Invariants for the Poincaré Group (TOPCB) ①

a) Poincaré Group

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(g_{\nu\rho} J_{\mu\sigma} - g_{\nu\sigma} J_{\mu\rho} - g_{\mu\rho} J_{\nu\sigma} + g_{\mu\sigma} J_{\nu\rho})$$

$$[P_\alpha, J_{\mu\nu}] = i(g_{\mu\alpha} P_\nu - g_{\nu\alpha} P_\mu)$$

$$[P_\alpha, P_\beta] = 0$$

To understand the meaning take

$$\vec{J} = (J^{23}, J^{31}, J^{12}) \quad ; \quad \vec{K} = (J^{01}, J^{02}, J^{03})$$

We then get

$$\begin{cases} [J^i, J^j] = i\varepsilon^{ijk} J^k & \leftarrow \text{Angular Momentum} \\ [K^i, K^j] = -i\varepsilon^{ijk} J^k & \\ [K^i, J^j] = i\varepsilon^{ijk} K^k & \leftarrow \text{transforms as a vector in 3 dim.} \end{cases}$$

using these vectors we can show that

$$W^0 = \vec{J} \cdot \vec{P} \quad ; \quad \vec{W} = P^0 \vec{J} + \vec{K} \times \vec{P}$$

In the rest frame $\vec{P} = (m, \vec{0})$ and $\vec{J}^2 = s(s+1)$

So

$$W^2 = (W^0)^2 - \vec{W} \cdot \vec{W} = -m^2 \vec{J}^2 = -m^2 s(s+1)$$

b) Casimir invariants of the Poincaré Group

We use $$[A^2, B] = A[A,B] + [A,B]A$$

then:

$$[P^2, P_\alpha] = P^\beta [P_\beta, P_\alpha] + [P_\beta, P_\alpha] P^\beta = 0$$

For the angular momentum:

$$[P^2, J_{\mu\nu}] = P^\alpha [P_\alpha, J_{\mu\nu}] + [P_\alpha, J_{\mu\nu}] P^\alpha$$

$$= P^\alpha i (g_{\mu\alpha} P_\nu - g_{\nu\alpha} P_\mu)$$

$$+ i (g_{\alpha\mu} P_\nu - g_{\nu\alpha} P_\mu) P^\alpha$$

$$= i (P_\mu P_\nu - P_\nu P_\mu + P_\nu P_\mu - P_\mu P_\nu)$$

$$= 0$$

For the Pauli-Lubanski:

$$[W_\mu, P_\alpha] = -\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} [J^{\nu\rho}, P_\alpha] P^\sigma$$

$$= -\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} (i P^\nu g^\rho_\alpha - i P^\rho g^\nu_\alpha) P^\sigma$$

$$= -\frac{i}{2} \varepsilon_{\mu\nu\alpha\sigma} P^\nu P^\sigma + \frac{i}{2} \varepsilon_{\mu\alpha\rho\sigma} P^\rho P^\sigma$$

$$= 0$$

therefore

$$[W^2, P_\alpha] = W^\mu [W_\mu, P_\alpha] + [W_\mu, P_\sigma] W^\mu = 0$$

for the angular momenta we have

$$[W_\mu, J_{\alpha\beta}] = -\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} [J^{\nu\rho} P^\sigma, J_{\alpha\beta}]$$

$$= -\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} J^{\nu\rho} [P^\sigma, J_{\alpha\beta}]$$

$$- \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} [J^{\nu\rho}, J_{\alpha\beta}] P^\sigma$$

$$= -\frac{i}{2} \varepsilon_{\mu\nu\rho\alpha} J^{\nu\rho} P_\beta + \frac{i}{2} \varepsilon_{\mu\nu\rho\beta} J^{\nu\rho} P_\alpha$$

$$- \frac{i}{2} \varepsilon_{\mu\nu\alpha\sigma} J^\nu{}_\beta P^\sigma + \frac{i}{2} \varepsilon_{\mu\nu\beta\sigma} J^\nu{}_\alpha P^\sigma$$

$$+ \frac{i}{2} \varepsilon_{\mu\alpha\rho\sigma} J^\rho{}_\beta P^\sigma - \frac{i}{2} \varepsilon_{\mu\beta\rho\sigma} J^\rho{}_\alpha P^\sigma$$

$$= -\frac{i}{2} \varepsilon_{\mu\nu\rho\alpha} J^{\nu\rho} P_\beta + \frac{i}{2} \varepsilon_{\mu\nu\rho\beta} J^{\nu\rho} P_\alpha$$

$$- i \varepsilon_{\mu\nu\alpha\sigma} J^\nu{}_\beta P^\sigma + i \varepsilon_{\mu\nu\beta\sigma} J^\nu{}_\alpha P^\sigma$$

$$[W_0, J_{0i}] = + \frac{i}{2} \varepsilon_{0jki} J^{jk} P^0 + i \varepsilon_{0jik} J^{j0} P^k$$

$$= i W_i$$

$$[W_0, J_{ij}] = 0$$

$$[W_i, J_{0j}] = -i\, g_{ij}\, P_0$$

$$[W_i, J_{jk}] = i\,(g_{ij} W_k - g_{ik} W_j)$$

and finally

$$[W_\mu, J_{\alpha\beta}] = i\,(g_{\mu\alpha} W_\beta - g_{\mu\beta} W_\alpha)$$

showing that W_μ is a 4-vector. Now

$$[W^2, J_{\alpha\beta}] = W^\mu [W_\mu, J_{\alpha\beta}] + [W_\mu, J_{\alpha\beta}] W^\mu$$

$$= W^\mu\, i\,(g_{\mu\alpha} W_\beta - g_{\mu\beta} W_\alpha)$$

$$+ i\,(g_{\mu\alpha} W_\beta - g_{\mu\beta} W_\alpha) W^\mu)$$

$$= i\,(W_\alpha W_\beta - W_\beta W_\alpha + W_\beta W_\alpha - W_\alpha W_\beta)$$

$$= 0$$