi(p)$ must be calculated depends on the renormalization scheme. As we will only be interested in the dependence on the new Physics scale, we will identify

$$\boxed{\delta m^2 = \Pi(0)}$$

2) The one-loop self-energy

$$-i\,\Pi(p) = -i\lambda\left(\frac{1}{2}\right)\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2 - m^2}$$

and therefore

$$\Pi(0) = i\,\frac{\lambda}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2-m^2}$$

Now we introduce the notation for the one loop integrals

$$I_{m,n}(C) \equiv \mu^{\varepsilon}\int\frac{d^dk}{(2\pi)^d}\,\frac{k^{2m}}{[k^2-C]^n}$$

to get

$$\Pi(0) = i\,\frac{\lambda}{2}\,I_{0,1}(m^2)$$

In dimensional regularization the integrals $I_{m,n}$ can be evaluated in closed form. For zero external momenta and for loops with the same particle in the loop, $C=m^2$, and they can also be easily done in the cutoff method. We have

$$I_{0,1}(m^2) = -i\,\frac{2\pi^{4/2}}{\Gamma(2)}\,\frac{1}{16\pi^4}\int_0^{\Lambda} d\bar{p}\,\frac{\bar{p}^3}{\bar{p}^2+m^2}$$

$$= -\frac{i}{16\pi^2}\int_0^{\Lambda^2} du\,\frac{u}{u+m^2}$$

$$= \frac{i}{16\pi^2}\left[-\Lambda^2 + 2m^2\ln\left(\frac{\Lambda}{m}\right)\right]$$

$$I_{0,2}(m^2) = i \frac{2\pi^{4/2}}{\Gamma(2)} \frac{1}{16\pi^4} \int_0^{\Lambda} d\bar{p} \frac{\bar{p}^3}{(\bar{p}^2+m^2)^2}$$

$$= \frac{i}{16\pi^2} \int_0^{\Lambda^2} du \frac{u}{(u+m^2)^2}$$

$$= \frac{i}{16\pi^2} \left[-1 + 2 \ln \frac{\Lambda}{m} \right]$$

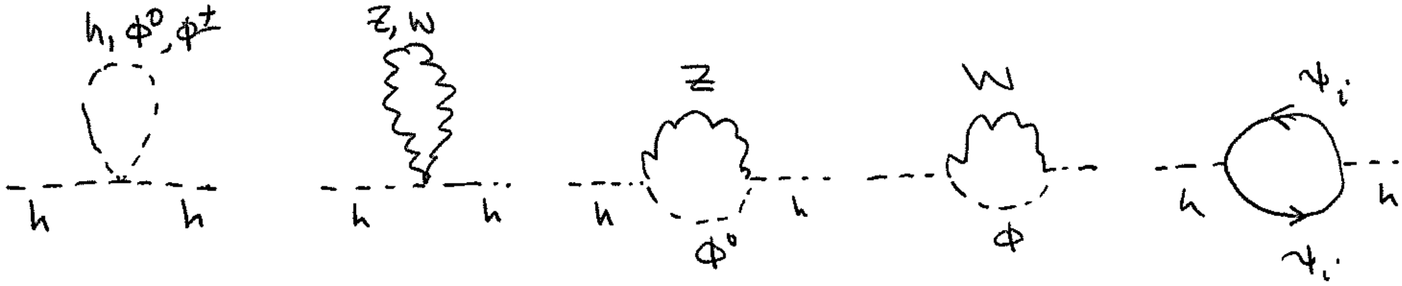
Therefore

$$\delta m^2 = \Pi(0) = \frac{\lambda}{32\pi^2} \Lambda^2$$

If m is at weak scale and Λ is the Gut scale we have the fine-tuning problem called hierarchy pro blem.

3) The hierarchy problem in the Standard Model

In the SM, in the Feynman-t'Hooft gauge we have the following diagrams:



As all the scalars contribute with the same sign, and the fermions with the opposite sign, we can ask if the contributions cancel. A not so complicated calculation gives

$$\delta m_h^2 = \frac{3g^2}{32\pi^2 M_W^2} \Lambda^2 \left[m_h^2 + M_Z^2 + 2 M_W^2 - 4 m_t^2 \right]$$

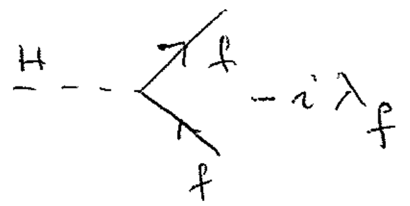
$$\simeq 0.04 \, \Lambda^2$$

(we kept only the top because the contributions are proportional to the particle mass, hence negligible for the other fermions)

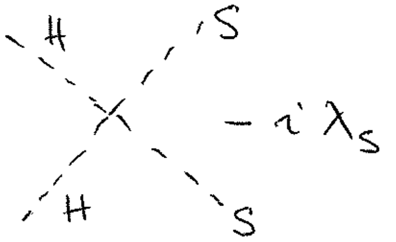
So, in conclusion, the fermions of the SM do not solve the hierarchy problem.

