Notation (Aitchison)

1) Weyl representation for Dirac matrices

$$\vec{\gamma} = \begin{pmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad ; \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \; \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$P_L = \frac{1-\gamma_5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad ; \quad P_R = \frac{1+\gamma_5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

If $\quad \Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix} \Rightarrow P_R \Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix} \; ; \; P_L \Psi = \begin{pmatrix} 0 \\ \chi \end{pmatrix}$

2) Dotted notation

$$\Psi \equiv \begin{pmatrix} \bar{\psi}^{\dot{\alpha}} \\ \chi_\alpha \end{pmatrix} \qquad \chi^\alpha \equiv \varepsilon^{\alpha\beta} \chi_\beta \; ; \; \chi_\alpha \equiv \varepsilon_{\alpha\beta} \chi^\beta$$

$$\bar{\psi}_{\dot{\alpha}} \equiv \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{\psi}^{\dot{\beta}} \; ; \quad \bar{\psi}^{\dot{\alpha}} \equiv \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\psi}_{\dot{\beta}}$$

with

$$\varepsilon_{\alpha\beta} = \varepsilon_{\dot{\alpha}\dot{\beta}} = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\varepsilon^{\alpha\beta} = \varepsilon^{\dot{\alpha}\dot{\beta}} = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\varepsilon_{\alpha\beta}\,\varepsilon^{\beta\gamma} = \delta_\alpha^{\ \gamma} \; ; \quad \varepsilon_{\dot{\alpha}\dot{\beta}}\,\varepsilon^{\dot{\beta}\dot{\gamma}} = \delta_{\dot{\alpha}}^{\ \dot{\gamma}}$$

3) <u>Lorentz invariants:</u>

with - L spinors:

$$\chi\xi \equiv \chi^\alpha \xi_\alpha = \xi^\alpha \chi_\alpha \equiv \xi\chi$$

with - R spinors

$$\bar\Psi\bar\theta \equiv \bar\Psi_{\dot\alpha}\bar\theta^{\dot\alpha} = \bar\theta_{\dot\alpha}\bar\Psi^{\dot\alpha} \equiv \bar\theta\bar\Psi$$

for instance:

$$\chi\xi \equiv \chi^\alpha\xi_\alpha = \varepsilon^{\alpha\beta}\chi_\beta \xi_\alpha = -\varepsilon^{\alpha\beta}\xi_\alpha\chi_\beta = \varepsilon^{\beta\alpha}\xi_\alpha\chi_\beta =$$
$$= \xi^\beta\chi_\beta \equiv \xi\chi$$

4) <u>Complex conjugation</u>

$$(\chi_\alpha)^* \equiv \bar\chi_{\dot\alpha} \quad ; \quad (\bar\chi^{\dot\alpha})^* \equiv \chi^\alpha$$

but the "$*$" mean complex conjugation for name functions and "$+T$" for operators. this is a subtle point.

5) <u>Dirac equation with Weyl - two component spinors</u>

$$\mathcal{L} = i\,\bar\Psi\gamma^\mu\partial_\mu\Psi - m\,\bar\Psi\Psi \qquad \Psi = \begin{pmatrix} \bar\Psi^{\dot\alpha} \\ \chi_\alpha \end{pmatrix}$$

<u>mass term:</u>

$$\bar\Psi\Psi = \left((\bar\Psi^{\dot\alpha})^*, (\chi_\alpha)^*\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\alpha\beta}\begin{pmatrix} \bar\Psi^{\dot\beta} \\ \chi_\beta \end{pmatrix}$$

$$= (\psi^\alpha, \bar\chi_{\dot\alpha}) \begin{pmatrix}\chi_\alpha \\ \bar\psi^{\dot\alpha}\end{pmatrix} = \psi^\alpha\chi_\alpha + \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha}$$

therefore

$$\bar\Psi\Psi = \psi\chi + \bar\psi\bar\chi$$

<u>Derivative term</u>:

$$i\,\bar\Psi\gamma^\mu\partial_\mu\Psi = i(\psi^\alpha, \bar\chi_{\dot\alpha}) \overbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}^{\gamma^0}\overbrace{\begin{pmatrix}0&\bar\sigma^\mu\\ \sigma^\mu&0\end{pmatrix}}^{\gamma^\mu}\begin{pmatrix}\partial_\mu\bar\psi^{\dot\beta}\\ \partial_\mu\chi_\beta\end{pmatrix}$$

