Chiral-Superfields (I. Aitchison + Bailin and Love)

1) A review of the P^μ operator

Consider a translation

$$x'^\mu = x^\mu + a^\mu$$

with a constant 4-vector a^μ. We should have for states in S'

$$|\alpha\rangle' = U|\alpha\rangle$$

where U is an unitary transformation such that

$$'\langle\beta|\Phi(x')|\alpha\rangle' = \langle\beta|U^{-1}\Phi(x')U|\alpha\rangle = \langle\beta|\Phi(x)|\alpha\rangle$$

As this should be valid for all states we infer

$$U^{-1}\Phi(x')U = \Phi(x) \quad \text{or} \quad \boxed{\Phi(x') = U\,\Phi(x)\,U^{-1}}$$

For infinitesimal transformations

$$x'^\mu = x^\mu + \varepsilon^\mu$$

We write

$$U = 1 + i\varepsilon_\mu P^\mu$$

where P^μ are the generators of the translation. We then have (to lowest order in ε)

$$(1 + i\varepsilon_\mu P^\mu)\,\Phi(x)\,(1 - i\varepsilon_\mu P^\mu) = \Phi(x^\mu + \varepsilon^\mu) = \Phi(x) + \varepsilon^\mu\partial_\mu\Phi$$

or

$$i\,\varepsilon_\mu[P^\mu, \Phi(x)] = \varepsilon^\mu\partial_\mu\Phi$$

$$\delta_\varepsilon\Phi(x) \equiv i\varepsilon_\mu[P^\mu, \Phi(x)] = \varepsilon^\mu\partial_\mu\Phi$$

As ε_μ is an arbitrary parameter we have the fundamental relation in QFT

$$i\,[P^\mu, \phi(x)] = \partial^\mu \phi$$

we can look at this in a different way. We can say the $\hat{P}^\mu \equiv i\partial^\mu$ is a differential representation of the momentum operator and therefore

$$\delta\phi = \varepsilon_\mu \partial^\mu \phi \equiv -i\,\varepsilon_\mu \hat{P}^\mu \phi$$

Before ending this section we note that for finite translations we can write

$$\phi(x) = U(x)\phi(0)\,U^{-1}_{(x)} = e^{i x_\mu P^\mu}\,\phi(0)\,e^{-i x_\mu P_\mu}$$

Now, due to the fact the $[P_\mu, P_\nu] = 0$, we have

$$U(x)\,U(a^\mu) = e^{i x_\mu P^\mu}\,e^{i a_\mu P_\mu} = e^{i(x+a)_\mu P^\mu}$$

and therefore

$$\phi(x+a) = U(x+a)\,\phi(0)\,U^{-1}(x+a)$$

2) <u>Superfield representation of the SUSY algebra</u>

the SUSY algebra is generated by P^μ, Q_α, $\overline{Q}_{\dot\alpha}$ such that

$$[Q_\alpha, P_\mu] = [\overline{Q}_{\dot\alpha}, P_\mu] = [P_\mu, P_\nu] = 0$$

$$\{Q_\alpha, Q_\beta\} = \{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} = 0$$

$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2\,(\sigma^\mu)_{\alpha\dot\beta}\,P_\mu$$

Let us represent an element of the corresponding group by

$$U(x^\mu, \theta, \bar{\theta}) = e^{i(x_\mu P_\mu + \theta Q + \bar{\theta}\bar{Q})}$$

$$= e^{i x_\mu P^\mu} e^{i(\theta Q + \bar{\theta}\bar{Q})}$$

where the last step follows from $[P, Q] = [P, \bar{Q}] = 0$. Notice however, that

$$e^{i(\theta Q + \bar{\theta}\bar{Q})} \neq e^{i\theta Q} e^{i\bar{\theta}\bar{Q}} \neq e^{i\bar{\theta}\bar{Q}} e^{i\theta Q}$$

because $\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} \neq 0$. (see Aitchison)

