Homework assignment

Hand in by the end of the first week of January, that is 6/1/2012.

1 Relativity and Cosmology

Problem 1 (total: 4 points)

For radial infalls (i.e. when the orbital angular momentum $\tilde{L} = 0$) the geodesic equation reduces to

$$\frac{dr}{d\tau} = -\left(\tilde{E}^2 - 1 + \frac{2M}{r}\right)^{1/2} . \tag{1}$$

By considering the limit $r \to \infty$ we see that there are three cases:

1) $\tilde{E} < 1$: in this case the particle falls from rest at (say) $r = R$, where $1 - \tilde{E}^2 = 2M/R$. Show that if $\tau = 0$ at $r = R$, then

$$\tau(r) = \left(\frac{R^3}{8M}\right)^{1/2} \left[2\left(\frac{r}{R} - \frac{r^2}{R^2}\right)^{1/2} + \arccos\left(\frac{2r}{R} - 1\right)\right] . \tag{2}$$

Now introduce the "cycloid parameter" η such that $r = \frac{R}{2}(1 + \cos\eta)$ and show that

$$\tau(\eta) = \left(\frac{R^3}{8M}\right)^{1/2} (\eta + \sin\eta) . \tag{3}$$

Finally, integrate

$$\frac{dt}{d\tau} = \frac{\tilde{E}}{1 - 2M/r} \tag{4}$$

in terms of η to get

$$\frac{t(\eta)}{2M} = \ln\left|\frac{(R/2M - 1)^{1/2} + \tan(\eta/2)}{(R/2M - 1)^{1/2} - \tan(\eta/2)}\right| + \left(\frac{R}{2M} - 1\right)^{1/2} \left[\eta + \frac{R}{4M}(\eta + \sin\eta)\right] . \tag{5}$$

What can you say about the *proper* time it takes for a particle to fall from rest down to $r = 2M$? What is the proper time for a particle to fall from rest down to $r = 0$? What is the *coordinate* time to fall from rest down to $r = 2M$? Make a plot of r/M as a function of t/M and τ/M.

2) $\tilde{E} = 1$: show that in this case

$$\tau(r) = -\frac{2}{3}\left(\frac{r^3}{2M}\right)^{1/2} + \text{constant} , \tag{6}$$

$$t(r) = -\frac{2}{3}\left(\frac{r^3}{2M}\right)^{1/2} - 4M\left(\frac{r}{2M}\right)^{1/2} + 2M\ln\left|\frac{(r/2M)^{1/2} + 1}{(r/2M)^{1/2} - 1}\right| + \text{constant} . \tag{7}$$

3) $\tilde{E} > 1$: $\tau(r)$, $r(\eta)$, $\tau(\eta)$ and $t(\eta)$ can be found by defining $2M/R = \tilde{E}^2 - 1$ and changing the sign of R in the equations listed above. Show that in this case $2M/R = v_\infty^2/(1 - v_\infty^2)$.

Problem 2 (4 points)

Find the radius as well as the ratio E/L of energy to orbital angular momentum for null circular geodesics in the equatorial plane of an extreme ($M = a$) Kerr geometry. In Boyer-Lindquist coordinates, the rotating Kerr (Roy Kerr, 1963) metric reads

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4aMr\sin^2\theta}{\rho^2} d\phi dt + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2 , \tag{8}$$

where

$$\rho^2 \equiv r^2 + a^2\cos^2\theta , \tag{9}$$

$$\Delta \equiv r^2 - 2Mr + a^2 , \tag{10}$$

$$a \equiv \frac{J}{M} . \tag{11}$$

Additional points if:

a) you find (E, L) as a function of the orbital radius for a non-extremal $(a < M)$ Kerr geometry,

b) you prove that equatorial geodesics in the Kerr geometry remain equatorial.

2 Astroparticles

Write a short text for each of the following three topics.

1. The GZK cutoff
2. Transparency of the Universe to gamma rays in the TeV region: main targets, physical processes and critical energies
3. Cherenkov light output from electromagnetic showers close to the shower axis

3 Particle Physics

1. Give two reasons for the need of having at least two Higgs doublets in supersymmetric extensions of the Standard Model. Explain the reasons in such a way that another student not having attending the classes may understand.
2. Consider the process $\nu_e + \overline{\nu}_e \to W_L^- + W_L^+$. Reproduce the plots for the cross section that I gave in class showing the cancellation of the bad high-energy behaviour. You can use the expressions for the square of the amplitudes that I gave in class and that you can find in the article in my page

   `http://porthos.ist.utl.pt/CTQFT/files/HiggsAndSM.pdf`