PART II

QUANTIZATION AND RENORMALIZATION OF GAUGE THEORIES

11. Path integral quantization

> One feels as Cavalieri must have felt calculating the volume of a pyramid before the invention of the calculus.
>
> R.P. Feynman

In this section we develop the quantization procedure based on the notion of path integration. The first hint of this procedure appeared in a paper by Dirac in 1933; the method was perfected by Feynman in 1948. We shall first consider a quantum mechanical system with one degree of freedom, and generalize to quantum field theory in the next section.

Let $|q, t\rangle_H$ be the Heisenberg picture state vector describing a state which at time t is an eigenstate of the coordinate Q_H with eigenvalue q:

$$Q_H(t)|q, t\rangle_H = q|q, t\rangle_H ,$$

$$Q_H(t) = e^{iHt} Q_S e^{-iHt}, \tag{11.1}$$

where Q_S is the time-independent position operator in the Schroedinger picture, and H in the exponent is the Hamiltonian. The state

$$|q\rangle = e^{-iHt}|q, t\rangle_H$$

is an eigenstate of Q_S with eigenvalue q:

$$Q_S|q\rangle = q|q\rangle$$

and

$$|q, t\rangle_H = e^{+iHt}|q\rangle . \tag{11.2}$$

The transformation matrix element

$$F(q', t'; q, t) = {}_H\langle q', t'|q, t\rangle_H = \langle q'|\exp\{-iH(t'-t)\}|q\rangle \tag{11.3}$$

plays a fundamental role in quantum mechanics. We are going to express $F(q', t'; q, t)$ as a path integral. We shall subdivide the time interval into $n+1$ equal segments, and define

$$t_l = l\epsilon + t, \qquad t' = (n+1)\epsilon + t. \tag{11.4}$$

We make use of the completeness of the state vectors $|q_l, t_l\rangle$ to write

$$F(q', t'; q, t) = \int dq_1(t_1) \int dq_2(t_2) \int \dots \int dq_n(t_n) \langle q', t'|q_n, t_n\rangle\langle q_n, t_n|q_{n-1}, t_{n-1}\rangle \dots \langle q_1, t_1|q, t\rangle . \tag{11.5}$$

Here and in the following, we shall drop the subscript H and understand the state $|q, t\rangle$ to mean that in the Heisenberg picture. For sufficiently large n, the time interval $t_l - t_{l-1}$ can be made as small as one likes, and we may write

$$\langle q', \epsilon | q, 0\rangle = \langle q' | e^{-i\epsilon H} | q\rangle = \delta(q - q') - i\,\epsilon\,\langle q' | H | q\rangle + O(\epsilon^2) \tag{11.6}$$

where the first equality follows from (11.3).

The Hamiltonian $H = H(P, Q)$ is a function of the operators P and Q. Consider the case when H is of the form

$$H = \tfrac{1}{2}P^2 + V(Q). \tag{11.7}$$

In this case

$$\langle q' | H(P, Q) | q\rangle = \int \frac{dp}{2\pi} \exp\{ip(q' - q)\}\,[\tfrac{1}{2}p^2 + V(q)] = \int \frac{dp}{2\pi} \exp\{ip(q' - q)\} H(p, \tfrac{1}{2}(q+q')) \tag{11.8}$$

where $H(p, q)$ is the classical Hamiltonian. We can write eq. (11.6) correct up to first order in ϵ, as

$$\langle q_l, t_l | q_{l-1}, t_{l-1}\rangle \approx \int \frac{dp}{2\pi} \exp[i\{p(q_l - q_{l-1}) - \epsilon\, H(p, \tfrac{1}{2}(q_l + q_{l-1}))\}]. \tag{11.9}$$

Substituting (11.9) into (11.5), we obtain for the amplitude to find q' at time t' from a state which was an eigenstate of the coordinate with eigenvalue q at an earlier time t,

$$F(q', t'; q, t) = \lim_{n\to\infty} \prod_{i=1}^{n} dq_i \prod_{l=1}^{n+1} \frac{dp_l}{2\pi} \exp\left\{ i \sum_{j=1}^{n+1} [p_j(q_j - q_{j-1}) - H(p_j, \tfrac{1}{2}(q_j + q_{j-1}))\,(t_j - t_{j-1})]\right\} \tag{11.10}$$

with $q_0 = q$ and $q_{n+1} = q'$.

We shall streamline our notation a little bit. We write (11.10) as

$$F(q', t'; q, t) = \int \left[\frac{dq\,dp}{2\pi\hbar}\right] \exp\left\{\frac{i}{\hbar}\int_t^{t'} (p\dot{q} - H(p, q))d\tau\right\} \tag{11.11}$$

which is a suggestive shorthand notation for the operation implied by the right-hand side of eq. (11.10). In eq. (11.11)

$$\int \left[\frac{dq\,dp}{2\pi\hbar}\right] = \int \prod_{\tau} \frac{dq(\tau)dp(\tau)}{2\pi\hbar}. \tag{11.12}$$

We have restored briefly $\hbar = 1$ to indicate that the functional integration is over all phase space volume $\int(\Delta q\,\Delta p/h)$ for all times between t and t'.

When the Hamiltonian has the form of eq. (11.7), the p-integration on the right-hand-side of eq. (11.10) can be performed explicitly by making use of the formula

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp\{i(p\dot{q} - \tfrac{1}{2}p^2)\} = [2\pi i\epsilon]^{-1/2} \exp(\tfrac{1}{2}\, i\epsilon\dot{q}^2). \tag{11.13}$$

The result is

$$F(q', t'; q, t) = \lim_{n\to\infty} \int \prod_{i=1}^{n} \frac{\mathrm{d}q_i}{[2\pi i\epsilon]^{1/2}} \exp\left\{ i \sum_{j=1}^{n+1} \epsilon \left[\tfrac{1}{2}\left(\frac{q_j - q_{j-1}}{\epsilon}\right)^2 - V\left(\frac{q_j + q_{j+1}}{2}\right)\right]\right\}$$

$$= \int \left[\frac{\mathrm{d}q}{[2\pi i\epsilon]^{1/2}}\right] \cdot \exp\left\{ i \int_t^{t'} L(q, \dot{q})\mathrm{d}t\right\} \tag{11.14}$$

where L is the Lagrangian,

$$L = \tfrac{1}{2}\dot{q}^2 - V(q) \tag{11.15}$$

and $q_0 = q(t)$ and $q_{n+1} = q'(t'_{n+1})$.

The quantity

$$S = \int L(q, \dot{q})\mathrm{d}t \tag{11.16}$$

is the action which generates the temporal development of the quantum mechanical system described by the Lagrangian (11.15).

We derived eq. (11.14) from the usual formalism of quantum mechanics. Alternatively, one can start from eq. (11.14) and derive the Schroedinger equation. All this and many other related matters were discussed in Feynman's original paper. In a few simple cases, the functional integrations in eq. (11.14) can be carried out explicitly.

When the Hamiltonia is not in the form of eq. (11.7), we must be careful about specifying the ordering of the operators P and Q. We shall *assume* that there is a way of ordering the operators in the quantum mechanical Hamiltonian $H(P, Q)$ so that the transformation matrix $F(q', t'; q, t)$ is correctly given by eq. (11.10) for this Hamiltonian, with the understanding that whenever there is an ambiguity, the integrals over p_i are to be performed before the q-integrations When the Hamiltonian is not of the form of eq. (11.7), we must use eq. (11.10) to find the "effective action", S_{eff}, i.e., the quantity which, after the p_i-integrations are performed, replaces the action in eq. (11.14). In general, S_{eff} is not given by (11.16).

As an illustration of this prescription, we apply it to the non-linear Lagrangian

$$L = \tfrac{1}{2}\dot{q}^2 f(q) \tag{11.17}$$

where $f(q)$ is a non-singular function of q. Eq. (11.17) describes a particular class of systems with velocity-dependent potentials. The momentum p canonically conjugate to q is

$$p = \partial L/\partial\dot{q} = \dot{q}f(q)$$

and the Hamiltonian is

$$H(p, q) = p\dot{q} - L = \tfrac{1}{2}p^2[f(q)]^{-1}.$$

Now, from eq. (11.10), the transformation matrix element is

$$F(q', t'; q, t) = \int \prod_{i=1}^{n} \mathrm{d}q_i \prod_{i=2}^{n} \frac{\mathrm{d}p_i}{2\pi}\, \delta(q_1 - q)\delta(q_n - q')\exp\left[i \sum_{i=2}^{n} \left\{ p_i(q_i - q_{i-1}) - \epsilon p_i^2 \left[f\left(\frac{q_i + q_{i-1}}{2}\right)\right]^{-1}\right\}\right]. \tag{11.18}$$

The p-integrations can be performed as before, and we obtain

$$F(q', t'; q, t) = \prod_{i=1}^{n} \frac{\mathrm{d}q_i}{[2\pi \mathrm{i}\epsilon]^{1/2}} \delta(q_1-q)\delta(q_n-q')$$

$$\times \left[\exp\left\{\mathrm{i} \sum_{i=2}^{n} \frac{\epsilon}{2}\left(\frac{q_i-q_{i-1}}{\epsilon}\right)^2 f\left(\frac{q_i+q_{i-1}}{2}\right)\right\}\right] \prod_{i=2}^{n} \left[f\left(\frac{q_i+q_{i-1}}{2}\right)\right]^{1/2} \Bigg] . \qquad (11.19)$$

The last factor can be written as

$$\prod_i \left[f\left(\frac{q_i-q_{i-1}}{2}\right)\right]^{1/2} = \exp\left\{\tfrac{1}{2}\sum_i \ln f\left(\frac{q_i+q_{i-1}}{2}\right)\right\}$$

$$= \exp\left\{\frac{1}{2\epsilon} \sum_i \epsilon \log f\left(\frac{q_i+q_{i-1}}{2}\right)\right\} \to \exp \tfrac{1}{2}\, \delta(0) \int \mathrm{d}t \ln f(q) \qquad (11.20)$$

where we have used the limits

$$\sum_i \epsilon \to \int \mathrm{d}t, \qquad \frac{1}{\epsilon}\delta_{ij} \to \delta(t_i - t_j). \qquad (11.21)$$

Finally, therefore, we can write eq. (11.16) as

$$F(q', t'; q, t) = \lim_{n\to\infty} \int \prod_{i=1}^{n} \frac{\mathrm{d}q_i}{[2\pi \mathrm{i}\epsilon]^{1/2}} \delta(q_1-q)\delta(q_n-q')$$

$$\times \exp\left[\mathrm{i} \sum_{i=1}^{n} \epsilon \left\{\tfrac{1}{2}\left(\frac{q_i-q_{i-1}}{\epsilon}\right)^2 f\left(\frac{q_i+q_{i-1}}{2}\right) - \frac{\mathrm{i}}{2\epsilon} \ln f\left(\frac{q_i+q_{i-1}}{2}\right)\right\}\right]$$

$$= \int \left[\frac{\mathrm{d}q}{[2\pi \mathrm{i}\epsilon]^{1/2}}\right] \exp(\mathrm{i}S_{\mathrm{eff}}) \qquad (11.22)$$

where

$$S_{\mathrm{eff}} = \int \mathrm{d}t [L(q, \dot{q}) - \mathrm{i}/2\delta(0) \ln f(q)] = \int \mathrm{d}t\, L_{\mathrm{eff}}(q, \dot{q}). \qquad (11.23)$$

This result was first obtained by Lee and Yang.

If S_{eff} is used to calculate transformation function $F(q', t'; q, t)$ or the scattering matrix for a particle with this Lagrangian L, an infinite term will appear to cancel the explicit term we have symbolically written $\delta(0)$. To do the calculation, one may go back to the explicit form in (11.22) before the limit $n \to \infty$ is taken, do the q_i integrations, then take the limit $n \to \infty$.

The advantage, or even the rationale, of following the prescription which led to eq. (11.23) is that the result written in the form

$$F(QT; qt) = \int \left[\frac{f^{1/2}(q)\mathrm{d}q}{\sqrt{2\pi \mathrm{i}\epsilon}}\right] \exp\{\mathrm{i}S(q, \dot{q})\}$$

is manifestly invariant under point transformations of the coordinate. In general, *writing the transformation function as a path-integral enables us to express quantum-mechanical quantities in terms of the classical Lagrangian, so that we can study the effects on quantum-mechanical quantities of various symmetries present in the classical Lagrangian.*

We develop a few properties of path integrals which will be useful in a generalization of the method to quantum field theory.

First of all, the generalization of eq. (11.11) to systems with more than one degree of freedom is straightforward. If there are N degrees of freedom, eq. (11.11) becomes

$$\langle q_1', q_2', \ldots q_N', t' | q_1, q_2, \ldots q_n, t\rangle = \int \prod_{n=1}^{N} \left[\frac{\mathrm{d}q_n\,\mathrm{d}p_n}{2\pi\hbar}\right] \exp\left\{\frac{\mathrm{i}}{\hbar}\int_t^{t'} \left[\sum_{n=1}^{N} p_n \dot{q}_n - H(p_i, q_i)\right] \mathrm{d}\tau\right\} \quad (11.24)$$

with $q_n(t) = q_n$, $q_n(t') = q_n'$.

For the rest of this section we restrict ourselves to $N = 1$; we shall use eq. (11.24) in the development of field theory.

Next, instead of the simple transformation function $\langle q', t' | q, t\rangle$, let us consider the matrix element of the co-ordinate operator Q evaluated at time t_0, between $\langle q', t'|$ and $|q, t\rangle$. We restrict t_0 to lie in the interval

$$t' > t_0 > t.$$

Now let us write $\langle q', t' | Q(t_0) | q, t\rangle$ as in eq. (11.5), selecting t_0 to be one of the t_i, say t_{i_0}. Thus

$$\langle q', t' | Q(t_0) | q, t\rangle = \int \prod_i \mathrm{d}q_i \langle q', t' | q_n, t_n\rangle\langle q_n, t_n | q_{n-1}, t_{n-1}\rangle \ldots$$

$$\langle q_{i_0+1}, t_{i_0+1} | q_{i_0}, t_{i_0}\rangle\langle q_{i_0}, t_{i_0} | Q(t_0) | q_{i_0-1}, t_{i_0-1}\rangle \ldots \langle q_1, t_1 | q, t\rangle.$$

In eq. (11.24), we have placed the operator $Q(t_0)$ next to one of its eigenstates, so $\langle q_{i_0}, t_{i_0} | Q(t_0) | q_{i_0-1}, t_{i_0-1}\rangle$ becomes $q_{i_0}\langle q_{i_0}, t_{i_0} | q_{i_0-1}, t_{i_0-1}\rangle$. The argument leading to eq. (11.10) can now proceed; nothing is changed except that an extra factor of q_{i_0} will appear under the integral on the right-hand side. Instead of eq. (11.11) we now obtain

$$\langle q', t' | Q(t_0) | q, t\rangle = \int \left[\frac{\mathrm{d}q\,\mathrm{d}p}{2\pi}\right] q(t_0) \exp\left\{\mathrm{i}\int_t^{t'} [p\dot{q} - H(p, q)]\,\mathrm{d}\tau\right\} . \quad (11.25)$$

Next, suppose we want to express

$$\langle q', t' | Q(t_1) Q(t_2) | q, t\rangle$$

as a path integral. We proceed as above, choosing t_1 and t_2 to be two of the times which bound the small intervals into which the interval $t' - t$ is broken. If $t_1 > t_2$, we can write

$$\langle q', t' | Q(t_1) Q(t_2) | q, t\rangle = \int \pi \mathrm{d}q_i \langle q', t' | q_n, t_n\rangle\langle q_n, t_n | q_{n-1}, t_{n-1}\rangle \ldots$$

$$\langle q_{i_1}, t_{i_1} | Q(t_1) | q_{i_1-1}, t_{i_1-1}\rangle \ldots \langle q_{i_2}, t_{i_2} | Q(t_2) | q_{i_2-1}, t_{i_2-1}\rangle \ldots \langle q_1, t_1 | q, t\rangle. \quad (11.26)$$

After going through a series of steps analogous to those which led to eq. (11.25), we obtain

$$\langle q', t'|Q(t_1)Q(t_2)|q, t\rangle = \int \left[\frac{\mathrm{d}q\,\mathrm{d}p}{2\pi}\right] q(t_1)q(t_2)\exp\left\{\mathrm{i}\int_t^{t'} [p\dot{q} - H(p, q)]\,\mathrm{d}\tau\right\}. \tag{11.27}$$

Eq. (11.27) holds only if $t_1 > t_2$. If $t_2 > t_1$, we could not have derived eqs. (11.26) and (11.27) the way we did. In fact, it is easy to see that if $t_2 > t_1$, the right-hand side of eq. (11.27) is equal to

$$\langle q', t'|Q(t_2)Q(t_1)|q, t\rangle.$$

Therefore the path integral in eq. (11.27) is the matrix element of the time-ordered product

$$T[Q(t_1)Q(t_2)].$$

The result generalizes immediately to the product of any number of Q's

$$\langle q', t'|T[Q(t_1)Q(t_2)...Q(t_N)]|q, t\rangle = \int \left[\frac{\mathrm{d}p\,\mathrm{d}q}{2\pi}\right] q(t_1)q(t_2)...q(t_N)\exp\left\{\mathrm{i}\int_t^{t'} [p\dot{q} - H]\,\mathrm{d}\tau\right\}. \tag{11.28}$$

Next we want to demonstrate a crucial theorem. Let L be a Lagrangian which does not depend explicitly on time, and let $\phi_n(q) = \langle q|n\rangle$ be the wave function of the energy eigenstate $|n\rangle$. In particular, let $\phi_0(q)$ be the ground state. If the system is in the ground state at a time T in the distant past, we want to calculate the amplitude for it to be in that state at a time T' in the distant future, when an arbitrary external source term $J(t)q(t)$ is added to L between T and T'.

To do this, consider

$$\langle Q', T'|Q, T\rangle^J = \int \left[\frac{\mathrm{d}p\,\mathrm{d}q}{2\pi}\right] \exp\left\{\mathrm{i}\int_T^{T'} [p\dot{q} - H(p, q) + Jq]\,\mathrm{d}\tau\right\} \tag{11.29}$$

where J is an arbitrary function of t, except that we restrict it to be non-vanishing only between t and t', where $T' > t' > t > T$. We can write eq. (11.29) as

$$\langle Q', T'|Q, T\rangle^J = \int \mathrm{d}q' \int \mathrm{d}q\, \langle Q', T'|q', t'\rangle\langle q', t'|q, t\rangle^J \langle q, t|Q, T\rangle. \tag{11.30}$$

Now $\langle q, t|Q, T\rangle$ and $\langle Q', T'|q', t'\rangle$ are given by formulae like (11.29) without the $J(\tau)q(\tau)$ term.

Let us insert a complete set of energy eigenstates in $\langle q, t|Q, T\rangle$:

$$\langle q, t|Q, T\rangle = \langle q|\exp\{-\mathrm{i}H(t - T)\}|Q\rangle = \sum_n \phi_n(q)\phi_n^*(Q)\exp\{-\mathrm{i}E_n(t - T)\}. \tag{11.31}$$

The T-dependence in (11.31) is known explicitly because we have required $J(\tau) = 0$ between T and t. Therefore, we can continue T along the positive imaginary axis. In that limit, all the terms with $n > 0$ drop out, as $T \to \mathrm{i}\infty$, and

$$\lim_{T\to \mathrm{i}\infty} \exp(-\mathrm{i}E_0T)\langle q, t|Q, T\rangle = \phi_0(q, t)\phi_0^*(Q),$$

$$\phi_0(q, t) = \phi_0(q)\exp(-\mathrm{i}E_0 t). \tag{11.32}$$

We can do the same analysis for $\langle Q'T'|q't'\rangle$. Therefore, provided Q and Q' approach some constants in the limit, we have

$$\lim_{\substack{T'\to -\mathrm{i}\infty \\ T\to \mathrm{i}\infty}} \frac{\langle Q',T'|Q,T\rangle^J}{\exp\{-\mathrm{i}E_0(T'-T)\}\phi_0^*(Q)\phi_0(Q')} = \int \mathrm{d}q \int \mathrm{d}q'\, \phi_0^*(q',t')\langle q',t'|q,t\rangle^J \phi_0(q,t) \tag{11.33}$$

which is the theorem we set out to prove. The right-hand side of (11.33) is just the ground state to ground state amplitude of interest, since t' and $-t$ can be taken as large as one pleases. Let us denote it, symbolically, as $W[J]$. Then eq. (11.33) tells us how to calculate $W[J]$.