Let us see the result of the successive action of U. More precisely we have

$$U(a^\mu, \varepsilon, \bar{\varepsilon})\, U(x^\mu, \theta, \bar{\theta})$$

$$= e^{i a^\mu P_\mu} e^{i(\varepsilon Q + \bar{\varepsilon}\bar{Q})} e^{i x^\mu P_\mu} e^{i(\theta Q + \bar{\theta}\bar{Q})}$$

$$= e^{i(x+a)^\mu P_\mu} e^{i(\varepsilon Q + \bar{\varepsilon}\bar{Q})} e^{i(\theta Q + \bar{\theta}\bar{Q})}$$

to calculate this we use the Baker-Hausdorff formula

$$e^A e^B = e^{(A + B + \frac{1}{2}[A,B] + \cdots)}$$

We have

$$[\varepsilon Q + \bar\varepsilon \bar Q,\ \theta Q + \bar\theta \bar Q]$$

$$= [\varepsilon Q, \bar\theta \bar Q] + [\bar\varepsilon \bar Q, \theta Q]$$

$$= \varepsilon^\alpha Q_\alpha \bar\theta_{\dot\beta} \bar Q^{\dot\beta} - \bar\theta_{\dot\beta} \bar Q^{\dot\beta} \varepsilon^\alpha Q_\alpha \quad - (\varepsilon \Leftrightarrow \theta)$$

$$= \varepsilon^\alpha \bar\theta^{\dot\beta} Q_\alpha \bar Q_{\dot\beta} + \varepsilon^\alpha \bar\theta^{\dot\beta} \bar Q_{\dot\beta} Q_\alpha \quad - (\varepsilon \Leftrightarrow \theta)$$

$$= \varepsilon^\alpha \bar\theta^{\dot\beta} \{ Q_\alpha, \bar Q_{\dot\beta} \} - (\varepsilon \Leftrightarrow \theta)$$

$$= 2 (\varepsilon \sigma^\mu \bar\theta) P_\mu - 2 (\theta \sigma^\mu \bar\varepsilon) P_\mu$$

Notice: As P_μ commutes with P, Q and $\bar Q$ the other terms in the Baker-Hausdorff relation vanish. Also the end result is a translation. We get

$$e^{i(\varepsilon Q + \bar\varepsilon \bar Q)}\, e^{i(\theta Q + \bar\theta \bar Q)}$$

$$= e^{\left[i(\varepsilon + \theta) Q + i(\bar\varepsilon + \bar\theta) \bar Q - (\varepsilon \sigma^\mu \bar\theta - \theta \sigma^\mu \bar\varepsilon) P_\mu \right]}$$

and therefore

$$U(a^\mu, \varepsilon, \bar\varepsilon)\, U(x^\mu, \theta, \bar\theta) =$$

$$= U\left(x^\mu + a^\mu + i \varepsilon \sigma^\mu \bar\theta - i \theta \sigma^\mu \bar\varepsilon,\ \theta + \varepsilon,\ \bar\theta + \bar\varepsilon\right)$$

that is

$$\theta \to \theta + \varepsilon$$
$$\bar\theta \to \bar\theta + \bar\varepsilon$$
$$x^\mu \to x^\mu + a^\mu + i \varepsilon \sigma^\mu \bar\theta - i \theta \sigma^\mu \bar\varepsilon$$

Exercise: Prove that $i\,\varepsilon\sigma^\mu\bar\theta - i\,\theta\sigma^\mu\bar\varepsilon$ is real

We therefore define a superfield by the action of $U(x,\theta,\bar\theta)$, that is

$$S(x,\theta,\bar\theta) = U(x,\theta,\bar\theta)\, S(0,0,0)\, U^{-1}(x,\theta,\bar\theta)$$

Consider now an infinitesimal transformation on the SUSY part (we take $a^\mu = 0$). We have

$$\delta S = i(\varepsilon\sigma^\mu\bar\theta - \theta\sigma^\mu\bar\varepsilon)\partial_\mu S + \varepsilon^\alpha \frac{\partial S}{\partial\theta^\alpha} + \bar\varepsilon_{\dot\alpha}\frac{\partial S}{\partial\bar\theta_{\dot\alpha}}$$