where $\sigma^\mu \equiv (1,\vec\sigma)$; $\bar\sigma^\mu\equiv(1,-\vec\sigma)$

$$= i(\psi^\alpha, \bar\chi_{\dot\alpha}) \begin{pmatrix}\sigma^\mu&0\\0&\bar\sigma^\mu\end{pmatrix}\begin{pmatrix}\partial_\mu\bar\psi^{\dot\beta}\\ \partial_\mu\chi_\beta\end{pmatrix}$$

$$= i\,\psi^\alpha(\sigma^\mu)_{\alpha\dot\beta}\partial_\mu\bar\psi^{\dot\beta} + i\,\bar\chi_{\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\beta}\partial_\mu\chi_\beta$$

$$\equiv i\,\psi\sigma^\mu\partial_\mu\bar\psi + i\,\chi\bar\sigma^\mu\partial_\mu\chi$$

notice that $(\sigma^\mu)_{\alpha\dot\beta}$ and $(\bar\sigma^\mu)^{\dot\alpha\beta}$, and

contractions are

${}^{\alpha}\searrow_{\alpha}$ and $\dot\alpha\leftrightarrow{}^{\dot\alpha}$

6) Other relations and definitions

6.1 $\sigma^\mu, \bar\sigma^\nu, \bar\sigma^{\mu\nu}, \sigma^{\mu\nu}$

- $\sigma^{\mu\nu} \equiv \frac{1}{2}\left(\sigma^\mu \bar\sigma^\nu - \sigma^\nu \bar\sigma^\mu\right)$; note $(\sigma^{\mu\nu})_\alpha{}^\beta$

- $\bar\sigma^{\mu\nu} \equiv \frac{1}{2}\left(\bar\sigma^\mu \sigma^\nu - \bar\sigma^\nu \sigma^\mu\right)$; note $(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}$

- $\sigma^\mu \bar\sigma^\nu \equiv \eta^{\mu\nu} + \sigma^{\mu\nu}$

- $\bar\sigma^\mu \sigma^\nu = \eta^{\mu\nu} + \bar\sigma^{\mu\nu}$

- $\sigma_2 \sigma^\mu \sigma_2 = \bar\sigma^{\mu T}$ $\qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

- $\sigma_2 \bar\sigma^\mu \sigma_2 = \sigma^{\mu T}$

- $\sigma_2 \sigma^{\mu\nu} \sigma_2 = -\sigma^{\mu\nu T}$

- $\sigma_2 \bar\sigma^{\mu\nu} \sigma_2 = -\bar\sigma^{\mu\nu T}$

- $\bar\chi \bar\sigma^\mu \varepsilon = -\varepsilon \sigma^\mu \bar\chi$

- $\chi \sigma^{\mu\nu} \psi = -\psi \sigma^{\mu\nu} \chi$

- $\bar\chi \bar\sigma^{\mu\nu} \bar\psi = -\bar\psi \bar\sigma^{\mu\nu} \bar\chi$

6.2 Converting two component into four components

Define

$$\bar\Psi_1 \equiv \begin{pmatrix} \bar\psi_1 \\ \chi_1 \end{pmatrix} \qquad \Psi_2 = \begin{pmatrix} \bar\psi_2 \\ \chi_2 \end{pmatrix}$$

then

$$\bar\chi_1 \bar\psi_2 = \bar\Psi_1 P_R \Psi_2$$

$$\chi_2 \psi_1 = \overline{\Psi}_1 P_L \Psi_2$$

$$\overline{\chi}_1 \bar{\sigma}^\mu \chi_2 = \overline{\Psi}_1 \gamma_\mu P_L \Psi_2$$

$$\psi_1 \sigma^\mu \overline{\psi}_2 = \overline{\Psi}_1 \gamma_\mu P_R \Psi_2$$

$$\overline{\psi}_2 \bar{\sigma} \psi_1 = -\overline{\Psi}_1 \gamma_\mu P_R \Psi_2$$

6.3 Other useful relations

- $\lambda_\alpha (\chi \psi) + \chi_\alpha (\psi \lambda) + \psi_\alpha (\lambda \chi) = 0$

- $\theta^\alpha \theta^\beta = -\frac{1}{2} \theta\theta \, \varepsilon^{\alpha\beta}$; $\theta_\alpha \theta_\beta = \frac{1}{2} \theta\theta \, \epsilon_{\alpha\beta}$

- $\bar{\theta}^{\dot\alpha} \bar{\theta}^{\dot\beta} = \frac{1}{2} \bar{\theta}\bar{\theta} \, \varepsilon^{\dot\alpha\dot\beta}$; $\bar{\theta}_{\dot\alpha} \bar{\theta}_{\dot\beta} = -\frac{1}{2} \bar{\theta}\bar{\theta} \, \varepsilon_{\dot\alpha\dot\beta}$