Why is $W[J]$ of interest? In (11.33), $\langle q',t'|q,t\rangle^J$ is given by a form like eq. (11.29), with t and t' replacing T and T'. The effect of varying W with respect to $J(t_0)$ is to bring down a factor $\mathrm{i}\phi(t_0)$ in front of the exponential. Let us do this n times, and then set $J=0$,

$$\lim_{J\to 0} \frac{\delta^n W[J]}{\delta J(t_1)\delta J(t_2)...\delta J(t_n)} = \mathrm{i}^n \int \mathrm{d}q \int \mathrm{d}q'\, \phi_0^*(q',t')\phi_0(q,t) \int \left[\frac{\mathrm{d}p\mathrm{d}q}{2\pi}\right] \exp\left\{\mathrm{i}\int_t^{t'} [p\dot{q} - H(p,q)]\,\mathrm{d}\tau\right\}$$

$$\times\; q(t_1)q(t_2)...q(t_n), \qquad t' > t_1, t_2..., t_n > t. \tag{11.34}$$

Comparing with eq. (11.28), we see that this expression is just the matrix element of the time ordered product $T(Q(t_1)Q(t_2)...Q(t_n))$ between the ground state at t and the ground state at t'. Therefore the expression (11.34) is the ground state expectation value of a time-ordered product of co-ordinates. In field theory, these will become the Green's functions.

We shall indicate how $W[J]$ can be evaluated from eq. (11.33). To within a multiplicative factor independent of J

$$W[J] \sim \lim_{\substack{T'\to -\mathrm{i}\infty \\ T\to \mathrm{i}\infty}} \langle Q',T'|Q,T\rangle^J$$

or

$$W[J] \sim \lim_{\substack{T'\to -\mathrm{i}\infty \\ T\to \mathrm{i}\infty}} \int [\mathrm{d}q] \exp\left\{\mathrm{i}\int_T^{T'} \mathrm{d}t\, [L_{\mathrm{eff}}(q,\dot{q}) + J(t)q(t)]\right\}. \tag{11.35}$$

In field theoretic applications, the multiplicative factors independent of J never matters, and we are allowed to be cavalier about it.

From eq. (11.34) and the remark following it, we have

$$\langle T(Q(t_1)...Q(t_n))\rangle_0 \sim \lim_{\substack{T'\to -\mathrm{i}\infty \\ T\to \mathrm{i}\infty}} \int \mathrm{d}q_1...\mathrm{d}q_n \langle Q',T'|q_1,t_1\rangle q_1 \langle q_1,t_1|q_2,t_2\rangle q_2 ... q_n \langle q_n,t_n|Q,T\rangle,$$

where $t_1 > t_2 ... > t_n$, and $\langle\,\rangle_0$ denotes the ground state expectation value. Let us consider continuing $\langle T(Q(t_1)...Q(t_n))\rangle$ in t_i analytically, from real to imaginary values $t_i = -\mathrm{i}\tau_i$. Since

$$\langle q,t|q',t'\rangle = \lim_{n\to\infty} \int \prod_i \frac{\mathrm{d}q_i}{\sqrt{2\pi\mathrm{i}\epsilon}} \exp\left\{\mathrm{i}\sum_i \epsilon\, L_{\mathrm{eff}}\left(\frac{q_i+q_{i-1}}{2}, \frac{q_i-q_{i-1}}{\epsilon}\right)\right\}$$

depends on $t - t'$ only through ϵ:

$$\epsilon = (t - t')/(n + 1),$$

the analytic continuation is effected by writing

$$\langle q, t|q', t'\rangle\Big|_{\substack{t = -\mathrm{i}\tau \\ t' = -\mathrm{i}\tau'}} = \lim_{n\to\infty} \int \prod_i \frac{\mathrm{d}q_i}{\sqrt{2\pi\epsilon'}} \exp\left\{\sum_i \frac{\mathrm{d}q_i}{\sqrt{2\pi\epsilon'}}\right\} \exp\left\{\sum \epsilon' \, L_{\mathrm{eff}}\left(\frac{q_i + q_{i-1}}{2}, \frac{q_i - q_{i-1}}{-\mathrm{i}\epsilon'}\right)\right\}$$

where

$$\epsilon' = (\tau - \tau')/(n + 1).$$

Thus the analytic continuation of $\langle T(Q(t_1)...Q(t_n))\rangle_0$ may be written as

$$\langle T(Q(t_1)...Q(t_n))\rangle_0 \,|_{t_i = -\mathrm{i}\tau_i} \sim \lim_{\substack{\tau_f\to\infty \\ \tau_i\to-\infty}} \int [\mathrm{d}q]\, q(\tau_1)q(\tau_2)...q(\tau_n) \exp\left\{\int_{\tau_i}^{\tau_f} L_{\mathrm{eff}}\left(q, \mathrm{i}\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)\right\} .$$

This suggests going over to an imaginary time, or Euclidean, formulation and defining

$$W_{\mathrm{E}}[J] = \int [\mathrm{d}q] \, \exp\left\{ \int_{-\infty}^{\infty} \mathrm{d}\tau \left[L_{\mathrm{eff}}\left(q, \mathrm{i}\frac{\mathrm{d}q}{\mathrm{d}\tau}\right) + J(\tau)q(\tau)\right]\right\} . \tag{11.36}$$

The boundary condition to be imposed on (11.36) is that q approaches some constants as $\tau \to \pm\infty$. It is convenient, but not necessary, to take these constants to be zero. The connection between $W[J]$ and $W_{\mathrm{E}}[J]$ is that

$$\frac{1}{W[J]} \frac{\delta^n W[J]}{\delta J(t_1)...\delta J(t_n)}\bigg|_{J=0} = (\mathrm{i})^n \frac{1}{W_{\mathrm{E}}[J]} \frac{\delta^n W_{\mathrm{E}}[J]}{\delta J(\tau_1)...\delta J(\tau_n)}\bigg|_{J=0,\, \tau_i = \mathrm{i}t_i} \tag{11.37}$$

where analytic continuation is implied on the right-hand side. Equation (11.37) is manifestly independent of the overall normalizations of $W[J]$ and $W_{\mathrm{E}}[J]$ which are independent of J.

Finally, in order to illustrate the formal discussion, and especially the Euclidean formulation, we discuss a simple example. Consider a simple harmonic oscillator in one dimension, whose Lagrangian is

$$L(q, \dot{q}) = \tfrac{1}{2}(\dot{q}^2 - \omega^2 q). \tag{11.38}$$

The transformation matrix in the presence of external source J can be computed from eq. (11.29):

$$\langle q', t'|q, t\rangle^J = \lim_{n\to\infty} \int \prod_{i=1}^{n} \frac{\mathrm{d}q(t_i)}{\sqrt{2\pi\mathrm{i}\epsilon}} \exp\left\{\mathrm{i}\int_t^{t'} \mathrm{d}\tau [L(q(\tau), \dot{q}(\tau)) + J(\tau)q(\tau)]\right\} \tag{11.39}$$

with the boundary condition $q(t') = q'$, $q(t) = q$. The integral can be worked out explicitly. The calculation is posed as a problem, with enough hints, in Feynman and Hibbs, "Quantum Mechanics and Path Integrals", p. 64. The answer is

$$\langle q', t'|q, t\rangle^J = [\omega/2\pi \mathrm{i} \sin \omega T]^{1/2} \exp\{\mathrm{i}Q(q', t', q, t)\} \tag{11.40}$$

where

$$T = t' - t$$

and

$$\begin{aligned} Q(q', t', q, t) = &\frac{\omega}{2 \sin \omega T} [(q^2 + q'^2) \cos \omega T - 2qq'] \\ &+ \frac{q'}{\sin \omega T} \int_t^{t'} J(\tau) \sin \omega(\tau - t)\mathrm{d}\tau + \frac{q}{\sin \omega T} \int_t^{t'} J(\tau) \sin \omega(t' - \tau)\mathrm{d}\tau \\ &- \frac{1}{\omega \sin \omega T} \int_t^{t'} \mathrm{d}\sigma \int_t^{\sigma} J(\sigma)J(\tau) \sin \omega(t' - \tau) \sin \omega(\sigma - t)\mathrm{d}\tau. \end{aligned} \tag{11.41}$$

We leave the derivation of eq. (11.41) as an exercise.

The quantity $W[J]$ defined in the remark following (11.33) is

$$W[J] = \langle 0, t'|0, t\rangle^J = \int \mathrm{d}q' \, \phi_0^*(q', t') \int \mathrm{d}q \, \phi_0(q, t) \langle q', t'|q, t\rangle^J \tag{11.42}$$

where "0" in (11.42) means the ground state, not the state with eigenvalue 0 for the coordinate; ϕ_0 is the ground state wave function of the simple harmonic oscillator:

$$\phi_0(q, \tau) = (\omega/\pi)^{1/4} \exp(-\tfrac{1}{2}\omega q^2) \exp(-\mathrm{i}\tfrac{1}{2}\omega\tau) \tag{11.43}$$

so that the integrals over q and q' are just Gaussian integrals. The result is

$$\langle 0, t'|0, t\rangle^J = \exp\left\{ \mathrm{i} \int_t^{t'} \mathrm{d}\sigma \int_t^{\sigma} \mathrm{d}\tau J(\sigma) \left[\frac{\mathrm{i}}{2\omega} \exp\{-\mathrm{i}\omega(\sigma - \tau)\}\right] J(\tau) \right\}. \tag{11.44}$$

We will make the result more general by extending the limits on the integrals from $-\infty$ to $+\infty$. Thus, if we are interested in the effect on the oscillator of the force term for just the period t to t', we may restrict $J(\tau)$ to vanish outside of this interval. Finally, we shall write eq. (11.44) as [see, R.P. Feynman, Phys. Rev. 80 (1950) 440]

$$W[J] = \exp\left\{ \frac{-\mathrm{i}}{2} \int_{-\infty}^{\infty} \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\sigma \, J(\tau) D_+(\tau - \sigma) J(\sigma) \right\} \tag{11.45}$$

where

$$D_+(t) = \frac{1}{2\mathrm{i}\omega} [\theta(t)\mathrm{e}^{-\mathrm{i}\omega t} + \theta(-t)\mathrm{e}^{\mathrm{i}\omega t}]. \tag{11.46}$$

Notice that

$$D_+(t-t') = \frac{1}{W[J]}\left.\frac{\delta^2 W[J]}{\delta J(t)\delta J(t')}\right|_{J=0} = \frac{\mathrm{i}\delta^2}{\delta J(t)\delta J(t')} \ln W[J]\bigg|_{J=0} . \tag{11.47}$$

In the Euclidean formulation (11.36), we have

$$W_{\mathrm{E}}[J] = \int [\mathrm{d}q] \exp\{-S_{\mathrm{E}}[J]\} \tag{11.48}$$

where

$$S_{\mathrm{E}}[J] = \int_{-\infty}^{\infty} \mathrm{d}\tau\, L_{\mathrm{E}}(\tau),$$

$$L_{\mathrm{E}} = \tfrac{1}{2}(\mathrm{d}q/\mathrm{d}\tau)^2 + \tfrac{1}{2}\omega^2 q^2 - J(\tau)q(\tau). \tag{11.49}$$

We expand $q(\tau)$ around $q(\tau_0)$, writing $q(\tau) = q_0(\tau) + y(\tau)$, and then expand S_{E} in powers of y,

$$S_{\mathrm{E}}(q) = S_{\mathrm{E}}(q_0) + \int_{-\infty}^{\infty} \left\{\frac{\delta L_{\mathrm{E}}}{\delta \dot{q}}\frac{\mathrm{d}y}{\mathrm{d}\tau} + \frac{\delta L_{\mathrm{E}}}{\delta q} y\right\}_{q=q_0} \mathrm{d}\tau + \int_{-\infty}^{\infty} \left\{\tfrac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \frac{\omega^2}{2} y^2\right\} \mathrm{d}\tau. \tag{11.50}$$

We wish to choose $q_0(\tau)$ so that the term in (11.50) linear in y vanishes. If the boundary condition is taken to be $q(\tau = \pm\infty) = 0$, we require $q_0(\tau = \pm\infty) = 0$ also. Then the surface terms vanish when the term $(\delta L_{\mathrm{E}}/\delta q)(\mathrm{d}y/\mathrm{d}\tau)$ is integrated by parts, and we require that $q_0(\tau)$ satisfy the classical equations of motion:

$$\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\delta L_{\mathrm{E}}}{\delta \dot{q}(\tau)} - \frac{\delta L_{\mathrm{E}}}{\delta q(\tau)}\right\}_{q=q_0(\tau)} = 0. \tag{11.51}$$

[If q is allowed to approach non-zero constants $q_\pm$ in the limits $\tau \to \pm\infty$, we may require $q_0 \to 0$ and $y \to q_\pm$. Then there will be surface terms equal to $\dot{q}_0(\pm\infty)q_\pm$ in (11.50). However, from the general solution below it is evident that $\dot{q}_0(\pm\infty) = 0$ if $q_0(\pm\infty)$ vanishes.]

Now we insert eq. (11.50) into eq. (11.48) and perform the integral over paths. The term linear in y has disappeared, so we write

$$W_{\mathrm{E}}[J] \sim \exp\{-S_{\mathrm{E}}(q_0)\} \int \prod_i \mathrm{d}y(\tau_i) \exp\left\{-\int_{-\infty}^{\infty}\left[\frac{\dot{y}^2}{2} + \frac{\omega^2 y^2}{2}\right]\right\} \mathrm{d}\tau.$$

The integration over $y(\tau_i)$ is just a number, independent of J, so we are left with

$$W_{\mathrm{E}}[J] \sim \exp\{-S_{\mathrm{E}}(q_0)\}. \tag{11.52}$$

Let us evaluate $q_0(J)$. From eq. (11.51)

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} - \omega^2\right] q_0(\tau) = -J(\tau). \tag{11.53}$$

Define a Euclidean Green's function $D_E(\tau)$ by

$$\left[\frac{d^2}{d\tau^2} - \omega^2\right] D_E(\tau) = \delta(\tau) \tag{11.54}$$

with the boundary condition $\lim_{\tau\to\pm\infty} D_E(\tau) = 0$. The solution is

$$D_E(\tau) = -\int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \frac{e^{-i\nu\tau}}{\nu^2 + \omega^2} = -\frac{e^{-\omega|\tau|}}{2\omega} \tag{11.55}$$

and therefore

$$q_0(\tau) = -\int_{-\infty}^{\infty} D_E(\tau - \sigma)\, J(\sigma)\, d\sigma. \tag{11.56}$$

[With other boundary conditions, the most general solution is (11.56) plus the general solution to the homogeneous equation, namely, $Ae^{-\omega t} + Be^{\omega t}$. If q_0 approaches a constant at both $+\infty$ and $-\infty$, $A = B = 0$, and it follows from (11.56) that $\dot{q}_0(\pm\infty) = 0$ also.]

Now in the definition (11.49) of S_E, we substitute $\omega^2 q_0(\tau)$ from eq. (11.53), integrate by parts, to obtain, using eq. (11.56)

$$S_E(q_0) = -\tfrac{1}{2}\int_{-\infty}^{\infty} J(\tau) q_0(\tau) d\tau = \tfrac{1}{2}\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\sigma\, J(\tau) D_E(\tau - \sigma) J(\sigma) \tag{11.57}$$

so that, from (11.52)

$$W_E[J] \sim \exp\left\{-\tfrac{1}{2}\int_{-\infty}^{\infty} d\tau d\sigma\, J(\tau) D_E(\tau - \sigma) J(\sigma)\right\} \tag{11.58}$$

or

$$D_E(\tau - \tau') = \frac{1}{W_E[J]} \left.\frac{-\delta^2 W_E[J]}{\delta J(\tau)\delta J(\tau')}\right|_{J=0}.$$

We can get the propagator $D_+(t)$ by analytic continuation in τ, by rotating counter-clockwise from real τ to imaginary τ:

$$D_+(t) = i\, D_E(it) \tag{11.59}$$

which yields eq. (11.46) immediately.

Note that the functional integral in (11.36) is a well behaved Gaussian (or more precisely, Wiener-Hopf) integral. Our notation may be simplified by writing the real time, ground-state to ground-state amplitude (11.49) as

$$W[J] \sim \int [dq(t)] \exp\left\{ i \int_{-\infty}^{\infty} \left[\tfrac{1}{2}\left(\frac{dq(t)}{dt}\right)^2 - \tfrac{1}{2}(\omega^2 - i\epsilon) q^2 + J(t) q(t) dt\right]\right\}. \tag{11.60}$$

Then we can repeat the imaginary time analysis using (11.60) directly, to obtain eqs. (11.45) and (11.46) the iϵ in (11.60) serving to select the correct boundary condition on the propagator:

$$D_+(t) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \frac{\exp(-i\nu t)}{\nu^2 - \omega^2 + i\epsilon}.$$

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This section incorporates several useful remarks of S. Coleman, D. Gross and S.B. Treiman.

12. Path integral formulation of field theory

Physics – Where the Action Is.

Anonymous

We have remarked that the generalization of the considerations in section 11 to many degrees of freedom is immediate. The transformation function is given by (11.24), which is a shorthand for

$$\lim_{n\to\infty} \prod_{\alpha=1}^{N} \prod_{i=1}^{n} dq_\alpha(t_i) \prod_{i=1}^{n+1} \frac{dp_\alpha(t_i)}{2\pi}$$

$$\times \exp\left[i \sum_{j=1}^{n} \left\{ \sum_{\alpha=1}^{N} p_\alpha(t_j)[q_\alpha(t_j) - q_\alpha(t_{j-1})] - \epsilon H\left(p(t_j), \frac{q(t_j)+q(t_{j-1})}{2}\right)\right\}\right] \qquad (12.1)$$

Eq. (12.1) can be applied to field theory. Consider a neutral scalar field $\phi(x)$. Let us subdivide space into cubes of dimension ϵ^3 and label them by an integer α. We define the αth coordinate $q_\alpha(t) = \phi_\alpha(t)$ by

$$\phi_\alpha(t) \equiv \frac{1}{\epsilon^3} \int_{V_\alpha} \mathrm{d}^3x \phi(x, t),$$

where the integration is over the αth cell of dimension ϵ^3. We can also rewrite the Lagrangian as

$$L = \int \mathrm{d}^3x \mathcal{L} \to \sum_\alpha \epsilon^3 \mathcal{L}_\alpha(\dot{\phi}_\alpha(t), \phi_\alpha(t), \phi_{\alpha \pm s}(t))$$

where $\dot{\phi}_\alpha(t)$ is the average of $\partial\phi(x, t)/\partial t$ over the αth cell and $\phi_{\alpha\pm s}$ is the average value of the field in the neighboring cell $\alpha \pm s$. The canonical momenta p_α conjugate to ϕ_α are

$$p_\alpha(t) = \frac{\partial L}{\partial \dot{\phi}_\alpha(t)} = \epsilon^3 \frac{\partial \mathcal{L}_\alpha}{\partial \dot{\phi}_\alpha(t)} \equiv \epsilon^3 \pi_\alpha(t).$$

The Hamiltonian is

$$H = \sum_\alpha p_\alpha \dot{\phi}_\alpha - L = \sum_\alpha \epsilon^3 \mathcal{H}_\alpha,$$

$$\mathcal{H}_\alpha = \pi_\alpha \dot{\phi}_\alpha - \mathcal{L}_\alpha = \mathcal{H}_\alpha(\pi_\alpha, \phi_\alpha, \phi_{\alpha\pm s}).$$

We may now write the expression (12.1) as

$$\lim_{\substack{n\to\infty \\ \epsilon\to 0}} \int \prod_\alpha \prod_{i=1}^{n} \mathrm{d}\phi_\alpha(t_i) \prod_{i=1}^{n+1} \frac{\epsilon^3}{2\pi} \mathrm{d}\pi_\alpha(t_i)$$

$$\times \exp\left[\mathrm{i} \sum_{j=1}^{n+1} \epsilon \sum_\alpha \epsilon^3 \left\{\pi_\alpha(t_j) \frac{\phi_\alpha(t_j) - \phi_\alpha(t_{j-1})}{\epsilon} - \mathcal{H}_\alpha\left(\pi_\alpha(t_j), \frac{\phi_\alpha(t_j)+\phi_\alpha(t_{j-1})}{2}, \frac{\phi_{\alpha\pm s}(t_j)+\phi_{\alpha\pm s}(t_{j-1})}{2}\right)\right\}\right]$$

$$\equiv \int [\mathrm{d}\phi] \left[\frac{\epsilon^3}{2\pi}\mathrm{d}\pi\right] \exp\left\{\mathrm{i}\int_t^{t'} \mathrm{d}\tau \int \mathrm{d}^3x \left[\pi(x, \tau) \frac{\partial\phi(x, t)}{\partial\tau} - \mathcal{H}(x, \tau)\right]\right\} \tag{12.2}$$

where we defined the momentum density conjugate to $\phi(x, t)$ by

$$\pi(x, t) = \partial\mathcal{L}/\partial\dot{\phi}(x, t).$$

Its cell average is just the $\pi_\alpha(t)$ defined above.

In field theory, all physical quantities are derivable from the vacuum-to-vacuum transition amplitude in the presence of external sources. The physical vacuum is the ground state, and plays the same role as the state whose wavefunction is $\phi_0(q)$ in eq. (11.33).