In analogy with

$$\delta\phi = -i\,\varepsilon_\mu \hat P^\mu \phi$$

we write

$$\delta S \equiv (-i\,\varepsilon\hat Q - i\,\bar\varepsilon\hat{\bar Q})S$$

this gives

$$\boxed{\hat Q_\alpha = i\frac{\partial}{\partial\theta^\alpha} - (\sigma^\mu\bar\theta)_\alpha\,\partial_\mu}$$

$$\hat{\bar Q}^{\dot\alpha} = i\frac{\partial}{\partial\bar\theta_{\dot\alpha}} - (\theta\sigma^\mu)_{\dot\beta}\,\varepsilon^{\dot\beta\dot\alpha}$$

or

$$\boxed{\hat{\bar Q}_{\dot\alpha} = -i\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + (\theta\sigma^\mu)_{\dot\alpha}\,\partial_\mu}$$

Exercise: Show that $\mathcal{Q}_\alpha, \bar{\mathcal{Q}}_{\dot\alpha}$ are a representation of the algebra

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3) Superfield expansion in component fields

the general superfield $S(x,\theta,\bar\theta)$ may be expanded in powers of $\theta, \bar\theta$. this expansion terminates because, for instance, $\theta_\alpha \theta_\beta \theta_\gamma = 0$. So the most general superfield is

$$S(x,\theta,\bar\theta) = f(x) + \theta\varphi(x) + \bar\theta\bar\chi(x) + \theta\theta m(x) + \bar\theta\bar\theta n(x) + \theta\sigma^\mu\bar\theta V_\mu(x) + \theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\psi(x) + \theta\theta\bar\theta\bar\theta d(x)$$

- $f(x)$, $m(x)$, $n(x)$, $d(x)$: Scalars
- $\varphi(x)$, $\bar\chi(x)$, $\bar\lambda(x)$, $\psi(x)$: Weyl spinors
- $V_\mu(x)$: vector field

Conclusion: the most general superfield is a reducible representation of the Susy algebra.

we can get irreducible representations by imposing constraints on $S(x,\theta,\bar\theta)$. For this we introduce the fermionic derivatives

$$\begin{cases} D_\alpha = \dfrac{\partial}{\partial\theta^\alpha} - i\,(\sigma^\mu\bar\theta)_\alpha\,\partial_\mu \\ \\ \bar D_{\dot\alpha} = -\dfrac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\,(\theta\sigma^\mu)_{\dot\alpha}\,\partial_\mu \end{cases}$$

Exercise: Show that D_α, $\bar D_{\dot\alpha}$ satisfy the SUSY algebra

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Exercise: Show that D_α and $\bar D_{\dot\alpha}$ anti-commute with the generator $\hat Q_\alpha$ and $\hat{\bar Q}_{\dot\alpha}$.

So D_α, $\bar D_{\dot\alpha}$ anticommute with $\hat Q_\alpha$ and $\hat{\bar Q}_{\dot\alpha}$ they commute with $\varepsilon\hat Q + \bar\varepsilon\bar Q$, so they can be used to impose constraints on superfields that are supersymmetric.

4) Chiral Superfields

Φ is a (left-handed) chiral superfield if satisfies

$$\bar D_{\dot\alpha}\Phi = 0$$

Noticing that

$$\bar D_{\dot\alpha}\theta = 0$$

and if $\quad y^\mu = x^\mu - i\,\theta\sigma^\mu\bar\theta$

$$\bar D_{\dot\alpha}\, y = 0$$

So any function of y and θ is a Chiral superfield. Its expansion is

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\,\theta\chi(y) + \theta\theta F(y)$$

this can be expanded to give

$$\Phi(x,\theta,\bar\theta) = \phi(x) + \sqrt{2}\,\theta\chi(x) + \theta\theta F$$
$$+ i\,\partial_\mu\phi\,\theta\sigma^\mu\bar\theta + \frac{i}{\sqrt{2}}\theta\theta\,\partial_\mu\chi\sigma^\mu\bar\theta$$
$$- \frac{1}{4}\Box\phi\,\theta\theta\,\bar\theta\bar\theta$$