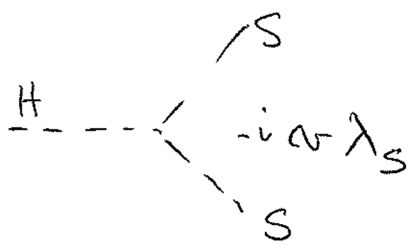
4) A Solution

Consider a theory with a number N_f of Dirac fermions with coupling to the Higgs $\lambda_f = \frac{\sqrt{2} m_f}{v}$

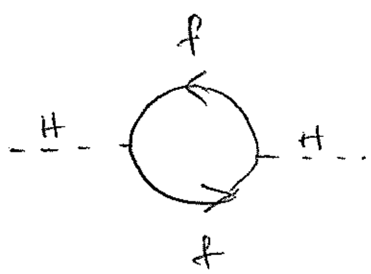


and a number N_s of complex scalars with the following couplings to the Higgs.



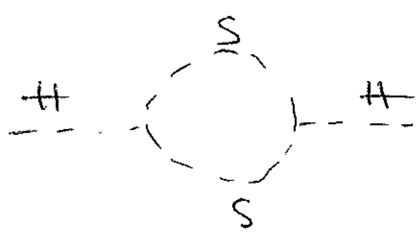


then the 1-loop contributions to the Higgs boson mass are given by the following diagrams



a)

b)



c)

$$-i\,\Pi^{(a)}_{HH}(0) = -\left(-i\frac{\lambda_f}{\sqrt{2}}\right)^2 i^2 \int \frac{d^4k}{(2\pi)^4}\, \frac{\mathrm{Tr}\left[(\not k + m_f)(\not k + m_f)\right]}{(k^2 - m_f^2)^2}$$

$$= -2\lambda_f^2 \int \frac{d^4k}{(2\pi)^4}\, \frac{k^2 + m_f^2}{k^2 - m_f^2}$$

$$= -2\lambda_f^2 \int \frac{d^4k}{(2\pi)^4}\left[\frac{1}{k^2 - m_f^2} + \frac{2m_f^2}{(k^2 - m_f^2)^2}\right]$$

$$= -2\lambda_f^2 \left[I_{0,1}(m_f^2) + 2m_f^2 I_{0,2}(m_f^2)\right]$$

$$= -\lambda_f^2 \frac{i}{8\pi^2}\left[-\Lambda^2 + 6m_f^2 \ln\frac{\Lambda}{m} - 2m_f^2\right]$$

$$\boxed{\delta m_H^{2\,(a)} = \Pi^{(a)}_{HH}(0) = N_f \frac{\lambda_f^2}{8\pi^2}\left[-\Lambda^2 + 6m_f^2 \ln\frac{\Lambda}{m_f} - 2m_f^2\right]}$$

For the other diagrams we have

$$-i\,\Pi^{(b)}_{HH}(0) = -i\lambda_S N_S \int \frac{d^4k}{(2\pi)^4}\, \frac{i}{k^2 - m^2}$$

or

$$\Pi^{(b)}_{HH}(0) = i\lambda_S N_S I_{0,1}(m^2)$$

$$= \frac{\lambda_S N_S}{16\pi^2}\left[\Lambda^2 - 2m_S^2 \ln\left(\frac{\Lambda}{m_S}\right)\right]$$

and

$$-i\Pi^{(c)}_{HH}(0) = N_S(\lambda_S v)^2\, i^4 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2)^2}$$

$$= \lambda_S^2 v^2 N_S\, I_{0,2}(m_S^2)$$

$$= i\, \frac{\lambda_S^2 v^2 N_S}{16\pi^2}\left[-1 + 2\ln\left(\frac{\Lambda}{m_S}\right)\right]$$

and finally

$$\delta m_H^{2\,(b+c)} = \Pi^{(b)}_{HH}(0) + \Pi^{(c)}_{HH}(0)$$

$$= \frac{\lambda_S N_S}{16\pi^2}\left[+\Lambda^2 - 2m_S^2 \ln\left(\frac{\Lambda}{m_S}\right)\right]$$

$$- \frac{\lambda_S^2 v^2 N_S}{16\pi^2}\left[-1 + 2\ln\left(\frac{\Lambda}{m_S}\right)\right]$$

Now IF $\boxed{N_S = 2N_f}$ and $\boxed{\lambda_f^2 = \frac{2m_f^2}{v^2} = \lambda_S}$

we get the final result

$$\boxed{\delta m_H^2 = \frac{\lambda_f^2 N_f}{4\pi^2}\left[(m_f^2 - m_S^2)\ln\left(\frac{\Lambda}{m_S}\right) + 3m_f^2 \ln\left(\frac{m_S}{m_f}\right)\right]}$$

the quadratic divergence cancells out