This amplitude, which we shall call $W[J]$, can be calculated from eq. (12.2) with a term $\int \mathrm{d}^3x J(x, t)\phi(x, t)$ added to the Lagrangian, in the limit $t' \to \infty$, $t \to -\infty$. That is

$$W[J] = \int [\mathrm{d}\phi]\left[\frac{\epsilon^3}{\pi}\mathrm{d}\pi\right] \exp\left[\,\mathrm{i}\int_{\infty}^{\infty} \mathrm{d}^4x\,\{\pi(x)\dot{\phi}(x) - \mathcal{H}(x) + \tfrac{1}{2}\mathrm{i}\epsilon\phi^2 + J(x)\phi(x)\}\right] . \qquad (12.3)$$

The extra term $\frac{1}{2}\mathrm{i}\epsilon\,\phi^2$ is simply a symbolic way of indicating how to rotate the time-integration contour to pick out the correct limit as indicated on the left-hand side of eq. (11.33). More on this later.

Now it follows from eq. (11.34) and the discussion following it that

$$\left.\frac{\delta^n W[J]}{\delta J(x_1)\delta J(x_2)\ldots\delta J(x_n)}\right|_{J=0} = \mathrm{i}^n \langle |T(\phi(x_1)\phi(x_2)\ldots.\phi(x_n))|\rangle = \mathrm{i}^n G(x_1\ldots x_n) \qquad (12.4)$$

where G is the n-point Green's function, the vacuum expectation value of the time-ordered product of n fields. The fact that the Green's functions may be defined by (12.4) was first discovered by Schwinger, and does not depend on the path-integral formula (12.3) for $W[J]$. However, eq. (12.3) provides not only a simple proof of (12.4), but an explicit formula for computing $W[J]$.

Eq. (12.4) gives the complete Green's functions. In general, these include some contributions from disconnected vacuum to vacuum diagram, which are simply products of lower order Green's functions.

The connected graphs are given by

$$G_c(x_1, x_2\ldots x_n) = \frac{(-\mathrm{i})^n}{W[J]}\;\frac{\delta^n W[J]}{\delta J_1(x_1)\delta J_2(x_2)\ldots\delta J_n(x)} \qquad (12.5)$$

or, writing

$$W[J] = \exp\{\mathrm{i}Z[J]\}, \qquad (12.6)$$

$$G_c(x_1\ldots x_n) = (-\mathrm{i})^{n-1}\,\frac{\delta^n Z[J]}{\delta J(x_1)\ldots\delta J(x_n)}. \qquad (12.7)$$

The proof that the connected parts of the n-point function is given by eq. (12.7) is left as an exercise.

When the Hamiltonian density takes the form

$$\mathcal{H}(x) = \tfrac{1}{2}\pi^2(x) + f[\phi(x), \nabla\phi(x)]. \qquad (12.8)$$

The π-integrations can be carried out explicitly, and we obtain

$$W[J] \sim \int [\mathrm{d}\phi]\exp\{\mathrm{i}\int [\mathcal{L}(x) + J(x)\phi(x)]\,\mathrm{d}^4x\} \qquad (12.9)$$

where $\mathcal{L}(x)$ is the Lagrangian density

$$\mathcal{L}(x) = \tfrac{1}{2}(\partial_0\phi)^2 - f[\phi(x), \nabla\phi].$$

When we discuss vector meson theories, the form (12.9) will be inadequate and we shall have to use the original form (12.3). As an example, however, let us first consider a case where (12.9) is applicable.

Let us concentrate, for definiteness, the case in which the Lagrangian is of the form

$$\mathcal{L} = \mathcal{L}_o + \mathcal{L}_I, \qquad \mathcal{L}_o = \tfrac{1}{2}[(\partial_\mu\phi)^2 - \mu^2\phi^2] \tag{12.10}$$

and

$$\mathcal{L}_I = \mathcal{L}_I(\phi).$$

The functional $W[J]$ of eq. (12.9) is in general an ill-defined integral even in the "lattice" approximation. Recent advances in axiomatic field theory indicate that if one can construct a well-behaved field theory in the Euclidean space (x, t), obeying certain appropriate axioms, then there is a corresponding field theory in the Minkowski space $(x_o, \boldsymbol{x})$ as the analytic continuation of the former as $\tau = \mathrm{i}x_o$, which obeys the Wightman axioms. Thus any ambiguities should be resolved by appealing to the *Euclidicity Postulate*, namely that the Green's functions (12.5) are the analytic continuation of those defined by the well-defined functional integral in the Euclidean field theory:

$$W_E[J] = \int [\mathrm{d}\phi] \exp\left\{-\int \mathrm{d}^3x\mathrm{d}\tau \left[\left(\frac{\partial\phi}{\partial\tau}\right) + (\nabla\phi)^2 + \mu^2\phi^2 - \mathcal{L}_I(\phi) - J\phi\right]\right\} .$$

Note that since $-\mathcal{L}_I$ is bounded from below the quantity in the square bracket in the exponent above is also bounded from below. As we anticipate, the Euclidicity postulate determines the boundary conditions to be imposed on propagators. For the present problem it means that we may provide a damping factor for the functional integration by adding a term in $\mathcal{L}_o$:

$$\mathcal{L}_o \to \tfrac{1}{2}[(\partial_\mu\phi)^2 - \mu^2\phi^2 + \mathrm{i}\epsilon\phi^2]$$

as we did in eq. (12.3).

First, consider the free field case:

$$W_o[J] = \int [\mathrm{d}\phi] \exp\{\mathrm{i}\int \mathrm{d}^4x[\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\mu^2\phi^2 + \tfrac{1}{2}\mathrm{i}\epsilon\phi^2 + J\phi]\}$$

$$= \lim_{\epsilon\to 0} \int \prod_\alpha \mathrm{d}\phi_\alpha \exp\left[\mathrm{i}\left\{\sum_\alpha \epsilon^4 \sum_\beta \epsilon^4 \tfrac{1}{2}\phi_\alpha K_{\alpha\beta}\phi_\beta + \sum_\alpha \epsilon^4 J_\alpha\phi_\alpha\right\}\right] . \tag{12.11}$$

Here, α labels space-time cells of dimension ϵ^4, and the matrix $K_{\alpha\beta}$ is such that

$$\lim_{\epsilon\to 0} K_{\alpha\beta} = (-\partial^2 - \mu^2 + \mathrm{i}\epsilon)\delta^4(x-y)$$

where $\alpha \to x$ and $\beta \to y$ as $\epsilon \to 0$. The ϕ-integrations in eq. (12.11) can be performed explicitly. We obtain

$$W_o[J] = \lim_{\epsilon\to 0} \frac{1}{\sqrt{\det K_{\alpha\beta}}} \prod_\alpha \sqrt{\frac{2\pi}{\mathrm{i}\epsilon^8}} \exp\left\{-\tfrac{1}{2}\mathrm{i}\pi \sum_\alpha \epsilon^4 \sum_\beta \epsilon^4 J_\alpha \frac{1}{\epsilon^8}(K^{-1})_{\alpha\beta} J_\beta\right\}$$

where, of course, K^{-1} is the inverse of K:

$$\sum_\gamma K_{\alpha\gamma}(K^{-1})_{\gamma\beta} = \delta_{\alpha,\beta}$$

or,

$$\sum_{\gamma} \epsilon^4 K_{\alpha\gamma} \left(\frac{1}{\epsilon^8} K^{-1}\right)_{\gamma\beta} = \frac{1}{\epsilon^4} \delta_{\alpha,\beta}. \tag{12.12}$$

As $\epsilon \to 0$, we have

$$\frac{1}{\epsilon^4} \delta_{\alpha,\beta} \to \delta^4(x-y), \qquad \sum_{\alpha} \epsilon^4 \to \int \mathrm{d}^4 x,$$

so, with the definition

$$\frac{1}{\epsilon^8} (K^{-1})_{\alpha\beta} \to \Delta_{\mathrm{F}}(x-y),$$

eq. (12.12) may be written, in the continuum limit $\epsilon \to 0$, as

$$(-\partial^2 - \mu^2 + \mathrm{i}\epsilon) \Delta_{\mathrm{F}}(x-y) = \delta^4(x-y). \tag{12.13}$$

Therefore, neglecting an inessential multiplicative factor, we can write

$$W_{\mathrm{o}}[J] = \exp(-\tfrac{1}{2}\mathrm{i}) \int \mathrm{d}^4 x \int \mathrm{d}^4 y \, J(x) \Delta_{\mathrm{F}}(x-y) J(y) \tag{12.14}$$

where

$$\Delta_{\mathrm{F}}(x-y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \, \frac{\exp\{-\mathrm{i}k(x-y)\}}{k^2 - \mu^2 + \mathrm{i}\epsilon} \tag{12.15}$$

is the Feynman propagator.

Now we are ready to discuss the interacting case. Returning to eqs. (12.10) and (12.11), we write

$$\begin{aligned} W[J] &\sim \int [\mathrm{d}\phi] \exp\{\mathrm{i} \int \mathrm{d}^4 x [\mathcal{L}_{\mathrm{o}} + \mathcal{L}_{\mathrm{I}}(\phi) + J\phi]\} \\ &= \exp\left[\mathrm{i} \int \mathrm{d}^4 x \, \mathcal{L}_{\mathrm{I}}\left(\frac{1}{\mathrm{i}} \frac{\delta}{\delta J(x)}\right)\right] \int [\mathrm{d}\phi] \exp\{\mathrm{i} \int \mathrm{d}^4 x [\mathcal{L}_{\mathrm{o}} + J\phi]\} \\ &\sim \exp\left[\mathrm{i} \int \mathrm{d}^4 x \, \mathcal{L}_{\mathrm{I}}\left(\frac{1}{\mathrm{i}} \frac{\delta}{\delta J(x)}\right)\right] \exp(-\tfrac{1}{2}\mathrm{i}) \int \mathrm{d}^4 x \int \mathrm{d}^4 y \, J(x) \Delta_{\mathrm{F}}(x-y) J(y). \end{aligned} \tag{12.16}$$

Equation (12.16) is the basis for the Feynman-Dyson expansion of the Green's functions of this theory, and when it is substituted in eq. (12.4), we obtain a formula which generates Green's functions. $W[J]$ can be expanded in powers of $\mathcal{L}_{\mathrm{I}}$, for example, by simply expanding the exponential factor

$$\exp\left\{\mathrm{i} \int \mathrm{d}^4 x \, \mathcal{L}_{\mathrm{I}}\left(\frac{1}{\mathrm{i}} \frac{\delta}{\delta J}\right)\right\} = \sum_{n=0}^{\infty} \frac{(\mathrm{i})^n}{n!} \left[\int \mathrm{d}^4 x \, \mathcal{L}_{\mathrm{I}}\left(\frac{1}{\mathrm{i}} \frac{\delta}{\delta J}\right)\right]^n.$$

What corresponds to Wick's theorem is simply the rule for functional differentiation:

$$\frac{\delta}{\delta J(x)} J(y) = \delta^4(x-y).$$

The student should convince himself the rules outlined here are in fact the Feynman rules discussed in the second volume of Bjorken and Drell. In fact, collateral reading of the first six sections of Chapter 17 of this book is urged.

In order to quantize fermion fields by the method of path integrations, it is necessary to introduce the concept of anticommuting c-numbers. We shall forgo this though, because the incorporation of fermion fields presents no special problem in quantizing a gauge theory.

In general, $\mathcal{L}_\mathrm{I}$ is a function of $\dot{\phi}$ as well as ϕ, and eq. (12.4) is inadequate. Just as in the one-dimensional example discussed in the preceding lecture, we shall see, the action of eq. (12.16) must then be replaced by an "effective action", which contains a correction to the integral over the Lagrangian. In that case, the correct Feynman rules are modified, and cannot be directly read off the Lagrangian.

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13. The Yang-Mills field in the Coulomb gauge

We wish to apply these path-integral methods to theories with gauge vector mesons. Indeed, it is in this case that the method becomes a powerful tool both to discover the correct Feynman rules and to study renormalization, while the canonical Wick theorem methods become awkward.

We shall study the three-component Yang-Mills field, although the generalization to other compact non-Abelian groups is immediate. In this section, we work out the canonical formalism in the Coulomb gauge, and construct the $W[J]$ function, starting from the basic equation (12.3). In later sections we shall study gauge-invariance and work out the Feynman rules in a more manifestly covariant gauge.

It is convenient to write out the Yang-Mills Lagrangian in the first-order formulation, in which $\mathbf{A}_\mu$ and $\mathbf{F}_{\mu\nu}$ are treated as independent co-ordinates:

$$\mathcal{L}_\mathrm{F} = \tfrac{1}{4}\mathbf{F}_{\mu\nu}\cdot\mathbf{F}^{\mu\nu} - \tfrac{1}{2}\mathbf{F}_{\mu\nu}\cdot(\partial^\mu\mathbf{A}^\nu - \partial^\nu\mathbf{A}^\mu + g\mathbf{A}^\mu\times\mathbf{A}^\nu). \tag{13.1}$$

(Bold-face symbols, dots and crosses all refer to isovectors and operations among them; we write out the space-time vector indices explicitly.)

The Lagrangian (13.1) is invariant under infinitesimal gauge transformations

$$\mathbf{A}_\mu(x) \to \mathbf{A}_\mu(x) + \mathbf{u}(x) \times \mathbf{A}_\mu(x) - \frac{1}{g}\partial_\mu \mathbf{u}(x)$$

$$\mathbf{F}_{\mu\nu} \to \mathbf{F}_{\mu\nu} + \mathbf{u} \times \mathbf{F}_{\mu\nu}. \tag{13.2}$$

The Euler-Lagrange equations,

$$\frac{\delta \mathcal{L}_\mathrm{F}}{\delta F^a_{\mu\nu}} = 0, \qquad \partial_\mu \left(\frac{\delta \mathcal{L}_\mathrm{F}}{\delta(\partial_\mu A^a_\nu)} \right) = \frac{\delta \mathcal{L}}{\delta A^a_\nu} \ ,$$

give

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + g\mathbf{A}_\mu \times \mathbf{A}_\nu \tag{13.3}$$

and

$$\partial^\mu \mathbf{F}_{\mu\nu} + g\mathbf{A}^\mu \times \mathbf{F}_{\mu\nu} = 0. \tag{13.4}$$

Equations (13.3) and (13.4) together are equivalent to the Euler-Lagrange equations of the second-order formulation, in which $\mathcal{L}_s$ is written in terms of A^μ and $\partial^\mu A^\nu$ only.

In classical field theory, one is given an initial configuration of fields in a space-like hyperplane and then one tries to determine the fields at later times. Equations (13.3) and (13.4) can be separated into two classes: those which specify the temporal evolutions of the fields are called equations of motion; the others are constraint equations. From (13.3) and (13.4), the equations of motion for $\mathbf{A}_i$ and $\mathbf{F}_{\mathrm{o}i}$ are

$$\partial_\mathrm{o} \mathbf{A}_i = \mathbf{F}_{\mathrm{o}i} + (\nabla_i + g\mathbf{A}_i \times)\mathbf{A}_\mathrm{o} \tag{13.5}$$

$$\partial_\mathrm{o} \mathbf{F}_{\mathrm{o}i} = (\partial_j + g\mathbf{A}_j \times)\mathbf{F}_{ji} - g\mathbf{A}_\mathrm{o} \times \mathbf{F}_{\mathrm{o}i}. \tag{13.6}$$

Next, let us determine the independent variables. Since

$$\delta\mathcal{L}/\delta(\partial_\mathrm{o}\mathbf{A}_\mu) = -\mathbf{F}^{\mathrm{o}\mu} \tag{13.7}$$

$\mathbf{F}_{\mathrm{o}i} = -\mathbf{F}^{\mathrm{o}i}$ are the momenta canonically conjugate to $\mathbf{A}_i$. Since $\mathcal{L}$ is independent of $\partial_\mathrm{o} A^\mathrm{o}$, A^o does not have a conjugate momentum, and must be treated as a dependent variable.

The constraint equations are

$$\mathbf{F}_{ij} = \partial_i \mathbf{A}_j - \partial_j \mathbf{A}_i + g\mathbf{A}_i \times \mathbf{A}_j \tag{13.8}$$

which defines $\mathbf{F}_{ij}$ in terms of $\mathbf{A}_i$ at equal times, and

$$(\nabla_k + g\mathbf{A}_k \times)\mathbf{F}_{k\mathrm{o}} = 0 \tag{13.9}$$

which tells us that not all the conjugate momenta $\mathbf{F}_{\mathrm{o}k}$ are independent (eq. (13.9) is analogous to $\nabla \cdot \mathbf{E} = 0$ in ordinary electrodynamics). It follows that not all the $\mathbf{A}_k$ can be treated as independent, and we are forced to impose a gauge condition. We choose the Coulomb gauge:

$$\nabla_k \mathbf{A}_k = 0. \tag{13.10}$$

This is always possible because of the gauge invariance of the second kind of the Lagrangian.

Eq. (13.10) means that A must be transverse. Therefore, the longitudinal component $\mathbf{F}^{\mathrm{L}}_{oi}$ of the canonical momentum $\mathbf{F}_{oi}$ is not independent, but depends on the other degrees of freedom through the constraint equation (13.9). F^{L}_{oi} and the transverse component F^{T}_{oi} can be defined

$$\mathbf{F}_{oi} = \mathbf{F}^{\mathrm{T}}_{oi} + \mathbf{F}^{\mathrm{L}}_{oi}, \qquad \nabla_i \mathbf{F}_{oi} = \nabla_i F^{\mathrm{L}}_{oi}, \qquad \epsilon^{ijk}\nabla_j F^{\mathrm{L}}_{ok} = 0. \tag{13.11}$$

Our task is now to express $\mathbf{A}_o$ and $\mathbf{F}^{\mathrm{L}}_{oi}$ in terms of the independent variables and construct the Hamiltonian. Let us write

$$\mathbf{F}^{\mathrm{L}}_{oi} = -\nabla_i \mathbf{f}, \qquad \mathbf{F}^{\mathrm{T}}_{oi} = \mathbf{E}_i, \qquad \nabla_i \mathbf{F}_{oi} = -\nabla^2 \mathbf{f} \tag{13.12}$$

where $\mathbf{E}_i$ is purely transverse. Therefore $\mathbf{E}_i$ and the transverse components of $\mathbf{A}_i$ are the independent variables conjugate to one-another. From the constraint eq. (13.9), we find that

$$(\nabla^2 + g\mathbf{A}_k \times \nabla_k)\mathbf{f} = g\mathbf{A}_i \times \mathbf{E}_i. \tag{13.13}$$

Equation (13.13) can be formally solved by introducing a Green's function $\mathcal{D}_c$, defined as the solution to

$$(\nabla^2 \delta^{ab} + g\,\epsilon^{acb} A^c_k \nabla_k)\mathcal{D}^{bd}_c(x, y; \mathrm{A}) = \delta^{ad}\delta_3(x-y). \tag{13.14}$$

Then f is a solution of (13.13) if

$$f^a(y, t) = g\int \mathrm{d}^3y\, \mathcal{D}^{ab}_c(x, y; \mathrm{A})\epsilon^{bcd} A^c_k(y, t) E^d_k(y, t). \tag{13.15}$$

Considering $\mathcal{D}_c$ to be an integral operator, we may write (13.15) as

$$\mathbf{f} = g\,\mathcal{D}_c \cdot \mathbf{A}_k \times \mathbf{E}_k .$$

The function $\mathcal{D}_c$ has no closed form, but can be expanded in a power series in g. The first approximation is just the Green's function for the Lagrangian, and

$$\mathcal{D}^{ab}_c(x, y; \mathrm{A}) = \frac{\delta^{ab}}{4\pi|x-y|} + g\int \mathrm{d}^3z\, \frac{1}{4\pi|x-z|}\,\epsilon^{acb} A^c_k \nabla_k \frac{1}{4\pi|y-z|} + \ldots \tag{13.16}$$

in analogy to the method for finding the Green's function for $H_o + H'$ where H' is small and the Green's function for H_o is known.