Exercise: Show this

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our notation indicates that this is just the W-Z free supermultiplet. Let us show that indeed we have the correct SUSY transformations. We have

$$\delta\Phi = (-i\,\varepsilon\hat{Q} - i\,\bar\varepsilon\hat{\bar{Q}})\,\Phi$$

$$= (-i\,\varepsilon^\alpha\hat{Q}_\alpha + i\,\bar\varepsilon^{\dot\alpha}\hat{\bar{Q}}_{\dot\alpha})\,\Phi$$

$$= \varepsilon^\alpha\left(\frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu\bar\theta)_\alpha\partial_\mu\right)\Phi + \bar\varepsilon^{\dot\alpha}\left(\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu\right)\Phi$$

$$= \sqrt{2}\, \varepsilon^\alpha \chi_\alpha + 2\, \varepsilon\theta f + i\, \partial_\mu \phi\, \bar{\varepsilon}^{\dot\alpha} (\theta\sigma^\mu)_{\dot\alpha}$$

$$+ \frac{i}{\sqrt{2}} \theta\theta\, \partial_\mu \chi\, \sigma^\mu \bar{\varepsilon} + i\, \partial_\mu \phi\, \bar{\varepsilon}^{\dot\alpha} (\theta\sigma^\mu)_{\dot\alpha}$$

$$+ i\sqrt{2}\, \theta\partial_\mu\chi\ \ \bar{\varepsilon}^{\dot\alpha}(\theta\sigma^\mu)_{\dot\alpha} + \cdots$$

$$= \sqrt{2}\, \varepsilon\chi - 2i\, \theta\sigma^\mu\bar{\varepsilon}\, \partial_\mu\phi + 2\varepsilon\theta f$$

$$- i\sqrt{2}\, \theta\theta\ \ \bar{\varepsilon}\bar{\sigma}^\mu \partial_\mu \chi + \cdots$$

If we compare with

$$\delta_\varepsilon \Phi = \delta_\varepsilon \phi + \sqrt{2}\, \theta\, \delta_\varepsilon \chi + \theta\theta\, \delta_\varepsilon f + \cdots$$

we get

$$\begin{cases} \delta_\varepsilon \phi = \sqrt{2}\, \varepsilon\chi \\ \delta_\varepsilon \chi_\alpha = -i\sqrt{2}\, (\sigma^\mu\bar{\varepsilon})_\alpha\, \partial_\mu\phi + \sqrt{2}\, \varepsilon_\alpha f \\ \delta_\varepsilon F = -i\sqrt{2}\ \ \bar{\varepsilon}\bar{\sigma}^\mu \partial_\mu \chi \end{cases}$$

exactly the SUSY transformation laws of the W-Z free supermultiplet. Very important for the following is that the f component (the Coefficient of $\theta\theta$) transforms like a total derivative and therefore Actions invariant under SUSY can be constructed from the F-component of appropriate Chiral Superfields

(5) Products of chiral Superfields: The Superpotential

Consider a Chiral superfield Φ_i such that

$$\Phi_i(y,\theta) = \phi_i(y) + \sqrt{2}\theta\chi_i(y) + \theta\theta F_i(y)$$

the product

$$\Phi_i \Phi_j$$

is again a Chiral Superfield. Therefore it can be expanded like

$$\Phi_i(y)\Phi_j(y) = \phi_{ij}(y) + \sqrt{2}\theta\chi_{ij} + \theta\theta F_{ij}.$$

As we are only interested in constructing actions invariant under SUSY from the products of chiral superfields, we only care about the term F_{ij}. We obtain

$$\boxed{F_{ij} = \phi_i F_j + \phi_j F_i - \chi_i\chi_j}$$

Similarly

$$\Phi_i \Phi_j \Phi_k = \phi_{ijk} + \sqrt{2}\,\theta \chi_{ijk} + \theta\theta F_{ijk}$$

where

$$F_{ijk} = \phi_i \phi_j F_k + \phi_j \phi_k F_i + \phi_k \phi_i F_j$$
$$- \chi_i \chi_j F_k - \chi_j \chi_k F_i - \chi_k \chi_i F_j$$

Now if we constructed

$$\boxed{W = \frac{1}{2} M_{ij} \Phi_i \Phi_j + \frac{1}{6} y_{ijk} \Phi_i \Phi_j \Phi_k}$$

where $M_{ij} = M_{ji}$ and y_{ijk} is completely symmetric under exchange of indices, we obtain

$$W\big|_F = M_{ij} \phi_i F_j - \frac{1}{2} M_{ij} \chi_i \chi_j$$
$$+ \frac{1}{2} y_{ijk} \phi_i \phi_j F_k - \frac{1}{2} y_{ijk} \chi_i \chi_j \phi_k$$

which has precisely the form of the WZ interactions Lagrangian. W is the superpotential and SUSY invariance is built in by construction.