We obtain an equation for $\mathbf{A}_o$ by taking the divergence of eq. (13.5) and using (13.10) and (13.11).

$$(\nabla^2 + g\,\mathbf{A}_i \times \nabla_i)\mathbf{A}_o = \nabla^2 \mathbf{f} \tag{13.17}$$

which can be solved using $\mathcal{D}_c$, since the operator in brackets is the same as in eq. (13.13):

$$A^a_o(x, t) = \int \mathrm{d}^3y\, \mathcal{D}^{ab}_c(x, y; \mathrm{A})\nabla^2 f^b(y, t)$$

or

$$\mathbf{A}_o = \mathcal{D}_c \nabla^2 \mathbf{f}. \tag{13.17a}$$

Now we construct the Hamiltonian density $\mathcal{H}$:

$$\mathcal{H} = \mathbf{E}_i \cdot \frac{\partial \mathbf{A}_i}{\partial t} - \mathcal{L} . \tag{13.18}$$

From (13.5), (13.11), (13.12) and (13.17a) we find that

$$\frac{\partial}{\partial t} \mathbf{A}_i = \mathbf{E}_i - \nabla_i \mathbf{f} + (\nabla_i + g\mathbf{A}_i \times) \mathcal{D}_c \cdot \nabla^2 \mathbf{f}$$

$$= \mathbf{E}_i - [\nabla_i - (\nabla_i + g\mathbf{A}_i \times) \mathcal{D}_c \cdot \nabla^2] \mathbf{f}. \tag{13.18a}$$

Because of (13.14), the operator in brackets operating on f is explicitly transverse. From (13.18a) and (13.13) or (13.15),

$$\int \mathrm{d}^3x \mathbf{E}_i \cdot \frac{\partial \mathbf{A}_i}{\partial t} = \int \mathrm{d}^3x [\mathbf{E}_i^2 + g(\mathbf{E}_i \times \mathbf{A}_i) \cdot \mathcal{D}_c \cdot \nabla^2 \mathbf{f}]$$

$$= \int \mathrm{d}^3x [\mathbf{E}_i^2 - \mathbf{f} \cdot \nabla^2 \mathbf{f}] = \int \mathrm{d}^3x [\mathbf{E}_i^2 + (\nabla_i \mathbf{f})^2]$$

and

$$\mathcal{L} = \tfrac{1}{4} \mathbf{F}_{\mu\nu} \cdot \mathbf{F}^{\mu\nu} - \tfrac{1}{2} \mathbf{F}_{\mu\nu} \cdot (\partial^\mu \mathbf{A}^\nu - \partial^\nu \mathbf{A}^\mu + g\mathbf{A}^\mu \times \mathbf{A}^\nu)$$

$$= \tfrac{1}{2}(\mathbf{F}_{ok})^2 - \tfrac{1}{2}(\mathbf{B}_i)^2 = \tfrac{1}{2}(\mathbf{E}_k - \nabla_k \mathbf{f})^2 - \tfrac{1}{2}(\mathbf{B}_k)^2 \tag{13.19}$$

where

$$\mathbf{B}_i = \tfrac{1}{2} \epsilon^{ijk} \mathbf{F}_{jk} .$$

So the Hamiltonian is

$$H = \tfrac{1}{2} \int \mathrm{d}^3x [\mathbf{E}_i^2 + \mathbf{B}_i^2 + (\nabla_i \mathbf{f})^2] . \tag{13.20}$$

The last term is like the familiar instantaneous Coulomb interaction which occurs in electrodynamics when quantized in this gauge.

Now we can write the Coulomb gauge generating functional $W_C[J]$ in terms of the independent co-ordinates and momenta, $\mathbf{A}_i$ and $\mathbf{E}_i$, where T stands for "Transverse", according to eq. (12.3):

$$W_C[J] = \int [\mathrm{d}\mathbf{E}_i^T][\mathrm{d}\mathbf{A}_i^T] \exp\{\mathrm{i} \int \mathrm{d}^4x [\mathbf{E}_k \cdot \dot{\mathbf{A}}_k - \tfrac{1}{2}\mathbf{E}_k^2 - \tfrac{1}{2}\mathbf{B}_k^2 - \tfrac{1}{2}(\nabla_k \cdot \mathbf{f}_k)^2\} - \vec{A}_k \cdot \vec{J}_k]\} \tag{13.21}$$

where f is a function of E_i^T and A_i^T as expressed in (13.15). [We write the source term with a negative sign, so that the covariant version below will have $+\mathbf{A}^\mu \cdot \mathbf{J}_\mu$.]

The transverse field $\mathbf{E}_i^T$ is difficult to compute with. Therefore we introduce a dummy variable $\mathbf{E}^L$ by

$$\int [\mathrm{d}\mathbf{E}_i^T] = \int [\mathrm{d}\mathbf{E}_i^T][\mathrm{d}\mathbf{E}^L] \prod_x \delta(\mathbf{E}^L) \tag{13.22}$$

and define three independent components $\mathbf{E}_i$ by

$$\mathbf{E}_i = \left(\delta_{ij} - \nabla_i \frac{1}{\nabla^2} \nabla_j\right) \mathbf{E}_j^T + \nabla_i \frac{1}{\nabla^2} \mathbf{E}^L \tag{13.23}$$

in an obvious notation. From (13.23).

$$\mathbf{E}^{\mathrm{L}} = \nabla_i \mathbf{E}_i$$

and therefore

$$\int [\mathrm{d}\mathbf{E}_i^{\mathrm{T}}] = \int [\mathrm{d}\mathbf{E}_i] \mathcal{J} \prod_x \delta(\nabla_j \mathbf{E}_j)$$

where $\mathcal{J}$ is the Jacobian of the transformation from the three $\mathbf{E}_i$ to $\mathbf{E}_i^{\mathrm{T}}$, $\mathbf{E}^{\mathrm{L}}$, and

$$[\mathrm{d}\mathbf{E}_i] \equiv \prod_x \prod_{i=1}^{3} \prod_{a=1}^{3} \mathrm{d}E_i^a(x).$$

To give $\mathcal{J}$ a meaning, we should go back to the definition of $[\mathrm{d}\mathbf{E}_i]$ as a limit of an approximation with a finite number of lattice points. In the limit, $\mathcal{J} \to \infty$, but in a way independent of the fields, so it is just a multiplicative factor in $W_{\mathrm{C}}[J]$ which doesn't matter. The same construction works for $\mathbf{A}_i^{\mathrm{T}}$. Therefore,

$$W_{\mathrm{C}}[J] = \int [\mathrm{d}\mathbf{E}_i][\mathrm{d}\mathbf{A}_i] \prod_x \delta(\nabla_k \mathbf{E}_k)\delta(\nabla_k \mathbf{A}_k) \exp\{i \int \mathrm{d}^4x [\mathbf{E}_k \cdot \dot{\mathbf{A}}_k - \tfrac{1}{2}\mathbf{E}_k^2 - \tfrac{1}{2}\mathbf{B}_k^2 - \tfrac{1}{2}(\nabla_k \mathbf{f})^2 - \mathbf{A}_k \cdot \mathbf{J}_k]\}. \tag{13.24}$$

At this point, we could examine (13.24) and obtain the Feynman rules in the Coulomb gauge. But they wouldn't be covariant, and the Lorentz covariance of the S-matrix will not be obvious throughout the calculation. It isn't useful to do calculations in the Coulomb gauge; the Coulomb gauge is the one in which the form $W[J]$ is most easily obtained from first principle.

The S-matrix, of course, is covariant and gauge invariant, so it must be possible to find a more covariant-looking form of $W_{\mathrm{C}}[J]$ than (13.24). In (13.24), $\mathbf{f}$ is a function of $\mathbf{E}$ and $\mathbf{A}$ given by eq. (13.15). We introduce f as a dummy variable by multiplying eq. (3.24) by the constant

$$\int [\mathrm{d}\mathbf{f}] \delta(\mathbf{f} - g\, \mathcal{D}_{\mathrm{c}} \cdot \mathbf{A}_i \times \mathbf{E}_i) \tag{13.25}$$

where by $\mathcal{D}_{\mathrm{c}} \cdot$ we mean the operation in (13.15). Since (13.15) is equivalent to (13.13), we write (13.25) as

$$\int [\mathrm{d}\mathbf{f}] \det M_{\mathrm{c}} \delta((\nabla^2 + g\, \mathbf{A}_i \times \nabla_i)\mathbf{f} - g\, \mathbf{A}_i \times \mathbf{E}_i) \tag{13.26}$$

where $\det M_{\mathrm{c}}$ is the Jacobian of the transformation from f to $(\nabla^2 + g\,\mathbf{A}_i \times \nabla_i)f$. M_{c} is a matrix in $x - y$ space as well as isospin space:

$$M_{\mathrm{c}}^{ab}(x, y) = (\nabla^2 \delta^{ab} + g\, \epsilon^{abc} A_i^c(y) \nabla_i) \delta^4(x - y)$$

$$= \nabla^2 [\delta^{ab} \delta_3(x - y) + g\, \epsilon^{abc} G(x, y) A_i^c(y) \nabla_i] \delta(x_{\mathrm{o}} - y_{\mathrm{o}}) \tag{13.27}$$

where $\nabla^2 G(x, y) = \delta_3(x - y)$. Now, eq. (13.24) becomes

$$W_{\mathrm{C}}[J] = \int [\mathrm{d}\mathbf{A}_i][\mathrm{d}\mathbf{E}_i][\mathrm{d}\mathbf{f}] \det M_{\mathrm{c}} \prod_x \delta(\nabla_i \mathbf{A}_i) \prod_x \delta(\nabla_i \mathbf{E}_i) \delta[\nabla^2 + g\, \mathbf{A}_i \times \nabla_i)\mathbf{f}$$

$$- g\, \mathbf{A}_i \times \mathbf{E}_i] \exp[i \int \{\mathbf{E}_k \cdot \dot{\mathbf{A}}_k - \tfrac{1}{2}(\mathbf{E}_k^2 + \mathbf{B}_k^2 + (\nabla_k \mathbf{f})^2) - \mathbf{J}_i \cdot \mathbf{A}_i\} \mathrm{d}^4x]. \tag{13.28}$$

Next we change variables from $\mathbf{E}_i$ to $\mathbf{F}_{oi}$, defined by

$$\mathbf{F}_{oi} = \mathbf{E}_i - \nabla_i \mathbf{f}. \qquad (13.29)$$

Then, in (13.28), we write

$$[\mathrm{d}\mathbf{E}_i]\,[\mathrm{d}\mathbf{f}] \prod_x \delta(\nabla_i \mathbf{E}_i)\delta[(\nabla^2 + g\,\mathbf{A}_i \times \nabla_i)\mathbf{f} - g\,\mathbf{A}_i \times \mathbf{E}_i]$$

$$= [\mathrm{d}\mathbf{F}_{oi}]\,[\mathrm{d}\mathbf{f}] \prod_x \delta(\nabla_i \mathbf{F}_{oi} + \nabla^2 \mathbf{f})\delta[\nabla^2 \mathbf{f} - g\,\mathbf{A}_i \times \mathbf{F}_{oi}]$$

$$= [\mathrm{d}\mathbf{F}_{oi}]\,[\mathrm{d}\mathbf{f}] \prod_x \delta(\nabla_i \mathbf{F}_{oi} + g\,\mathbf{A}_i \times \mathbf{F}_{oi})\delta(\nabla^2 \mathbf{f} - g\,\mathbf{A}_i \times \mathbf{F}_{oi}). \qquad (13.30)$$

Now we consider the [df] integration, using the last δ-function in (13.30). The Jacobian is just det Δ^2, an infinite constant which we drop (or absorb into the definition of M_C). Thus

$$W_C[J] = \int [\mathrm{d}\mathbf{A}_i]\,[\mathrm{d}\mathbf{F}_{oi}] \det M_C \prod_x \delta(\nabla_i \mathbf{A}_i)\delta(\nabla_i \mathbf{F}_{oi} + g\,\mathbf{A}_i \times \mathbf{F}_{oi})$$

$$\times \exp\{\mathrm{i}\int \mathrm{d}^4x [\mathbf{F}_{oi} \cdot \partial_o \mathbf{A}_i - \tfrac{1}{2}\mathbf{F}^2_{oi} - \tfrac{1}{4}(\nabla_i \mathbf{A}_j - \nabla_j \mathbf{A}_i + g\,\mathbf{A}_i \times \mathbf{A}_j)^2 - \mathbf{J}_i \cdot \mathbf{A}_i]\}. \qquad (13.31)$$

To obtain the exponent in (13.31), we have written in the exponent in (13.28)

$$[E^2_k + (\nabla_k f)]^2 = (E_k - \nabla_k f)^2 = F^2_{oi}$$

omitting the cross-term which vanishes upon integration over x.

Next we write the δ-function as an integral, using $\mathbf{A}_o$ as the dummy variable:

$$\prod_x \delta(\nabla_i \mathbf{F}_{oi} + g\,\mathbf{A}_i \times \mathbf{F}_{oi}) = \prod_x \int \frac{\mathrm{d}\mathbf{A}_o}{2\pi} \exp\{\mathrm{i}\mathbf{A}_o \cdot (\nabla_i \mathbf{F}_{oi} - g\,\mathbf{A}_i \times \mathbf{F}_{oi})\}$$

$$\sim \int [\mathrm{d}\mathbf{A}_o] \exp\{\mathrm{i}\int \mathrm{d}^4x\, \mathbf{F}_{oi} \cdot (g\,\mathbf{A}_o \times \mathbf{A}_i - \nabla_i \mathbf{A}_o)\}. \qquad (13.32)$$

Finally, we write the term $\frac{1}{4}(\nabla_i \mathbf{A}_j - \nabla_j \mathbf{A}_i + g\,\mathbf{A}_i \times \mathbf{A}_j)^2$ in the exponent in (13.31) as

$$\int [\mathrm{d}\mathbf{F}_{ij}] \exp\{\mathrm{i}[\tfrac{1}{4}\mathbf{F}_{ij} \cdot \mathbf{F}_{ij} - \tfrac{1}{2}\mathbf{F}_{ij} \cdot (\nabla_i \mathbf{A}_j - \nabla_j \mathbf{A}_i + g\,\mathbf{A}_i \times \mathbf{A}_j)]\} \qquad (13.33)$$

which is a standard Gaussian integral. Putting (13.33) and (13.32) into (13.31), and restricting J_o to be zero, we obtain

$$W_C[J] = \int [\mathrm{d}\mathbf{A}_\mu]\,[\mathrm{d}\mathbf{F}_{\mu\nu}] \det M_c \prod_x \delta(\nabla_i \mathbf{A}_i)$$

$$\times \exp\Big[\mathrm{i}\int \mathrm{d}^4x \{-\tfrac{1}{2}\mathbf{F}_{oi} \cdot \mathbf{F}_{oi} + \tfrac{1}{4}\mathbf{F}_{ij} \cdot \mathbf{F}_{ij} - \tfrac{1}{2}\mathbf{F}_{ij} \cdot (\nabla_i \mathbf{A}_j - \nabla_j \mathbf{A}_i + g\,\mathbf{A}_i \times \mathbf{A}_j) + \mathbf{F}_{oi}(\partial_o \mathbf{A}_i - \nabla_i \mathbf{A}_o + g\,\mathbf{A}_o \times \mathbf{A}_i)\}\Big]$$

$$= \int [\mathrm{d}\mathbf{A}_\mu]\,[\mathrm{d}\mathbf{F}_{\mu\nu}] \det M_C \prod_x \delta(\nabla_i \mathbf{A}_i) \exp\{\mathrm{i}\int \mathrm{d}^4x [\mathcal{L} + \mathbf{J}^\mu \cdot \mathbf{A}_\mu]\}. \qquad (13.34)$$

Were it not for the factor det M_C, (13.34) implies that one could get the Feynman rules directly from $\mathcal{L}$. The extra factor is analogous to the correction obtained in section 11 for a velocity-dependent potential.

How do we interpret $\det M_C$? From (13.27),

$$\det M_C = \det \nabla^2 \cdot \det [I + L] \tag{13.35}$$

where

$$L = g\, \epsilon^{abc} G(x, y) A_i^c(y) \cdot \nabla_i \delta(x_o - y_o), \qquad I = \delta^{ab} \delta^4(x - y). \tag{13.36}$$

Now $\det \nabla^2$ is an infinite constant, and

$$\det(I + L) = \exp \operatorname{Tr} \log(I + L)$$

$$= \exp \sum \frac{(-1)^{n-1}}{n} \int d^4x_1 \ldots d^4x_n \operatorname{Tr} L(x_1, x_2)\, L(x_2, x_3) \ldots L(x_n, x_1). \tag{13.37}$$

The trace inside the integral is over isospin indices only.

We shall encounter Jacobians like $\det M_C$ in the next few sections. Eq. (13.37) is a general formula for evaluating them. In our case

$$\det(I+L) = \exp\Big[\delta(0) \Big\{ -\sum \frac{g^n}{n} \int d^3x_1 \ldots d^3x_n \int dt \operatorname{Tr}\Big[\mathbf{T}\cdot \mathbf{A}_{i_1}(x_1, t) \nabla_{i_1} G(x_1, x_2) \mathbf{T}\cdot \mathbf{A}_{i_2}(x_2, t) \nabla_{i_2} G(x_2, x_3) \ldots$$
$$\ldots \mathbf{T}\cdot \mathbf{A}_{i_n}(x_n, t) \nabla_{i_n} G(x_n, x_1)\Big]\Big\}\Big] \tag{13.38}$$

where $(T^a)_{bc} = \epsilon^{abc}$ and Tr means trace over isospin indices.

Since (13.38) is a power series in the exponent, it is an effective correction in each order to the Feynman rules obtained from $\mathcal{L}$ alone.

Bibliography

The presentation in this lecture is similar to and inspired by

1. V.N. Popov and L.D. Fadde'ev, Perturbation Theory for Gauge Invariant Fields, Kiev ITP report (unpublished).

See also

2. L.D. Fadde'ev and V.N. Popov, Phys. Letters 25B (1967) 29.

14. Intuitive approach to the quantization of gauge fields

Equation (13.34) can be further simplified. We can perform the functional integration over $F^a_{\mu\nu}$ and obtain

$$W[J] = \int [d\mathbf{A}_\mu] \det M_C \prod_x \delta(\nabla_i \mathbf{A}_i(x)) \exp\{i \int d^4[\mathcal{L}(x) + \mathbf{J}_\mu(x) \cdot \mathbf{A}^\mu(x)]\} \tag{14.1}$$

where $\mathcal{L}(x)$ is the second-order Lagrangian:

$$\mathcal{L}(x) = -\tfrac{1}{4}(\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + g\mathbf{A}_\mu \times \mathbf{A}_\nu)^2 .$$

Except for the factor $\det M_C\ \Pi_x\ \delta(\nabla_i A_i(x))$, eq. (14.1) is in the standard form for simple field theories

$$W[J] \sim \int [\mathrm{d}\phi] \exp\{\mathrm{i}\int \mathrm{d}^4x [\mathcal{L}(x) + J(x)\phi(x)]\}. \tag{14.2}$$

The following intuitive argument due to Faddeev and Popov shows very clearly the raison d'etre for this extra factor.

The reason eq. (14.2) is not applicable to the gauge theory is that the quadratic part of the Lagrangian

$$L_0 = -\int \tfrac{1}{4}(\partial_\mu \mathrm{A}_\nu - \partial_\nu \mathrm{A}_\mu)^2\, \mathrm{d}^4x = \int \mathrm{d}^4x\, \mathrm{d}^4y\, \tfrac{1}{2}\mathrm{A}_\mu(x)\cdot K^{\mu\nu}(x,y)\cdot \mathrm{A}_\nu(y),$$

$$K^{\mu\nu}(x,y) = -(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)\delta^4(x-y)$$

is singular, in the sense that the operator $K^{\mu\nu}$ which defines the quadratic form is singular and cannot be inverted. In fact, the operator $K^{\mu\nu}$ is essentially a projection operator for the transverse components of A_μ. This means, in particular, that the Euclidean version of the functional integral of eq. (12.2) [see the discussion following eq. (12.10)] has no Gaussian damping factor with respect to the variation of the longitudinal component of A_μ, and eq. (14.2) is meaningless at this elementary level even in the Euclidean formulation. More generally, the action is invariant under the gauge transformation $\mathrm{A}_\mu \to \mathrm{A}_\mu^g$ where A_μ^g is the result of applying the element g of the gauge group G to the field A_μ:

$$\mathrm{A}_\mu^g \cdot \mathrm{L} = U(g)\left[\mathrm{A}_\mu \cdot \mathrm{L} + \frac{1}{\mathrm{i}g} U^{-1}(g)\partial_\mu U(g)\right] U^{-1}(g). \tag{14.3}$$

To put it differently, the action is constant on the orbits of the gauge group, which are formed by all A_μ^g for fixed A_μ and g ranging all over G. Thus, the path integral for the vacuum-to-vacuum amplitude $W[\mathrm{J}]$ diverges even in the Euclidean formulation, since for those variations of A_μ which are along the orbits, the action does not provide necessary damping. Faddeev and Popov pointed out that the amplitude $W[\mathrm{J}=0]$ is therefore proportional to the "volume" of orbits $\Pi_x \mathrm{d}g(x)$, and this factor should be extracted before defining $W[\mathrm{J}]$. In other words, for the gauge fields, the path integral is to be performed not over all variations of the gauge fields, but rather over distinct orbits of A_μ under the action of the gauge group.

To implement the above idea, we shall choose a "hypersurface" in the manifold of all fields which intersects each orbit only once. This means that if

$$f_a(\mathrm{A}_\mu) = 0, \qquad a = 1, 2, \ldots N \tag{14.4}$$

is the equation of the hypersurface, N being the dimension of the group, the equation

$$f_a(\mathrm{A}_\mu^g) = 0$$

must have a unique solution g for given A_μ. We are going to integrate over this hypersurface, instead of integrating over the manifold of all fields. The conditions $f_a(\mathrm{A}_\mu) = 0$ define a gauge; the Coulomb gauge $f_a(\mathrm{A}_\mu) = \nabla_i A_i^a$ is an example.

Before proceeding further, let us pause here to review briefly a few simple facts about group representations. Let $g, g' \in G$. Then $gg' \in G$, and

$$U(g)U(g') = U(gg').$$

The invariant Hurwitz measure over the group G is an integration measure on the group space which is invariant in the sense that

$$\mathrm{d}g' = \mathrm{d}(gg'). \tag{14.5}$$

If we parametrize $U(g)$ in the neighborhood of the identity as

$$U(g) = 1 + \mathrm{i}u \cdot \mathbf{L} + \mathrm{O}(u^2),$$

then in the neighborhood of the identity we may always choose

$$\mathrm{d}g = \prod_a \mathrm{d}u_a, \qquad g \approx 1. \tag{14.6}$$

Let us define the quantity $\Delta_f[\mathrm{A}_\mu]$ by

$$\Delta_f[\mathrm{A}_\mu] \int \prod_x \mathrm{d}g(x) \prod_{x,a} \delta[f_a(\mathrm{A}_\mu^g(x))] = 1. \tag{14.7}$$

The "naive" expression for the vacuum-to-vacuum amplitude is

$$\int [\mathrm{dA}_\mu] \exp\{\mathrm{i} \int \mathrm{d}^4 x \mathcal{L}(x)\}. \tag{14.8}$$

We may insert the left-hand side of eq. (14.7) into the integrand of eq. (14.8) without changing anything:

$$\int \prod_x \mathrm{d}g(x)[\mathrm{dA}_\mu]\Delta_f[\mathrm{A}_\mu] \prod_{x,a} \delta[f_a(\mathrm{A}_\mu^g(x))] \exp\{\mathrm{i}\int \mathrm{d}^4 x \mathcal{L}[\mathrm{A}_\mu(x)]\}. \tag{14.9}$$

Now, in the integrand of eq. (14.9) we can perform a gauge transformation on $\mathrm{A}_\mu(x)$: $\mathrm{A}_\mu(x) \to [\mathrm{A}_\mu(x)]^{g^{-1}}$. Under the gauge transformation (14.3) the action and the metric are invariant, and one can verify easily from eqs. (14.5) and (14.7) that $\Delta_f[\mathrm{A}_\mu]$ is gauge invariant:

$$\begin{aligned}\Delta_f^{-1}[\mathrm{A}_\mu^g] &= \int \prod_x \mathrm{d}g'(x) \prod_{x,a} \delta[f_a(\mathrm{A}_\mu^{gg'}(x))] \\ &= \int \prod_x \mathrm{d}(g(x)g'(x)) \prod_{x,a} \delta[f_a(\mathrm{A}_\mu^{gg'}(x))] \\ &= \int \prod_x \mathrm{d}g''(x) \prod_{x,a} \delta[f_a(\mathrm{A}_\mu^{g''}(x))] = \Delta_f^{-1}[\mathrm{A}_\mu],\end{aligned}$$

or

$$\Delta_f[\mathrm{A}_\mu^g] = \Delta_f[\mathrm{A}_\mu]. \tag{14.10}$$

So, eq. (14.9) is equal to

$$\prod_x \mathrm{d}g(x) \int [\mathrm{dA}_\mu]\Delta_f[\mathrm{A}_\mu] \prod_{x,a} \delta[f_a(\mathrm{A}_\mu)] \exp\{\mathrm{i}\int \mathrm{d}^4 x \mathcal{L}[\mathrm{A}_\mu(x)]\}$$

and we find that the integrand of the group integration is independent of $g(x)$. This is the observation of Fadeev and Popov, who saw that $\int\prod_x \mathrm{d}g(x)$ is simply an infinite factor independent of fields. Therefore, it can be divided out, and $W[\mathrm{J}]$ may be defined as

$$W_f[\mathrm{J}] = \int [\mathrm{dA}_\mu]\Delta_f[\mathrm{A}_\mu] \prod_{x,a} \delta[f_a(\mathrm{A}_\mu)] \exp\{\mathrm{i}\int \mathrm{d}^4 x [\mathcal{L}(x) + \mathrm{J}^\mu(x)\cdot \mathrm{A}_\mu(x)]\}. \tag{14.11}$$

It is to the credit of Faddeev and Popov that they also gave the canonical derivation of eq. (13.34) as discussed in the preceeding section, as well as this elegant argument. Before demonstrating the connection between eqs. (14.1) and (14.11) above, we shall compute $\Delta_f[\mathrm{A}_\mu]$.

Since the factor $\Delta_f[\mathrm{A}_\mu]$ is multipled by $\Pi_x\,\delta[f(\mathrm{A}_\mu(x))]$ in eq. (14.11), it suffices to compute the former only for A_μ which satisfies eq. (14.4). Now define M_f by

$$f_a(\mathrm{A}_\mu^g(x)) = f_a(\mathrm{A}_\mu(x)) + \int \mathrm{d}^4 y \sum_b [M_f(x,y)]_{ab} u_b(y) + \mathrm{O}(u^2). \tag{14.12}$$

Then from eq. (14.7) we find that

$$\Delta_f^{-1}[\mathrm{A}_\mu] = \int \prod_x \prod_a \{\mathrm{d}u_a(x)\delta[f_a(\mathrm{A}_\mu^g(x))]\} = \int \prod_x \prod_a \{\mathrm{d}u_a(x)\delta(M_f u)\}$$

for A_μ satisfying $f_a(\mathrm{A}_\mu) = 0$, so that

$$\Delta_f[\mathrm{A}_\mu] = \det M_f = \exp\{\mathrm{Tr}\ln M_f\}. \tag{14.13}$$

The hypersurface equation $f_a = 0$ is just the gauge condition, and for the Coulomb gauge adopted in the preceeding section, we have

$$f_a(\mathrm{A}_\mu) = \nabla_i A_i^a = 0$$

and

$$f_a(\mathrm{A}_\mu^g) = \nabla_i A_i^a + \frac{1}{g}(\nabla^2\delta^{ab} - g\epsilon^{abc}A_i^c\nabla_i)u_b(x) + \mathrm{O}(u^2)$$

so

$$[M_f(x,y)]_{ab} \sim \frac{1}{g}\nabla^2\left(\delta^{ab} - g^{abc}\frac{1}{\nabla^2}A_i^c\nabla_i\right)\delta^4(x-y) \sim [M_c(x,y)]_{ab}, \tag{14.14}$$

which shows that eq. (14.1) is indeed a special case of eq. (14.11) above for $f_a = \nabla_i A_i^a$.

The form of eq. (14.11) suggests using a wide range of gauges order than the Coulomb gauge. For the moment, we shall not ask what relations the Green's functions generated in such a gauge bear to those defined in the Coulomb gauge, but merely note the explicit form of Δ_f when the manifestly covariant Laudau gauge condition

$$\partial^\mu \mathrm{A}_\mu(x) = 0$$

is chosen. Equation (14.12) takes the form

$$\partial^\mu \mathrm{A}_\mu^g(x) = \partial^\mu \mathrm{A}_\mu(x) + \frac{1}{g}[\partial^2 \mathrm{u} + g\partial^\mu(\mathrm{A}_\mu \times \mathrm{u})] + \mathrm{O}(u^2)$$

so that M_f is given by

$$[M_\mathrm{L}(x,y)]_{ab} = \frac{1}{g}(\partial^2\delta_{ab} - g\epsilon_{abc}A_\mu^c\partial^\mu)\delta^4(x-y) \tag{14.15}$$

when A_μ is restricted to $\partial^\mu \mathrm{A}_\mu = 0$. Therefore removing the trivial factor $(1/g)\partial^2$, we have

$$\Delta_\mathrm{L} \equiv \det M_\mathrm{L} \sim \exp\{\mathrm{Tr}\,\ln(1+L)\} \tag{14.16}$$

where

$$\langle x, a|L|y, b\rangle = g\epsilon_{abc}\int D_F(x-z)A^c_\mu(z)\frac{\partial}{\partial z_\mu}\delta^4(z-y)\mathrm{d}^4z.$$

More explicitly we can write

$$\Delta_L = \exp\left\{-\sum\frac{(-g)^n}{n}\int \mathrm{d}^4x_1\ldots\mathrm{d}^4x_n\mathrm{Tr}[\partial^\lambda D_F(x_1-x_2)\mathbf{t}\cdot\mathbf{A}_\mu(x_2)\partial^\mu D_F(x_2-x_3)\right.$$
$$\left.\ldots D_F(x_n-x_1)\mathbf{t}\cdot\mathbf{A}_\lambda(x_1)]\right\}. \quad (14.17)$$

Here we have used the conventional notation

$$(-\partial^2+\mathrm{i}\epsilon)D_F(x-y)=\delta^4(x-y).$$

The iϵ, $\epsilon > 0$, is chosen according to the Euclidicity postulate.

The necessity of having the extra factor $\Delta_f\Pi_x\,\delta[f(\mathbf{A}_\mu(x)]$ was first noted by Feynman. We can write eq. (14.11) as

$$W_f[\mathbf{J}] = \int[\mathrm{d}\mathbf{A}_\mu]\prod_x \delta[f(\mathbf{A}_\mu(x))]\exp\{\mathrm{i}[S_{\mathrm{eff}}+\int\mathrm{d}^4x\mathbf{J}^\mu(x)\cdot\mathbf{A}_\mu(x)]\} \quad (14.18)$$

where

$$S_{\mathrm{eff}} = \int\mathrm{d}^4x\mathcal{L}(x) - \mathrm{i}\,\mathrm{Tr}\ln M_f.$$

In the case of the Landau gauge, it has been observed that the additional term in the effective action $-\mathrm{i}\,\mathrm{Tr}\ln M_L$ can be viewed as arising from loops generated by a fictitious isotriplet of complex scalar fields c obeying Fermi statistics, whose presence and interactions can be described by the action

$$S_c = -\int\mathrm{d}^4x[\partial^\mu\mathbf{c}^\dagger(x)\cdot\partial_\mu\mathbf{c}(x)+g\partial^\mu\mathbf{c}^\dagger(x)\cdot\mathbf{A}_\mu(x)\times\mathbf{c}(x)]$$
$$\sim \int\mathrm{d}^4x\mathrm{d}^4y\sum_{a,b}c^\dagger_a(x)[M_L(x,y)]_{ab}c_b(y).$$

That is, eq. (14.18) may be written as

$$W_L[J] = \int[\mathrm{d}\mathbf{A}_\mu]\prod_x\delta[f(\mathbf{A}_\mu(x))]\int[\mathrm{d}\mathbf{c}^\dagger][\mathrm{d}\mathbf{c}]\exp\left\{\mathrm{i}[S+S_c+\int\mathrm{d}^4x\,\mathbf{J}^\mu(x)\cdot\mathbf{A}_\mu(x)]\right\}. \quad (14.19)$$

In fact it is not difficult to show that the c- and c†-integrations could be carried out trivially if they were commuting c-numbers, yielding

$$\int[\mathrm{d}c^\dagger][\mathrm{d}c]\exp(\mathrm{i}S_c)\sim(\det M_L)^{-1}=\exp\{-\mathrm{Tr}\ln M_L\},$$

and

$$\exp\{-\mathrm{Tr}\ln M_L\}\sim\exp[-\mathrm{Tr}\ln(1+L)]$$
$$=\exp\left\{-\mathrm{Tr}\,L+\tfrac{1}{2}\mathrm{Tr}\,L^2+\ldots(-)^n\frac{1}{n}\mathrm{Tr}\,L^n+\ldots\right\}$$

where the terms in the exponent may be viewed as arising from loops of the complex boson fields c. If the c are fermion fields, then the terms $\mathrm{Tr}L^n$ have to be multiplied by an extra $-$ sign, so that we have

$$\int [\mathrm{d}c^\dagger][\mathrm{d}c]\exp(\mathrm{i}S_c) \sim \exp\left\{+\mathrm{Tr}\,L - \tfrac{1}{2}\,\mathrm{Tr}\,L^2 \ldots + \frac{(-1)^{n+1}}{n}\,\mathrm{Tr}\,L^n \ldots\right\}$$

$$= \exp\{\mathrm{Tr}\,\ln(1+L)\} \sim \det M_{\mathrm{L}}.$$

The Feynman rules for $W_{\mathrm{L}}[J]$ of eq. (14.19) can be worked out in much the same way as we did for a scalar field theory in section 12. The gauge boson propagator is determined from

$$W_{\mathrm{L}}^{\mathrm{o}}[J] = \int [\mathrm{dA}_\mu] \prod_x \delta[\partial^\mu \mathrm{A}_\mu(x)] \exp\left[\mathrm{i}\int \mathrm{d}^4x \left\{-\tfrac{1}{4}(\partial_\mu \mathrm{A}_\nu - \partial_\nu \mathrm{A}_\mu)^2 + \mathrm{J}^\mu(x)\cdot \mathrm{A}_\mu(x)\right\}\right]. \qquad (14.20)$$

A convenient way of computing eq. (14.20) is to write

$$\prod_x \delta[\partial^\mu \mathrm{A}_\mu(x)] \sim \lim_{\alpha\to 0} \exp\left\{\frac{-\mathrm{i}}{2\alpha}\int \mathrm{d}^4x[\partial^\mu \mathrm{A}_\mu(x)]^2\right\}.$$

[We have discarded an infinite constant $\Pi_x\sqrt{2\pi\alpha}$.] Then we have

$$W_{\mathrm{L}}^{\mathrm{o}}[J] = \lim_{\alpha\to 0}\int [\mathrm{dA}_\mu] \exp\left[\mathrm{i}\left\{-\int \mathrm{d}^4x\ \mathrm{A}_\mu(x)\cdot\left[-\partial^2 g^{\mu\nu} + \partial^\mu\partial^\nu\left(1-\frac{1}{\alpha}\right)\right]\mathrm{A}_\nu(x) + \int \mathrm{d}^4x\ \mathrm{J}^\mu(x)\cdot \mathrm{A}_\mu(x)\right\}\right]$$

$$= \lim_{\alpha\to 0}\exp\left(\frac{-\mathrm{i}}{2}\right)\int \mathrm{d}^4x\,\mathrm{d}^4y\ \mathrm{J}_\mu(x)\cdot D_{\mathrm{F}}^{\mu\nu}(x-y;\alpha)\mathrm{J}_\nu(y) \qquad (14.21)$$

where the vector boson propagator $D_{\mathrm{F}}^{\mu\nu}$ in this gauge is

$$D_{\mathrm{F}}^{\mu\nu}(x-y;\alpha) = \frac{\mathrm{d}^4k}{(2\pi)^4}\exp\{\mathrm{i}k\cdot(x-y)\}\,\frac{1}{k^2+\mathrm{i}\epsilon}\left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}(1-\alpha)\right] \xrightarrow{\alpha\to 0}$$

$$D_{\mathrm{F}}^{\mu\nu}(x-y) = -\int \frac{\mathrm{d}^4k}{(2\pi)^4}\exp\{\mathrm{i}k\cdot(x-y)\}\,\frac{1}{k^2+\mathrm{i}\epsilon}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \qquad (14.22)$$

and is four-dimensionally transverse. The rest of the Feynman rules can be derived as in the scalar case. They are recorded in the following fig. 14.1. In addition, the following rules must kept in mind: the ghost-ghost-vector vertex is "dotted", the dot indicating which ghost line is differentiated; a ghost line cannot be dotted at both ends; a ghost loop carries an extra minus sign.

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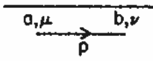
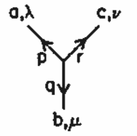
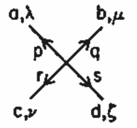
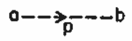
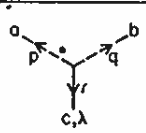
	VERTICES	BARE VERTICES
	$-i\delta^{ab}\Delta_{\mu\nu}(p)$	$-i\delta^{ab}[(g_{\mu\nu}-p_\mu p_\nu/p^2)1/p^2+\alpha p_\mu p_\nu(p^2)^{-2}]$
	$i\Gamma^{abc}_{\lambda\mu\nu}\ (p,q,r)$ $p+q+r=0$	$\epsilon^{abc}[(p-q)_\nu g_{\lambda\mu}$ $+(q-r)_\lambda g_{\mu\nu}+(r-p)_\mu g_{\nu\lambda}]$
	$i\Gamma^{abcd}_{\lambda\mu\nu\zeta}\ (p,q,r,s)$ $p+q+r+s=0$	$-i\epsilon^{abf}\epsilon^{cdf}(g_{\lambda\nu}g_{\mu\zeta}-g_{\lambda\zeta}g_{\mu\nu})$ $-i\epsilon^{acf}\epsilon^{bdf}(g_{\lambda\mu}g_{\nu\zeta}-g_{\lambda\zeta}g_{\mu\nu})$ $-i\epsilon^{adf}\epsilon^{cbf}(g_{\lambda\nu}g_{\mu\zeta}-g_{\lambda\mu}g_{\zeta\nu})$
	$-\delta^{ab}iG(p)$	$-i\delta^{ab}1/p^2$
	$i\gamma^{abc}_\lambda(p,q,r)$	$\epsilon^{abc}p_\lambda$

Fig. 14.1. Feynman rules in the Yang-Mills theory. Solid lines are vector mesons. Dashed lines are scalar ghosts.

15. Equivalence of the Landau and Coulomb gauges

Formally, the S-matrix computed in the Landau gauge is the same as that computed in the Coulomb gauge. An element of the unrenormalized S-matrix is obtained from the corresponding Green's functions by removing single particle propagators corresponding to external lines, taking the Fourier transform of the resulting "amputated" Green's function and placing external momenta on the mass shell. The demonstration we shall present is rigorous except that the S-matrix of a gauge theory is plagued by infrared divergences and may not even be defined. In fact this may be the reason why massless Yang-Mills particles are not seen in Nature. The point of presenting this demonstration is purely pedagogical: the spirit and the technique we espouse here will become useful when we discuss spontaneously broken versions of gauge theories.

We shall first establish the connection between $W_C[J]$ and $W_L[J]$. Recall that [eq. (14.1)]

$$W_C[J] = \int [dA_\mu]\Delta_C[A_\mu]\prod_x \delta(\nabla_i A_i(x))\exp\left\{iS[A_\mu]+i\int d^4x\, J^\mu\cdot A_\mu\right\} \tag{15.1}$$

where $\Delta_C = \det M_C$ and that

$$\Delta_L[A_\mu]\int \prod_x dg(x)\prod_x \delta(\partial^\mu A^g_\mu(x)) = 1. \tag{15.2}$$

Inserting the left-hand side of eq. (15.2) in the integrand of the functional integration in eq. (15.1) we write

$$W_C[J] = \int \prod_x dg(x) \int [dA_\mu]\Delta_C[A_\mu]\Delta_L[A_\mu] \prod_x \delta(\nabla_i A_i(x))$$
$$\times \prod_x \delta(\partial^\mu A_\mu^g(x)) \exp\{iS[A_\mu] + i\int d^4x\, J^\mu \cdot A_\mu\}.$$

We now make a gauge transformation of the integration variables $A_\mu(x)$: $A_\mu(x) \to [A_\mu(x)]^{g^{-1}}$. Recalling the gauge invariance of the action S, Δ_f and the metric $[dA_\mu(x)]$, we find that

$$W_C[J] = \int [dA_\mu]\Delta_L[A_\mu] \prod_x \delta(\partial^\mu A_\mu(x)) \exp(iS[A_\mu])$$
$$\times \Delta_C[A_\mu] \int \prod_x dg(x) \prod_x \delta(\nabla_i A_i^{g^{-1}}) \exp\left\{i\int d^4x\, J^\mu \cdot A_\mu^{g^{-1}}\right\}$$
$$= \int [dA_\mu]\Delta_L[A_\mu] \prod_x \delta(\partial^\mu A_\mu(x)) \exp\left\{iS[A_\mu] + i\int d^4x\, J^\mu \cdot A_\mu^{g_o}\right\} \qquad (15.3)$$

where $A_\mu^{g_o}$ is the gauge transform of A_μ, which satisfies $\partial^\mu A_\mu = 0$, such that

$$L \cdot \nabla_i A_i^{g_o} = \nabla_i \left\{U(g_o)\left[L \cdot A_i + \frac{1}{ig} U^{-1}(g_o)\nabla_i U(g_o)\right] U^{-1}(g_o)\right\} = 0. \qquad (15.4)$$

In deriving eq. (15.3), we have used the fact that

$$\Delta_C[A_\mu] \int \prod_x dg(x) \prod_x \delta(\nabla_i A_i^{g^{-1}}) = \Delta_C[A_\mu] \int \left\{\prod_x \prod_a du_a(x)\right\} \prod_x \delta\left(\nabla_i A_i^{g_o} - \frac{1}{g} M_C[A_\mu^{g_o}]u\right)$$
$$\sim \Delta_C[A_\mu]\Delta_C^{-1}[A_\mu^{g_o}] = 1.$$

Now, we must find out $A_\mu^{g_o}$ by solving eq. (15.4). It is possible to construct $A_\mu^{g_o}$ in a power series in A_μ. We leave it as an exercise to construct first few terms in this expansion. For our purpose it suffices to note that

$$A_i^{g_o} = \left(\delta_{ij} - \nabla_i \frac{1}{\nabla^2} \nabla_j\right) A_j + O(A_\mu^2).$$

The source J_μ in the Coulomb gauge shall be restricted to

$$J_o = 0, \qquad \nabla_i J_i = 0. \qquad (13.26)$$

Therefore, we may write

$$\int d^4x\, J^\mu \cdot A_\mu^{g_o} = \int d^4x\, J^\mu \cdot F_\mu(x; A_\lambda)$$

where

$$F_\mu(x; A_\lambda) = A_\mu(x) + O(A_\lambda^2). \qquad (15.5)$$

We can finally write down an equation for W_C in terms of W_L. It is

$$W_C[J] = \left[\exp\left\{i\int d^4x\, J^\mu(x) \cdot F_\mu\left(x; \frac{1}{i}\frac{\delta}{\delta j_\lambda}\right)\right\}\right] W_L[j]\Big|_{j=0} \qquad (15.6)$$

It is helpful to visualize eq. (15.3) or eq. (15.6) in terms of Feynman diagrams. These equations say that Green's functions in the Coulomb gauge are the same as those in the Landau gauge, when the source is suitably restricted [eq. (13.26)] except that one must take into account extra vertices between a source and fields, represented by the term

$$\int d^4x\, J^\mu \cdot (F_\mu - A_\mu) \tag{15.7}$$

when one tries to construct Coulomb gauge Green's functions by the Feynman rules of the Landau gauge. This connection becomes much simpler, if we go to the mass shell. In this case, we ought to compare only the terms having a pole in each of the external momenta, p_i, when $p_i^2 \to 0$. Of all the diagrams generated by the extra couplings of (15.7), only those in which the whole effect of the extra vertices can be reduced to a type of self energy insertion to the corresponding external line survive in this limit. The other corrections introduced by (15.7) will not contribute to poles of the Green's functions at $p_i^2 = 0$, and therefore not to the S-matrix. Therefore in the limit $p_i^2 \to 0$, the Coulomb gauge and the Landau gauge (unrenormalized) S-matrix elements will differ by a factor σ^n where n is the number of external lines and σ is a factor independent of n. Comparing the two-point Green's functions in the two gauges C and L:

$$\lim_{p^2\to 0} D'_{\mu\nu}(p;\mathrm{L}) = \frac{Z_\mathrm{L}}{p^2 + i\epsilon}(g_{\mu\nu} + \ldots), \qquad \lim_{p^2\to 0} D'_{\mu\nu}(p;\mathrm{C}) = \frac{Z_\mathrm{C}}{p^2 + i\epsilon}(g_{\mu\nu} + \ldots)$$

we find

$$\sigma^2 = Z_\mathrm{C}/Z_\mathrm{L}.$$

In general, unremormalized S-matrix elements in the two gauges C and L are related to each order by

$$S_\mathrm{C} = \sigma^n S_\mathrm{L} = (Z_\mathrm{C}/Z_\mathrm{L})^{n/2} S_\mathrm{L}$$

so that the renormalized S-matrix element

$$S_\mathrm{ren} \equiv Z_\mathrm{C}^{-n/2} S_\mathrm{C} = Z_\mathrm{L}^{-n/2} S_\mathrm{L}$$

is independent of the gauge chosen to compute it.