We will see next how to construct the kinetic part of W-Z in terms of superfields.

6. Kinetic W-Z free Action from Superfields

up to now we only considered chiral superfields that satisfy

$$\bar{D}_{\dot\alpha} \Phi = 0$$

the superfield $\Phi^\dagger$ satisfies

$$D_\alpha \Phi^\dagger$$

and it is called antichiral superfield. It's expansion is

$$\Phi^\dagger(y,\bar\theta) = \phi^\dagger(y) + \sqrt{2}\,\bar\theta\,\bar\chi(y) + \bar\theta\bar\theta F^\dagger(y)$$

or expanding

$$\Phi^\dagger = \phi^\dagger(x) + \sqrt{2}\,\bar\theta\,\bar\chi(x) + \bar\theta\bar\theta\, F^\dagger(x)$$
$$+ i\,\partial_\mu\phi^\dagger\,\theta\sigma^\mu\bar\theta - \frac{i}{\sqrt{2}}\bar\theta\bar\theta\,\theta\sigma^\mu\partial_\mu\bar\chi$$
$$- \frac{1}{4}\Box\phi^\dagger\,\theta\theta\,\bar\theta\bar\theta$$

Now we take the product

$$\Phi_i^\dagger(y)\Phi_j(y) = \phi_i^\dagger(y)\,\phi_j(y) + \sqrt{2}\,\theta\chi_j(y)\phi_i^\dagger(y) + \sqrt{2}\,\bar\theta\bar\chi_i(y)\phi_j(y)$$
$$+ 2\,\bar\theta\bar\chi_i\,\theta\chi_j + F_j\,\phi_i^\dagger\,\theta\theta + F_i^\dagger\,\phi_j\,\bar\theta\bar\theta$$
$$+ \sqrt{2}\,\theta\theta\,\bar\theta\bar\chi_i\,F_j + \sqrt{2}\,\bar\theta\bar\theta\,\theta\chi_j\,F_i^\dagger$$
$$+ \theta\theta\,\bar\theta\bar\theta\,F_i^\dagger F_j$$

this superfield is not chiral, is in fact a particular case of a general superfield with $S = S^*$, that is a real superfield. It can be shown that the coefficient of $\theta\theta\bar\theta\bar\theta$ transforms like a total derivative, so it is a candidate for an invariant Lagrangian. After some calculations we get

$$\Phi_i^\dagger \Phi_j \Big|_D = F_i^\dagger F_j + \frac{1}{2} \partial_\mu \phi_i^\dagger \partial^\mu \phi_j - \frac{1}{4} \phi_i^\dagger \Box \phi_j$$
$$- \frac{1}{4} \Box \phi_i^\dagger \phi_j + \frac{i}{2} \bar\chi_i \bar\sigma^\mu \partial_\mu \chi_j$$
$$- \frac{i}{2} \partial_\mu \bar\chi_i \bar\sigma^\mu \chi_j$$

$$= \partial_\mu \phi_i^\dagger \partial^\mu \phi_j + i \bar\chi_i \bar\sigma^\mu \partial_\mu \chi_j + F_i^\dagger F_j$$
$$+ \partial_\mu X^\mu$$

which, up to a total derivative gives the free WZ Lagrangian. We can then write

$$\boxed{\begin{aligned} \mathcal{L}_{WZ} &= \Phi_i^\dagger \Phi_j \Big|_D + \left(W\big|_F + h.c. \right) \\ W &= \frac{1}{2} M_{ij} \phi_i \phi_j + \frac{1}{6} y_{ijk} \phi_i \phi_j \phi_k \end{aligned}}$$