In sum, what we have shown here is that $W_\mathrm{C}[J]$ is equal to the expression (15.3) which would be $W_\mathrm{L}[J]$ except that the coefficient of J^μ is A_μ^{go} instead of A_μ. For the S-matrix, the only consequence of this difference is that the renormalization constants attached to each external line depend on the gauge.

Thus we have shown that the S-matrix can be calculated from $W_\mathrm{L}[J]$, not just by the intuitive argument of section 14, but more formally, by obtaining $W_\mathrm{C}[J]$ from first principles, and then demonstrating the equivalence of S_C and S_L.

As pointed out earlier, the only flaw in the above argument is that the singularity at $p_i^2 = 0$ is not in general a simple role.

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16. Generating functionals for Green's functions and proper vertices

In this section we develop the formalism of generating functionals of connected Green's functions and of proper vertices. This topic is slightly out of the main line of development of this review. However, many recent papers on spontaneously broken symmetry make use of this elegant formalism for a very good reason: this formalism allows the discussion of the conditions for spontaneous breakdown of symmetry which goes beyond the one based on the classical Lagrangian and which is valid to all orders in perturbation theory.

Let us go back to the discussion of section 12 on scalar fields. We define the generating functional $Z[J]$ of connected Green's functions by

$$W[\mathrm{J}] = \exp\{\mathrm{i}\, Z[\mathrm{J}]\} = \int [\mathrm{d}\phi]\, \exp\{\mathrm{i} \int \mathrm{d}^4 x [\mathcal{L}[\phi(x)] + \mathrm{J}(x)\cdot \boldsymbol{\phi}(x)]\} \tag{16.1}$$

where $\boldsymbol{\phi}$ and J are multicomponent fields and sources, respectively.

The first derivative of $Z[\mathrm{J}]$ with respect to J_i is

$$\frac{\delta Z[\mathrm{J}]}{\delta J_i(x)} = \frac{1}{W[\mathrm{J}]} \int [\mathrm{d}\phi]\, \phi_i(x) \exp\{\mathrm{i} \int \mathrm{d}^4 x [\mathcal{L}(x) + \mathrm{J}(x)\cdot \boldsymbol{\phi}(x)]\}\,. \tag{16.2}$$

We give it a special name, $\phi_i(x)$:

$$Z[\mathrm{J}]/\delta J_i(x) = \Phi_i(x). \tag{16.3}$$

$\Phi_i(x)$ is the vacuum expectation value of $\phi_i(x)$ in the presence of $J(x)$; i.e., it is the classical field.

The value of eq. (16.2) when the external source is turned off ($\mathrm{J}(x) = 0$) is the vacuum expectation value of the field ϕ:

$$\delta Z[J]/\delta J_i(x)\,|_{\mathrm{J}=0} = v_i. \tag{16.4}$$

Note that v is independent of space-time, since in the limit $\mathrm{J} = 0$, the left-hand side of (16.4) is translationally invariant.

It turns out that higher derivatives of $Z[\mathrm{J}]$ at $\mathrm{J} = 0$ are *Green's functions of the field $\bar{\phi} = \phi - v$ whose vacuum expectation value vanishes.* For example

$$\left.\frac{\delta^2 Z[\mathrm{J}]}{\delta J_i(x)\delta J_j(y)}\right|_{J=0} = \mathrm{i}\,\frac{1}{W[0]} \int [\mathrm{d}\phi]\, [\phi(x) - v]_i [\phi(y) - v]_j \exp(\mathrm{i} \int \mathrm{d}^4 x \mathcal{L})$$

$$= \mathrm{i}\,\frac{1}{W[0]} \int [\mathrm{d}\phi]\, \bar{\phi}_i(x) \bar{\phi}_j(y) \exp\{\mathrm{i} \int \mathrm{d}^4 x \mathcal{L}(x)\} \tag{16.5}$$

as can be verified by differentiating eq. (16.2) with respect to J_j and letting $\mathrm{J} \to 0$. More generally we have

$$\left.\frac{\delta^n Z[J]}{\delta J_{i_1}(x_1)\ldots\delta J_{i_n}(x_n)}\right|_{J=0} = (\mathrm{i})^{n-1} \langle T(\bar{\phi}_{i_1}(x_1)\ldots\bar{\phi}_{i_n}(x_n))\rangle^{\mathrm{c}} \tag{16.6}$$

(where the superscript c denotes the connected part of the Green's function) as can be shown by induction.

We shall now define the Legendre transform $\Gamma[\Phi]$ of $Z[\mathrm{J}]$. It is defined as

$$\Gamma[\Phi] = Z[\mathrm{J}] - \int \mathrm{d}^4x\, \mathrm{J}(x)\cdot\Phi(x), \qquad \delta Z[\mathrm{J}]/\delta J_i = \Phi_i. \tag{16.7}$$

The meaning of eq. (16.7) is this: Γ is a functional of $\Phi(x)$ as defined by the right-hand side of the first equality. In it, J is to be expressed in terms of Φ by inverting eq. (16.3), which defines Φ as a function of J. The Legendre transform (16.7) is a functional version of the well-known transformation familiar in classical mechanics and thermodynamics. By differentiating eq. (16.7) with respect to Φ_i, we find that

$$\delta\Gamma[\Phi]/\delta\Phi_i(x) = \sum_j \int \mathrm{d}^4y\, \{\delta Z[\mathrm{J}]/\delta J_j(y)\}\{\delta J_j(y)/\delta\Phi_i(x)\} - J_i(x) - \sum_j \int \mathrm{d}^4y\, \Phi_j(y)\{\delta J_j(y)/\delta\Phi_i(x)\},$$

or

$$\delta\Gamma[\Phi]/\delta\Phi_i(x) = -J_i(x). \tag{16.8}$$

Equation (16.8) is dual to eq. (16.3): by this we mean that the relation (16.3) which expresses Φ in terms of J is the inverse of eq. (16.8) which expresses J in terms of Φ. This, in particular, means that eq. (16.4) can be written as

$$\delta\Gamma[\Phi]/\delta\Phi_i(x)|_{\Phi=\mathrm{v}} = 0, \tag{16.9}$$

i.e., when $\mathrm{J} = 0$, Φ takes the value v, and vice versa. Equation (16.9) is very important. It expresses the vacuum expectation value v of the field ϕ as the solution to a variational problem: v is the value of Φ which extremizes $\Gamma[\Phi]$.

What is the physical significance of Γ? To streamline our discussion, let us agree on the following convention: We will denote by subscripts i, j ..., any labels J or Φ carry, including the space-time variable x. We will adopt the convention that summations and integrations are always to be carried out over repeated indices. Differentiating eq. (16.3) with respect to Φ, we obtain

$$\frac{\delta^2 Z[\mathrm{J}]}{\delta J_i \delta J_j}\,\frac{\delta J_j}{\delta \Phi_k} = \delta_{ik}. \tag{16.10}$$

From eq. (16.8) we learn that

$$\delta J_i/\delta\Phi_j = -\delta^2\Gamma[\Phi]/\delta\Phi_i\delta\Phi_j. \tag{16.11}$$

Define

$$\{X^{-1}[\mathrm{J}]\}_{ij} = -\delta^2 Z[\mathrm{J}]/\delta J_i\delta J_j \tag{16.12}$$

and

$$\{X[\Phi]\}_{ij} = \delta^2\Gamma[\Phi]/\delta\Phi_i\delta\Phi_j. \tag{16.13}$$

Equations (16.10) and (16.11) mean that

$$(X^{-1})_{ij}(X)_{jk} = \delta_{ik}. \tag{16.14}$$

Since

$$\{X^{-1}[\mathrm{J}=0]\}_{ij} = -\delta^2 Z[\mathrm{J}]/\delta J_i \delta J_j|_{\mathrm{J}=0} = +[\Delta'_{\mathrm{F}}]_{ij}$$

is the full propagator for the barred field, and $\mathrm{J} = 0$ implies $\mathbf{\Phi} = \mathrm{v}$, it follows that

$$\{X[\mathbf{\Phi}=\mathrm{v}]\}_{ij} = \delta^2\Gamma[\mathbf{\Phi}]/\delta\Phi_i\delta\Phi_j|_{\mathbf{\Phi}=\mathrm{v}}$$

is the inverse of the full propagator.

Next differentiate eq. (16.10) with respect to J_l. We obtain

$$-\frac{\delta^3 Z[\mathrm{J}]}{\delta J_i \delta J_j \delta J_l} X_{j\bar{k}}(X^{-1})_{ij} \frac{\delta^3\Gamma}{\delta\Phi_j\delta\Phi_k\delta\Phi_m}(X^{-1})_{lm} = 0$$

or

$$\frac{1}{\mathrm{i}^2}\frac{\delta^3 Z[\mathrm{J}]}{\delta J_i\delta J_j\delta J_k} = (\mathrm{i}\,X^{-1})_{il}(\mathrm{i}\,X^{-1})_{jm}(\mathrm{i}\,X^{-1})_{kn}\left[\mathrm{i}\frac{\delta^3\Gamma[\mathbf{\Phi}]}{\delta\Phi_l\delta\Phi_m\delta\Phi_n}\right]. \qquad (16.15)$$

Now take the limit $\mathrm{J} = 0$, $\mathbf{\Phi} = \mathrm{v}$. In this limit $X^{-1}[\mathrm{J}=0]$ is the full propagator, so that

$$\delta^3\Gamma[\mathbf{\Phi}]/\delta\Phi_i\delta\Phi_j\delta\Phi_k|_{\mathbf{\Phi}=\mathrm{v}} \equiv \Gamma^{(3)}_{ijk} \qquad (16.16)$$

is the three-point proper vertex. A proper vertex (or one-particle irreducible vertex) is a Green's function which cannot be made disconnected by cutting a single internal propagator, and from which (by convention) full propagators corresponding to external lines are removed. The three-point function has no such disconnected graphs except corrections to the propagators, which are explicitly removed in (16.15).

In general, the nth derivative of Γ at $\mathbf{\Phi} = \mathrm{v}$ is the n-point proper vertex:

$$\delta^n\Gamma/\delta\Phi_i\delta\Phi_j \ldots = \Gamma^{(n)}_{ij}\ldots .$$

The proof of this statement proceeds inductively. Assume that $\delta^n Z[\mathrm{J}]/\delta J_i\delta J_j \ldots$ can be expressed as a sum of tree diagrams, each diagram consisting of proper vertices corresponding to $\delta^m\Gamma[\mathbf{\Phi}]/\delta\Phi_i\delta\Phi_j \ldots$, internal lines corresponding to Δ'_{F} connecting pairs of proper vertices, and external lines. In particular,

$$\frac{1}{\mathrm{i}^{n-1}}\frac{\delta^n Z[\mathrm{J}]}{\delta J_i\delta J_j\ldots} = (\mathrm{i}\,X^{-1})_{il}(\mathrm{i}\,X^{-1})_{jm}\ldots\left[\mathrm{i}\frac{\delta^n\Gamma[\mathbf{\Phi}]}{\delta\Phi_l\delta\Phi_m\ldots}\right] + \text{one-particle reducible terms.} \qquad (16.17)$$

Now, differentiate eq. (16.17) with respect to J_k. Recall that

$$\frac{\delta}{\delta J_k} = \frac{\delta\Phi_r}{\delta J_k}\frac{\delta}{\delta\Phi_r} = \frac{\delta^2 Z}{\delta J_k\delta J_r}\frac{\delta}{\delta\Phi_r} = -(X^{-1})_{kr}\delta/\delta\Phi_r . \qquad (16.18)$$

The differential operator $\delta/\delta\Phi_r$, when applied to the right-hand side of eq. (16.17) can act either on some X^{-1}, or on some $\delta^m\Gamma/\delta\Phi_i\delta\Phi_j\ldots$. In the former case, we have

$$\frac{1}{\mathrm{i}}\frac{\delta}{\delta J_l}(\mathrm{i}\,X^{-1})_{kl} = (\mathrm{i}\,X^{-1})_{km}(\mathrm{i}\,X^{-1})_{ln}(\mathrm{i}\,X^{-1})_{ij}\,\mathrm{i}\frac{\delta^3\Gamma}{\delta\Phi_m\delta\Phi_n\delta\Phi_j}$$

which amounts to adding a new external line to a newly created three point vertex, and in the latter

$$\frac{1}{\mathrm{i}}\frac{\delta}{\delta J_i}\frac{\delta^m \Gamma}{\delta\Phi_k\delta\Phi_l\ldots} = (\mathrm{i}\,X^{-1})_{ij}\;\mathrm{i}\,\frac{\delta^{m+1}\Gamma}{\delta\Phi_j\delta\Phi_k\delta\Phi_l\ldots}$$

which amounts to adding a new external line to what used to be an m-point proper vertex. In any case, when the differential operator of eq. (16.15) is applied to the right-hand side of eq. (16.14), we generate *all* tree diagrams for the $(n+1)$-point Green's function, and

$$\frac{1}{\mathrm{i}^n}\frac{\delta^{n+1}Z[\mathrm{J}]}{\delta J_i\delta J_j\ldots} = (\mathrm{i}\,X^{-1})_{il}(\mathrm{i}\,X^{-1})_{jm}\ldots\;\mathrm{i}\,\frac{\delta^{n+1}\Gamma[\Phi]}{\delta\Phi_l\delta\Phi_m\ldots} + \text{one-particle reducible terms.} \tag{16.19}$$

Therefore, in the limit $\mathrm{J} = 0$, $\Phi = \mathrm{v}$,

$$\delta^{n+1}\Gamma[\Phi]/\delta\Phi_l\delta\Phi_m\ldots\big|_{\Phi=\mathrm{v}} = \Gamma^{(n+1)}_{lm\ldots}$$

is the $(n+1)$-point proper vertex. Now our proof is complete, since the induction hypothesis is true for $n = 3$, as shown in eq. (16.15).

The generating functional of proper vertice $\Gamma[\Phi]$ has the representation:

$$\Gamma[\Phi] = \sum_{n=2}^{\infty}\frac{1}{n!}\Gamma^{(n)}_{i_1,i_2,i_3,\ldots,i_n}(\Phi-v)_{i_1}(\Phi-v)_{i_2}\ldots(\Phi-v)_{i_n} \tag{16.20}$$

with

$$\Gamma^{(2)}_{ij} = [\Delta_{\mathrm{F}}^{-1}]_{ij}. \tag{16.21}$$

Let us revert to the standard notation:

$$\Gamma^{(n)}_{i_1,i_2,\ldots,i_n} = \Gamma^{(n)}_{i_1,i_2\ldots i_n}(x_1, x_2, \ldots x_n).$$

Because of the translational invariance $\Gamma^{(n)}$ depends only on $n-1$ differences $x_i - x_j$, so that its Fourier transform $\widetilde{\Gamma}^{(n)}$ is defined as

$$\widetilde{\Gamma}^{(n)}_{i_1\ldots i_n}(p_1,\ldots p_n)(2\pi)^4\delta^4(p_1+\ldots+p_n) = \left(\prod_{i=1}^{n}\int \mathrm{d}^4x\exp(\mathrm{i}p_ix_i)\right)\Gamma^{(n)}_{i_1\ldots i_n}(x_1,\ldots,x_n). \tag{16.22}$$

This means that four-momentum must be conserved at vertices.

In discussing the implications of the condition (16.9), it is convenient to consider the case in which Φ is a constant ϕ independent of space-time. Define the super-potential $\mathcal{V}$ by

$$\Gamma[\Phi = \phi] = -(2\pi)^4\delta^4(0)\,\mathcal{V}(\phi),$$

$$\mathcal{V}(\phi) = -\sum_{n=2}^{\infty}\frac{1}{n!}\widetilde{\Gamma}^{(n)}_{i_1,i_2\ldots i_n}(0,0,\ldots,0)(\phi-v)_{i_1}(\phi-v)_{i_2}\ldots(\phi-v)_{i_n}, \tag{16.23}$$

so that

$$\left.\frac{\mathrm{d}^N \mathcal{V}(\phi)}{\delta\phi_{i_1}\delta\phi_{i_2}\ldots\mathrm{d}\phi_{i_n}^N}\right|_{\phi=v} = -\widetilde{\Gamma}^{(N)}_{i_1,i_2\ldots i_n}(0,0,\ldots,0)$$

is the negative of the N-point proper vertex evaluated at the point where all external momenta vanish. The condition (16.9) translates into

$$\left.\frac{\mathrm{d}\mathcal{V}(\phi)}{\mathrm{d}\phi_i}\right|_{\phi=\mathrm{v}} = 0. \tag{16.24}$$

Furthermore,

$$\frac{\mathrm{d}^2\mathcal{V}(\phi)}{\delta\phi_i \mathrm{d}\phi_j} = -[\widetilde{\Delta}_\mathrm{F}^{-1}(0)]_{ij} \tag{16.25}$$

is positive semi-definite, since $\widetilde{\Delta}_\mathrm{F}^{-1}$ behaves like $(p^2 - m^2)$ near $p^2 \approx m^2$ and it cannot have any other zero for $p^2 < m^2$. Thus the vacuum expection value $\phi = \mathrm{v}$ is the value of ϕ which minimizes $\mathcal{V}(\phi)$. The discussion in section 2 suggests that $\phi = \mathrm{v}$ must be the absolute minimum of $\mathcal{V}$, but we do not prove it here.

When $\mathcal{L}$ is invariant under

$$\phi_i \to \phi_i - \mathrm{i}\theta^\alpha L^\alpha_{ij}\phi_j$$

it follows from the structure of eq. (16.1) that $Z[J]$ is invariant under

$$J_i \to J_i - \mathrm{i}\theta^\alpha L^\alpha_{ij} J_j,$$

and so on, and finally the superpotential $\mathcal{V}(\phi)$ is an invariant function of ϕ under the above transformation. The analysis of section 2 on the potential V can now be applied verbatim to the superpotential $\mathcal{V}$, with $-[\Delta'_\mathrm{F}(0)]_{ij}$ of eq. (16.25) taking the place of M^2_{ij} of eq. (2.19). We find therefore that the occurrence and the number of the Goldstone bosons discussed there are true to all orders of perturbation theory.

We can construct $Z[\mathrm{J}]$, $\Gamma[\Phi]$ and $\mathcal{V}[\phi]$ in perturbation theory. For simplicity we shall consider the case of a single-component field. An effective way of expanding these quantities in a series is to write eq. (16.1) with a fictitious parameter a:

$$\exp\{\mathrm{i}Z[\mathrm{J}]\} = \int[\mathrm{d}\phi]\exp\left[\mathrm{i}\int \mathrm{d}^4x\left\{\frac{1}{a}\mathcal{L}(x) + \mathrm{J}(x)\cdot\phi(x)\right\}\right]$$

$$\sim \exp\left\{\mathrm{i}\int \mathrm{d}^4x\,\frac{1}{a}\,\mathcal{L}_\mathrm{I}\left[\frac{1}{\mathrm{i}}\,\frac{\delta}{\delta \mathrm{J}(x)}\right]\right\}\exp\left\{\tfrac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,a\mathrm{J}(x)\Delta_\mathrm{F}(x-y)\mathrm{J}(y)\right\}, \tag{16.26}$$

and expand $Z[\mathrm{J}]$ in powers of a and let $a = 1$ afterwards. Since each propagator is multiplied by a and each vertex by a^{-1} when we use eq. (16.26) as the definition of Z, it follows that a Feynman diagram with E external lines, I internal lines and V vertices is multiplied by the factor, a^{E+I-V}. There is a topological relation that holds for any Feynman diagram. It is

$$L = I - V + 1$$

where L is the number of loops (i.e., the number of independent four-momentum integrations) in the diagram. Therefore the expansion in this fictitious parameter a corresponds to expanding a Green's function in the number of loops in the Feynman diagrams. The reason this expansion is preferable over the expansion in powers of some coupling constant is that in the former any symmetry of the Lagrangian is preserved in each order of perturbation theory since, effectively, a multiplies the whole Lagrangian. In contrast, if we were to split up the Yang-Mills Lagrangian into a free and perturbing parts to develop a perturbation expansion, for example, each part would not be separately gauge invariant and the consequences of gauge invariance of the Lagrangian might not manifest themselves in each order of perturbation series. (Recall that non-Abelian gauge transformations depend on the coupling constant.)

In the following we shall discuss explicit constructions of Z, Γ and $\mathcal{V}$ in the first two orders of loop expansion for a simple model:

$$\mathcal{L} = \tfrac{1}{2}(\partial^\mu \phi)^2 - \tfrac{1}{2}\mu_0^2\phi^2 - \tfrac{1}{4}\lambda_0 \phi^4 . \tag{16.27}$$

The method can be generalized easily to other models. Our discussion will not show that our construction is in fact the expansion in the number of loops, but the interested student can convince himself of this fact by first referring to Nambu's paper which shows that the loop expansion is also an expansion in the Planck constant h, and then noting that our method is an asymptotic evaluation of these quantities in h.

Imagine that eq. (16.1) is written in the Euclidean space as explained in section 12. Since the exponent in the right-hand side is bounded from above in this case, we are tempted to evaluate the functional integral by the method of steepest descent. We shall keep the Minkowsky notation for simplicity, but the ultimate justification of this method lies in the Euclidicity postulate.

We shall expand the exponent on the right-hand side of eq. (16.1):

$$S[\phi] + \int d^4x\, J(x)\phi(x) = \int d^4x \{\mathcal{L}(x) + J(x)\phi(x)\},$$

about a point $\phi(x) = \phi_0(x)$:

$$S[\phi] + \int d^4x J(x)\phi(x) = S[\phi_0] + \int d^4x J(x)\phi_0(x) + \int d^4x \left\{\frac{\delta S[\phi_0]}{\delta\phi_0(x)} + J(x)\right\} [\phi(x) - \phi_0(x)]$$

$$+ \frac{1}{2!}\int d^4x\, d^4y [\phi(x) - \phi_0(x)] [\phi(y) - \phi_0(y)] \frac{\delta^2 S[\phi_0]}{\delta\phi_0(x)\delta\phi_0(y)} + \ldots \tag{16.28}$$

and choose ϕ_0 so that the term linear in $\phi - \phi_0$ is missing from the expansion of eq. (16.28). This will be achieved provided

$$\delta S[\phi_0]/\delta\phi_0(x) = -J(x) \tag{16.29}$$

which means that ϕ_0 is the solution of the classical (non-quantized) field equation in the presence of the external source $J(x)$. For the Lagrangian (16.27), eq. (16.29) is

$$(\partial^2 + \mu^2)\phi_0(x) + \lambda\, \phi_0^3(x) = J(x). \tag{16.30}$$

In any case, ϕ_0 is obtained from eq. (16.29) as a functional of the external source J.

When eq. (16.28) is substituted in eq. (16.1), we obtain

$$\exp(\mathrm{i}\, Z[J]) = \exp\{\mathrm{i}\, S[\phi_0] + \mathrm{i}\int \mathrm{d}^4x J(x)\phi_0(x)\}$$

$$\times \int [\mathrm{d}\phi] \exp\left\{\mathrm{i}\left[\int \mathrm{d}^4x\, \mathrm{d}^4y \,\frac{1}{2!}\, \frac{\delta^2 S[\phi_0]}{\delta\phi(x)\delta\phi(y)} (\phi(x) - \phi_0(x))(\phi(y) - \phi_0(y)) + ...\right]\right\} \quad (16.31)$$

The lowest order approximation (which is one order lower than the steepest descent approximation) is obtained if we ignore the functional integral over $\phi(x)$ altogether and set

$$Z[J] \approx S[\phi_0] + \int \mathrm{d}^4x J(x)\phi_0(x) \equiv Z^0[J] \quad (16.32)$$

which is a functional of J only, because ϕ_0 is a functional of J. We can evaluate Z^0 explicitly by first solving for ϕ_0 in eq. (16.30) and then substituting that ϕ_0 in eq. (16.32). Equation (16.30) can be solved in powers of λ:

$$\phi_0(x) = -\int \mathrm{d}^4y\; \Delta_\mathrm{F}(x-y;\mu^2)J(y) - \lambda[\int \mathrm{d}^4y\; \Delta_\mathrm{F}(x-y;\mu^2)J(y)]^3 + ... \quad (16.33)$$

where the use of Δ_F is dictated by the Euclidicity postulate. When eq. (16.33) is substituted in eq. (16.32), one finds that $Z^0[J]$ is the generating functional of Green's functions in the tree- (i.e., no loop) approximation:

$$Z^0[J] = -\tfrac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y J(x)\Delta_\mathrm{F}(x-y;\mu^2)J(y) + \frac{\lambda}{4}\int \mathrm{d}^4u \prod_{i=1}^{4} \int \mathrm{d}^4x_i J(x_i)\Delta_\mathrm{F}(x_i - u;\mu^2) + ... \quad (16.34)$$

We can see more readily that Z^0 is the tree approximation to Z if we compute $\Gamma[\Phi]$ in this approximation. Since

$$\Phi(x) = \frac{\delta Z}{\delta J(x)} \approx \frac{\delta Z^0}{\delta J(x)} = \int \mathrm{d}^4y \left\{\frac{\delta S[\phi_0]}{\delta\phi_0(y)}\,\frac{\delta\phi_0(y)}{\delta J(x)} + J(y)\,\frac{\delta\phi_0(y)}{\delta J(x)}\right\} + \phi_0(x),$$

we have, to this order,

$$\Phi(x) = \phi_0(x). \quad (16.35)$$

Therefore, $\Gamma[\Phi]$ can be computed to this order:

$$\Gamma^0[\Phi] \approx Z^0[J] - \int \mathrm{d}^4x J(x)\Phi(x)$$

$$= \{S[\Phi] + \int \mathrm{d}^4x J(x)\Phi(x)\} - \int \mathrm{d}^4x J(x)\Phi(x) = S[\Phi]. \quad (16.36)$$

So, to this order, proper vertices are generated by the Lagrangian itself and Green's functions are built up of these unmodified vertices by the rules of tree graphs. The superpotential $\mathcal{V}$ [eq. (16.36)] is, to this order

$$\mathcal{V}(\phi) = -S(\phi) = V(\phi)$$

where ϕ is independent of space-time and V is the negative of the part of the Lagrangian which is independent of derivative of fields. That is, $V(\phi)$ is the potential of the field ϕ. This justifies the name "super-potential" for $\mathcal{V}$.

We can proceed further by applying the steepest descent method to the functional integral in eq. (16.31). This consists of neglecting terms higher than quadratic in $(\phi - \phi_0)$ in the exponent of the integrand and performing the functional Gaussian integration. In this way we obtain

$$\int[\mathrm{d}\phi]\exp\left\{\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\frac{1}{2!}\,\frac{\delta^2 S[\phi_0]}{\delta\phi_0(x)\delta\phi_0(y)}\,[\phi(x)-\phi_0(x)]\,[\phi(y)-\phi_0(y)]\right\}$$

$$\approx \frac{1}{\sqrt{\det \delta^2 S[\phi_0]/\delta\phi_0(x)\delta\phi_0(y)}} = \exp(-\tfrac{1}{2})\mathrm{Tr}\ln\frac{\delta^2 S[\phi_0]}{\delta\phi_0(x)\,\delta\phi_0(y)}\,,$$

so that

$$Z[J] \approx Z^0[J] + \tfrac{1}{2}\mathrm{i}\,\mathrm{Tr}\,\ln\{\delta^2 S[\phi_0]/\delta\phi_0(x)\delta\phi_0(y)\} \equiv Z^1[J]\,. \tag{16.37}$$

For the Lagrangian (16.27), for example,

$$\delta^2 S/\delta\phi(x)\delta\phi(y) = (-\partial^2 - \mu^2 - 3\lambda\phi^2(x))\delta^4(x-y),$$

so that

$$\frac{\mathrm{i}}{2}\mathrm{Tr}\,\ln\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)} \approx \frac{\mathrm{i}}{2}\mathrm{Tr}\,\ln\left(1 - 3\lambda\,\frac{1}{-\partial^2-\mu^2+\mathrm{i}\epsilon}\,\phi^2\right)$$

$$= -\frac{\mathrm{i}}{2}\sum_{n=1}\frac{(-3\mathrm{i}\lambda)^n}{n}\int \mathrm{d}^4x_1\ldots\mathrm{d}^4x_n\Delta_\mathrm{F}(x_1-x_2)\phi^2(x_2)\Delta_\mathrm{F}(x_2-x_3)\ldots\Delta_\mathrm{F}(x_n-x_1)\phi^2(x_1). \tag{16.38}$$

Let us now construct $\Gamma[\Phi]$ to this order:

$$\Gamma^1[\Phi] \equiv Z^1[J] - \int \mathrm{d}^4x J(x)\Phi(x), \tag{16.39}$$

where

$$\Phi(x) = \delta Z^1[J]/\delta J(x) \equiv \phi_0(x) + \epsilon(x) \tag{16.40}$$

and $\epsilon(x)$ is given by

$$\epsilon(x) = \frac{\delta}{\delta J(x)}\,\frac{\mathrm{i}}{2}\,\underset{\xi,\,\eta}{\mathrm{Tr}}\,\ln\frac{\delta^2 S[\phi_0]}{\delta\phi_0(\xi)\delta\phi_0(\eta)}\,.$$

Fortunately, it is not necessary to know the form of $\epsilon(x)$ to construct $\Gamma[\Phi]$ to first order in $\epsilon(x)$, as we shall demonstrate presently. First, note that

$$Z^0[J] = S[\phi_0] + \int \mathrm{d}^4x J(x)\phi_0(x)$$

$$= S[\Phi] + \int \mathrm{d}^4x J(x)\Phi(x) - \int \mathrm{d}^4y\left\{\frac{\delta S[\phi_0]}{\delta\phi_0(y)} + J\right\}\epsilon(y) + \mathrm{O}(\epsilon^2)$$

$$= S[\Phi] + \int \mathrm{d}^4x J(x)\Phi(x) + \mathrm{O}(\epsilon^2) \tag{16.41}$$

by virtue of eq. (16.29). Therefore to order ϵ, we have from eqs. (16.37), (16.39) and (16.41)

$$\Gamma^1[\Phi] = S[\Phi] + \frac{i}{2} \mathop{\mathrm{Tr}}_{\xi,\eta} \ln \frac{\delta^2 S[\Phi]}{\delta\Phi(\xi)\delta\Phi(\eta)}. \tag{16.42}$$

The second term is the one-loop correction to the generating functional of proper vertices.

The super-potential $\mathcal{V}$ can be evaluated explicitly from eqs. (16.38) and (16.42). Recalling the definition of $\mathcal{V}$ of eq. (16.23), we find

$$\mathcal{V}(\phi) = +\frac{\mu_0^2}{2}\phi^2 + \frac{\lambda_0}{4}\phi^4 + \frac{i}{2}\int \frac{d^4k}{(2\pi)^4} \sum_{N=1} \frac{1}{N}\left(\frac{-3\lambda\phi^2}{k^2-\mu^2+i\epsilon}\right)^N. \tag{16.43}$$

The terms for $N = 1$ and 2 are divergent. However these terms are proportional to ϕ^2 and ϕ^4 and the divergences in these terms can be amalgamated with μ_0^2 and λ_0.

We may write

$$\mathcal{V}(\phi) = \tfrac{1}{2}\mu^2\phi^2 + \tfrac{1}{4}\lambda\phi^4 + J(\phi^2)$$

where

$$J(\phi^2) = \frac{i}{2}\int \frac{d^4k}{(2\pi)^4} \sum_{N \geq 3} \frac{1}{N}\left(\frac{-3\lambda\phi^2}{k^2-\mu^2+i\epsilon}\right)^N \tag{16.44}$$

and μ^2 and λ are defined as the value of the two- and four-point vertices at the point where all external momenta vanish.

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17. Renormalization in the σ-model

The formalism developed in the preceeding section is useful in discussing renormalization of spontaneously broken symmetry models and, in particular, the σ-model. In the generic sense, the σ-model is a model in which a symmetry is broken by a term of dimension one, i.e., by a term proportional to a boson field.

A simple example of this kind of models is

$$\mathcal{L}=\tfrac{1}{2}[(\partial_\mu\pi)^2+(\partial_\mu\sigma)^2]-\tfrac{1}{2}\mu_0^2(\sigma^2+\pi^2)-\tfrac{1}{4}\lambda_0(\sigma^2+\pi^2)^2+c\sigma\equiv\mathcal{L}_{\rm sym}+c\sigma \tag{17.1}$$

which is a two-dimensional generalization of the model discussed in the preceeding section. Except for the last term $c\sigma$, the Lagrangian (17.1) is the one studied in section 2, and it was noted there that this Lagrangian is invariant under a U(1) transformation of the fields σ and π. The salient features of this model are that the "almost" conserved current

$$A_\mu=\pi\partial_\mu\sigma-\sigma\partial_\mu\pi \tag{17.2}$$

has a divergence proportional to the π-field

$$\partial^\mu A_\mu=c\pi \tag{17.3}$$

and that the σ-field acquires a nonvanishing vacuum expectation value thanks to the last term in eq. (17.1). Equation (17.3) is a version of the PCAC condition, and for this reason the model is of some physical interest.

It pays to study first the classical solution of the Lagrangian (17.1). The potential is given by

$$V(\sigma,\pi)=\tfrac{1}{4}\lambda(\sigma^2+\pi^2)^2+\tfrac{1}{2}\mu^2(\sigma^2+\pi^2)-c\sigma \tag{17.4}$$

(we drop the subscript 0 on λ and μ^2 for the moment). The minimum of the potential occurs at $\pi=0$ and $\sigma=u$ where

$$u(\mu^2+\lambda u^2)=c, \tag{17.5}$$

u being the vacuum expectation value of the σ-field in this approximation. If we displace the field σ by the amount u and define s by $s=\sigma-u$ eq. (17.1) takes the form

$$\mathcal{L}=\tfrac{1}{2}[(\partial_\mu s)^2+(\partial_\mu\pi)^2]-\tfrac{1}{2}\mu_\sigma^2 s^2-\tfrac{1}{2}\mu_\pi^2\pi^2-\tfrac{1}{4}\lambda(s^2+\pi^2)^2-\lambda u(s^2+\pi^2)s \tag{17.6}$$

so that in this approximation the s-field represents a particle of mass μ_σ^2:

$$\mu_\sigma^2=\mu^2+3\lambda u^2 \tag{17.7}$$

and the π-field a particle of mass μ_π^2:

$$\mu_\pi^2=\mu^2+\lambda u^2. \tag{17.8}$$

In this approximation, when $c=0$, i.e., when the Lagrangian is invariant under the U(1) transformation, either $u=0$ or $\mu_\pi^2=\mu^2+\lambda u^2=0$ according to eq. (17.5). If $u^2=0$, then $\mu^2\geqslant 0$ in order that $\mu_0^2=\mu_\pi^2=\mu^2\geqslant 0$. This is the "usual" way the symmetry of the Lagrangian manifests itself: the particles corresponding to the fields σ and π are degenerate. On the other hand, if $\mu_\pi^2=0$, we must have $\mu^2 < 0$ since $\lambda u^2 > 0$. The second case is the Goldstone mode of the symmetry with the field π playing the role of the Goldstone boson. In that case, $\mu^2 = -\lambda u^2$, and $\mu_\sigma^2 = -2\mu^2 > 0$. For more thorough discussion of the σ-model, see the monograph "Chiral Dynamics" by one of us.

We return to the discussion of the full solution, including radiative corrections. An important fact about the σ-model is that the Green's function of this model are generated by the generating functional of Green's functions of the symmetric theory. The latter is given by

$$\exp\{\mathrm{i}\, Z[\mathbf{J}]\} = \int [\mathrm{d}\sigma]\,[\mathrm{d}\pi]\exp\{\mathrm{i}\int \mathrm{d}^4x\,[\mathcal{L}_{\mathrm{sym}}(x) + J_\sigma(x)\sigma(x) + J_\pi(x)\pi(x)]\},$$

$$\mathbf{J} = (J_\sigma, J_\pi). \tag{17.9}$$

Now, expand $Z[\mathbf{J}]$ in $\mathbf{J}$ about $J_\sigma = c$ and $J_\pi = 0$. We have

$$\frac{1}{\mathrm{i}^{n+m-1}}\,\frac{\delta^{n+m+1}Z}{\delta J_\sigma(x_1)\ldots\delta J_\sigma(x_n)\delta J_\pi(y_1)\ldots\delta J_\pi(y_n)}\bigg|_{J_\sigma=c,\,J_\pi=0} \tag{17.10}$$

$$= \frac{1}{W[c,\,0]}\int[\mathrm{d}\sigma]\,[\mathrm{d}\pi]\,s(x_1)\ldots s(x_n)\pi(y_1)\ldots\pi(y_m)\,\exp\{\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}(x)\} - \text{disconnected pieces},$$

where $s = \sigma - u$, u being the vacuum expectation value of σ so that

$$\int[\mathrm{d}\sigma]\,[\mathrm{d}\pi]\,s(x)\exp\{\mathrm{i}\int \mathrm{d}^4y\,\mathcal{L}(y)\} = 0, \qquad s(x) \equiv \sigma(x) - u. \tag{17.11}$$

and

$$W[c,\,0] = \int[\mathrm{d}\sigma]\,[\mathrm{d}\pi]\exp\{\mathrm{i}\int \mathrm{d}^4x\,[\mathcal{L}_{\mathrm{sym}}(x) + c\sigma(x)]\} = \int[\mathrm{d}\sigma]\,[\mathrm{d}\pi]\exp\{\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}(x)\}, \tag{17.12}$$

is the vacuum-to-vacuum amplitude of the σ-model. To recapitulate; if we expand $[\mathbf{J}]$ about $\mathbf{J} = 0$, the expansion coefficients are the Green's functions of the symmetric model (i.e., the theory given by the Lagrangian $\mathcal{L}_{\mathrm{sym}}$); if we expand $Z[\mathbf{J}]$ about $\mathbf{J} = (c, 0)$, they are the Green's functions of the σ-model.

The point is simply that the symmetry-breaking term $c\sigma$ has the form of an external source term $J\sigma$ for constant $J = c$. This important theorem has an analog in terms of Γ. Since

$$\delta\Gamma[\Phi]/\delta\Phi_i(x) = -J_i(x), \qquad \Gamma[\Phi] = Z[\mathbf{J}] - \int \mathrm{d}^4x\,\mathbf{J}(x)\cdot\Phi(x) \tag{17.13}$$

where

$$\delta Z[\mathbf{J}]/\delta J_i(x) = \Phi_i(x) \tag{17.14}$$

we have from eq. (17.13)

$$\delta\Gamma[\Phi]/\delta\Phi_i(x)\,\big|_{\Phi=\mathbf{u}} = -c_i \tag{17.15}$$

which is the analog of eq. (16.9). Eq. (17.5) is the lowest order version of (17.15). Furthermore, we can repeat the analysis leading to eq. (16.17), but this time taking the limit $\mathbf{J} = \mathbf{c}$ and $\Phi = \mathbf{u}$, to find that

$$\frac{\delta^n\Gamma[\Phi]}{\delta\Phi_{i_1}\delta\Phi_{i_2}\ldots\delta\Phi_{i_n}}\bigg|_{\Phi=\mathbf{u}} \equiv \Gamma^{(n)}_{i_1,i_2\ldots i_n}(\mathbf{u}) \tag{17.16}$$

is the proper n-point vertex of the σ-model. (In eq. (17.16) we have reverted to the convention of representing the internal symmetry index and the space-time variable x collectively by an index i.) To recapitulate, the generating functional of proper vertices of the symmetric theory generates the Green's functions of the σ-model when it is expanded about $\Phi = \mathbf{u}(c)$, where $\mathbf{u}(c)$ is given by eq. (17.15). As was shown in the preceeding section, $\Gamma[\Phi] \sim S[\Phi]$ to lowest order, so that eq. (17.5) follows from eq. (17.15).

Let us now consider the limit $c \to 0$ of eq. (17.15). Equation (17.15) is really an equation which determines the vacuum expectation value u in terms of c. To study the ramifications of eq. (17.15) it suffices to consider the superpotential defined in eq. (16.20):

$$\Gamma[\Phi = \phi] = -(2\pi)^4 \delta^4(0) \mathcal{V}(\phi)$$

where ϕ is independent of space-time. Eq. (17.15) is equivalent to

$$\delta \mathcal{V}(\phi)/\delta\phi_i \Big|_{\phi = \mathrm{u}} = c_i \, . \tag{17.17}$$

The limit of $\mathrm{u}(c)$ as $c \to 0$ may or may not vanish, depending on the parameters of the symmetric Lagrangian. If it does not, i.e., $\mathrm{u}(0) = v \neq 0$, the symmetry of the Lagrangian is spontaneously broken.

Let us consider, however, the case in which the parameters of the symmetric Lagrangian are such that $\mathrm{u}(0) = 0$, that is, the case in which the symmetry is manifested in the usual way. From eq. (17.16) it follows that

$$\Gamma[\Phi] = \sum_{n=2} \frac{1}{n!} (\Phi - u)_{i_1} (\Phi - u)_{i_2} \ldots (\Phi - u)_{i_n} \Gamma^{(n)}_{i_1, i_2 \ldots i_n}(\mathrm{u}). \tag{17.18}$$

Further, the analog of the relation

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n f(x) \Bigg|_{x=a} = \sum_{m=0} \frac{a^m}{m!} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n+m} f(x) \Bigg|_{x=0}$$

gives

$$\Gamma^{(n)}_{i_1, i_2 \ldots i_n}(\mathrm{u}) = \sum_{m=0} \frac{1}{m!} u_{j_1} u_{j_2} \ldots u_{j_m} \Gamma^{(n+m)}_{i_1, i_2, \ldots i_n, j_1, j_2, \ldots j_m}(\mathrm{u} = 0)$$

or, in momentum space,

$$\widetilde{\Gamma}^{(n)}_{i_1, i_2 \ldots i_n}(p_1, p_2 \ldots, p_n; \mathrm{u}) = \sum_{m=0} \frac{1}{m!} u^m \widetilde{\Gamma}^{(n+m)}_{i_1, i_2, \ldots i_n, \underbrace{1, 1, \ldots 1}_{m}}(p_1, p_2, \ldots p_m, \underbrace{0, 0, \ldots 0}_{m}). \tag{17.19}$$

In eq. (17.19), the indices i's and j's stand for σ or π and $\widetilde{\Gamma}_{ij\ldots}(p, q, \ldots) \equiv \widetilde{\Gamma}_{ij\ldots}(p, q, \ldots; u = 0)$ is the momentum space proper vertex of the symmetric theory ($c = 0$, $\mathrm{u} = \mathrm{v} = 0$).

Equation (17.19) is important in that it affords us a handle for removing the divergences from the σ-model if we know how to renormalize the symmetric model, since eq. (17.19) expresses the proper vertex of the σ-model in terms of proper vertices of the symmetric theory. We shall give a brief review of the renormalization theory in the next section, but suffice it to say for the moment that if we write the Lagrangian of the symmetric theory as

$$\mathcal{L} = \tfrac{1}{2}[(\partial\sigma)^2 + (\partial\pi)^2 - \mu^2(\sigma^2 + \pi^2)] - \tfrac{1}{4}\lambda(\sigma^2 + \pi^2)^2$$
$$+ \tfrac{1}{2}(Z_3 - 1)[(\partial\sigma)^2 + (\partial\pi)^2 - \mu^2(\sigma^2 + \pi^2)] - \tfrac{1}{2}\delta\mu^2(\sigma^2 + \pi^2) - \tfrac{1}{4}\delta\lambda(\sigma^2 + \pi^2)^2 \qquad (17.20)$$

where μ^2 and λ are finite constants, and choose Z_3, $\delta\mu^2$, $\delta\lambda$ in an appropriate way, then all infinities of the theory can be removed. Thus starting from the Lagrangian it is possible to construct a *finite* generating functional $\Gamma[\Phi]$ for $\mathbf{u} = 0$. Once we have a renormalized (i.e., finite) expression for $\Gamma[\Phi]$, we can expand it about $\Phi = \mathbf{u}$, where u is determined from eq. (17.15), to recover the proper vertices of the σ-model characterized by the parameters λ, μ^2 and c.

Finally, we turn to the Ward-Takahashi identities of the model. Since $\mathcal{V}(\phi)$ is the generating function of zero-momentum proper vertices of the symmetric theory when we expand it about $\phi = 0$, it follows that $\mathcal{V}$ is a function of the invariant of ϕ, i.e., of $\phi^2 = \phi_\sigma^2 + \phi_\pi^2$. Thus eq. (17.17) takes the form

$$2\phi_\sigma \delta\mathcal{V}(\phi)/\delta(\phi^2)\Big|_{\phi_\sigma = u,\, \phi_\pi = 0} = c. \qquad (17.21)$$

Since the inverse π-propagator at zero momentum is given by [see (16.25)]

$$-\Delta_\pi^{-1}(0) = \delta^2\mathcal{V}(\phi)/\delta\phi_\pi\,\delta\phi_\sigma\Big|_{\phi_\pi = 0, \phi_\sigma = u} = 2\delta\mathcal{V}(\phi)/\delta(\phi^2)\Big|_{\phi = \mathbf{u}} \qquad (17.22)$$

it follows that

$$-u\Delta_\pi^{-1}(0) = c \qquad (17.23)$$

from which the value of u can be determined conveniently, if we know $\Delta_\pi^{-1}(0)$ in terms of λ, μ^2 and u.

The above prescription for constructing renormalized proper vertices of the σ-model works if $\mu^2 > 0$, since in that case there is a comparison symmetric theory that makes sense. However, once $\Gamma[\Phi]$ is constructed in terms of λ, μ^2 and c there is nothing that stops us from expressing $\Gamma[\Phi]$ in terms of λ, u and m_π^2, where the last is defined as

$$m_\pi^2 = -\Delta_\pi^{-1}(0) = 2\delta\mathcal{V}(\phi)/\delta\phi^2\Big|_{\phi = \mathbf{u}}$$

and taking the limit $m_\pi^2 \to 0$. Then eq. (17.23) reads

$$um_\pi^2 = c. \qquad (17.24)$$

Equation (17.24) is the renormalized Goldstone theorem: if $c = 0$ either $u = 0$, or $m_\pi^2 = 0$. The latter corresponds to the Goldstone mode. In this case the basic parameters of the theory can be taken to be λ and $u = v$, instead of λ and $-\mu^2$.

The moral of the above discussion is that the renormalizability of the σ-model in the Goldstone mode depends only on the renormalizability of the symmetric theory. The process of renormalization does not induce additional symmetry breaking, in the sense that the symmetric counterterms exhibited in (17.20) suffice to remove infinities from the theory whether or not the symmetry is broken externally ($c \neq 0$) or internally ($v \neq 0$).

Later we will discuss a way of renormalizing the σ-model without making explicit reference to the symmetric theory. This method makes use of the Ward-Takahashi identities. Let us derive them. The generating functional $Z[J]$ in eq. (17.9) is invariant under the U(1) transformation of the external sources:

$$\begin{pmatrix} J_\sigma \\ J_\pi \end{pmatrix}' = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} J_\sigma \\ J_\pi \end{pmatrix} \tag{17.25}$$

as can be seen by making the change of integration variables

$$\begin{pmatrix} \sigma \\ \pi \end{pmatrix}' = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sigma \\ \pi \end{pmatrix} \tag{17.26}$$

which leaves the scalar product $J_\sigma \sigma + J_\pi \pi$ invariant. Therefore,

$$\mathrm{d}Z/\mathrm{d}\theta = 0,$$

or

$$\int \mathrm{d}^4x \left\{ \frac{\delta Z[\mathrm{J}]}{\delta J_\sigma(x)} J_\pi(x) - \frac{\delta Z[\mathrm{J}]}{\delta J_\pi(x)} J_\sigma(x) \right\} = 0. \tag{17.27}$$

Substituting eqs. (17.23) and (17.24) into eq. (17.27), we find that

$$\int \mathrm{d}^4x \left\{ \Phi_\sigma(x) \frac{\delta\Gamma[\Phi]}{\delta\Phi_\pi(x)} - \Phi_\pi(x) \frac{\delta\Gamma[\Phi]}{\delta\Phi_\sigma(x)} \right\} = 0 \tag{17.28}$$

which shows that Γ is an invariant functional of Φ under the U(1) transformation:

$$\begin{pmatrix} \Phi_\sigma \\ \Phi_\pi \end{pmatrix}' = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \Phi_\sigma \\ \Phi_\pi \end{pmatrix}. \tag{17.29}$$

Note that the invariance of Γ under the transformation (17.29) is true whether $\mu^2 > 0$ or $\mu^2 < 0$. The renormalized Γ constructed according to the prescription above, thus satisfies eq. (17.28) as we continue m_π^2 to zero.

Equation (17.28) is the Ward-Takahashi identity for the generating functional of proper vertices. An infinite number of Ward-Takahashi identities is obtained if we differentiate eq. (17.28) with respect to Φ_π and Φ_σ repeatedly, and set $\Phi_\pi = 0$, $\Phi_\sigma = u$. If we differentiate eq. (17.28) with respect to Φ_π and set $\Phi_\pi = 0$, $\Phi_\sigma = u$, we obtain the "eigenvalue" equation for u, eq. (17.23). If we differentiate it with respect to Φ_π and Φ_σ and take the limit, we obtain

$$\Delta_\sigma^{-1}(p^2) - \Delta_\pi^{-1}(p^2) = u\Gamma_{\sigma\pi\pi}(p; 0, -p). \tag{17.30}$$

An important lesson to be learned here is that the Ward-Takahashi identity for the generating functional for proper vertices is the same, whether or not the symmetry is spontaneously broken. It is satisfied by the generating functional constructed first in the symmetric theory and then continued to the Goldstone mode by varying an appropriate parameter of the theory.

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18. BPHZ renormalization

In this section we will give a brief survey of renormalization theory developed and perfected in recent years by Bogoliubov, Parasiuk, Hepp and Zimmermann (BPHZ). Nothing will be proved, but we will try to give definitions and theorems in a precise manner.

First, we will give some definitions. The interaction Lagrangian is a sum of terms $\mathcal{L}_i$ which is a product of b_i boson fields and f_i fermion fields with d_i derivatives. The vertex of the ith type arising from $\mathcal{L}_i$ has the index δ_i defined as

$$\delta_i = b_i + \tfrac{3}{2} f_i + d_i - 4 = \dim \mathcal{L}_i - 4. \tag{18.1}$$

Let Γ be a one-particle irreducible (IPI) diagram (i.e., a diagram that cannot be made disconnected by cutting only one line). Let E_B and E_F be the numbers of external boson and fermion lines, I_B and I_F the numbers of internal boson and fermion lines, n_i the number of vertices of the ith type. Then

$$E_\mathrm{B} + 2I_\mathrm{B} = \sum_i n_i b_i \tag{18.2}$$

$$E_\mathrm{F} + 2I_\mathrm{F} = \sum_i n_i f_i . \tag{18.3}$$

The *superficial degree of divergence* of Γ is the degree of divergence one would naively guess by counting the powers of momenta in the numerator and denominator of the Feynman integral. It is

$$D(\Gamma) = \Sigma n_i d_i + 2I_\mathrm{B} + 3I_\mathrm{F} - 4V + 4 \tag{18.4}$$

the last two terms arising from the fact that at each vertex there is a four dimensional delta function which allows one to express one four-momentum in terms of other momenta, except that one delta function expresses the conservation of external momenta. Making use of eqs. (18.1), (18.2), and (18.3) we can write eq. (18.4) as

$$D = \Sigma n_i \delta_i - E_\mathrm{B} - \tfrac{3}{2} E_\mathrm{F} + 4, \tag{18.5}$$

or,

$$D + E_\mathrm{B} + \tfrac{3}{2} E_\mathrm{F} - 4 = \Sigma n_i \delta_i . \tag{18.6}$$

The purpose of renormalization theory is to give a definition of the *finite part* of the Feynman integral corresponding to Γ:

$$F_\Gamma = \lim_{\epsilon \to 0_+} \int \mathrm{d}k_1 \ldots \mathrm{d}k_L I_\Gamma \tag{18.7}$$

where I_Γ is a product of propagators Δ_F and vertices P:

$$I_\Gamma = \prod_{a,b,\sigma} \Delta_F^{ab\sigma} \prod_a P_a. \tag{18.8}$$

The finite part of F_Γ will be denoted by J_Γ and written

$$J_\Gamma = \lim_{\epsilon\to 0_+} \int \mathrm{d}k_1 \ldots \mathrm{d}k_L R_\Gamma. \tag{18.9}$$

We shall describe Bogoliubov's prescription of constructing R_Γ from I_Γ.

Let us first consider a simple case, in which Γ is primitively divergent. The diagram Γ is primitively divergent if it is proper (i.e., IPI), superficially divergent (i.e., $D(\Gamma) \geqslant 0$) and becomes convergent if any line is broken up. In this case, we may use the original prescription of Dyson. We write

$$J_\Gamma = \int \mathrm{d}k_1 \ldots \mathrm{d}k_L (1 - t^\Gamma) I_\Gamma,$$

i.e.,

$$R_\Gamma = (1 - t^\Gamma) I_\Gamma.$$

The operation t^Γ must be defined to cancel the infinity in J_Γ. I_Γ is a function of $E_F + E_B - 1 = E - 1$ external momenta $p_1, \ldots p_{E-1}$:

$$I_\Gamma = f(p_1, \ldots p_{E-1}).$$

The operation $(1 - t^\Gamma)$ on f is defined by subtracting from f the first $D(\Gamma) + 1$ terms in a Taylor expansion about $p_i = 0$:

$$t^\Gamma f(p_1, \ldots, p_{E-1}) = f(0, \ldots, 0) + \ldots + \frac{1}{d!} \sum_{j_1, \ldots, j_d = 1}^{E-1} (p_{j_1})_\lambda (p_{j_2})_\mu \ldots (p_{j_d})_\nu \times \frac{\partial^d f}{(\partial p_{j_1})_\lambda (\partial p_{j_2})_\mu \ldots (\partial p_{j_d})_\nu} \tag{18.10}$$

where $d = D(\Gamma)$. The operation $(1 - t^\Gamma)$ amounts to making subtractions in the integrand I_Γ, the number of subtractions being determined by the superficial degree of divergence of the integral.

Some more definitions: A *renormalization part* is a proper diagram which is superficially divergent ($D \geqslant 0$). Two diagrams (subdiagrams) are *disjoint*, $\gamma_1 \cap \gamma_2 \neq \emptyset$ if they have no lines or vertices in common. Let $\{\gamma_1, \ldots \gamma_c\}$ be a set of mutually disjoint connected subdiagrams of Γ. Then

$$F \equiv \Gamma / \{\gamma_1, \ldots \gamma_c\}$$

is defined by contracting each γ to a point and assigning the value 1 to the corresponding vertex.

We are now in a position to describe *Bogoliubov's R operation:*

(1) if Γ is not a renormalization part (i.e., $D(\Gamma) \leqslant -1$),

$$R_\Gamma = \bar{R}_\Gamma; \tag{18.11}$$

(2) if Γ is a renormalization part ($D(\gamma) \geqslant 0$),

$$R_\Gamma = (1 - t^\Gamma) \bar{R}_\Gamma, \tag{18.12}$$

where $\overline{R}_\Gamma$ is defined as

$$\overline{R}_\Gamma = I_\Gamma + \sum_{\{\gamma_1,\dots\gamma_c\}} I_{\Gamma/\{\gamma_1,\dots\gamma_c\}} \prod_{\tau=1}^{c} O_{\gamma_\tau} \qquad (18.13)$$

and $O_\gamma = -t^\gamma \overline{R}_\gamma$, where the sum is over all possible different sets of $\{\gamma_i\}$. This definition of $\overline{R}_\Gamma$ in terms of $\overline{R}_\gamma$ appears to be recursive; in perturbation theory there is no problem; the $\overline{R}_\gamma$ appearing in the definition of $\overline{R}_\Gamma$ is necessarily of lower order.

It is possible to "solve" eq. (18.13). We refer the interested reader to Zimmermann's lectures and merely present the result. Again we need some more definitions before we can do this. Two diagrams γ_1 and γ_2 said to *overlap*, $\gamma_1 \, 0 \, \gamma_2$, if none of the following holds:

$$\gamma_1 \cap \gamma_2 = \emptyset, \qquad \gamma_1 \supset \gamma_2, \qquad \gamma_2 \supset \gamma_1 .$$

A Γ-*forest* U is a hierachy of subdiagrams satisfying (a)–(c) below: (a) elements of U are renormalization parts; (b) any two elements of U, γ' and γ'' are nonoverlapping; (c) U may be empty. A Γ-forest U is full or normal respectively depending on whether U contains Γ itself or not. The *theorem due to Zimmermann* is

$$R_\Gamma = \sum_{\text{all } U} \prod_{\lambda \in U} (-t^\lambda) I_\Gamma \qquad (18.14)$$

where Σ extends over all possible (full, normal and empty) Γ-forests, and in the product $\Pi(-t^\lambda)$ the factors are ordered such that t^λ stands to the left of t^σ if $\lambda \supset \sigma$. If $\lambda \cap \sigma = \emptyset$, the order is irrelevant. A simple example is in order. Consider the diagram in fig. 18.1. The forests are $\emptyset$ (empty); γ_1 (full); γ_2 (normal); γ_1, γ_2 (full). Equation (18.14) can be written in this case as

$$R_\Gamma = (1 - t^{\gamma_1} - t^{\gamma_2} + t^{\gamma_1} t^{\gamma_2}) I_\Gamma = (1 - t^{\gamma_1})(1 - t^{\gamma_2}) I_\Gamma .$$

Note that in the BPH program, the R-operation is performed with respect to subdiagrams which consist of vertices and *all* propagators in Γ which connect these vertices. By the BPH definition, the subdiagram γ_2 above does not contain renormalization parts other than itself and in this sense the present treatment differs from Salam's discussion.

In formulating the BPH theorem it is necessary first to regularize the propagators in eq. (18.9) by some device such as by

$$\Delta_F(p) \underset{\text{reg}}{\rightarrow} \Delta_F(p; r, \epsilon) = -\mathrm{i} \int_r^\infty \mathrm{d}\alpha \, \exp\{\mathrm{i}\alpha(p^2 - m^2 + \mathrm{i}\epsilon)\}$$

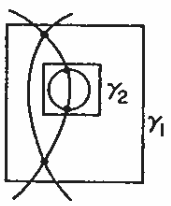

Fig. 18.1. Example of the BPHZ definition of subdiagrams in a particular contribution to the four-point function in a $\lambda\phi^4$ coupling theory.

and define $I_\Gamma(r, \epsilon)$ as in eq. (18.9) in terms of $\Delta_F(r, \epsilon)$, and then construct $R_\Gamma(r, \epsilon)$ by the R-operation. The BPH theorem states that R_Γ exists as $r \to 0$ and $\epsilon \to 0+$, as a boundary value of an analytic function in the external momenta. Another theorem, the proof of which can be found in the book by Bogoliubov and Shirkov, section 26, and which is combinatoric in nature, states that the subtractions implied by the $(1 - t^\Gamma)$ prescription in the R-operation can be formally implemented by adding counterterms in the Lagrangian.

A theory which has a finite number of renormalization parts is called renormalizable. A theory in which all δ_i are less than, or equal to zero is renormalizable. In this case the index of a subtraction term in the R-operation is bounded by $D + E_B + \frac{3}{2}E_F - 4$ which is at most equal to zero by eq. (18.5). In such a theory, only a finite number of renormalization counterterms to the Lagrangian suffice to implement the R-operation. In the σ model we considered in the preceeding section, all two-, three- and four-point proper vertices are superficially divergent. The two-point vertices (self-energy parts) are quadratically divergent so the R operation makes two subtractions in p^2 from the Feynman integrals. The other vertices are only logarithmically divergent.

The BHPZ renormalization can be combined with the Ward-Takahashi identities discussed in the preceeding section to produce a systematic scheme for renormalizing the σ-model without explicit reference to the symmetric theory. This was first worked out by Symanzik. Construction of a renormalized perturbation series according to the BPHZ prescription requires prescribing values of renormalization parts at subtraction points. Suppose these values are determined up to the $(n - 1)$ loop approximation, in such a way as to satisfy the Ward-Takahashi identities, and we are to construct proper vertices up to the n-loop approximation. Suppose further that we have a regularization scheme so that the Ward-Takahashi identities hold for regularized proper vertices. For example, we have, from eq. (17.30),

$$\Delta_\sigma^{-1}(p^2; r) - \Delta_\pi^{-1}(p^2; r) = u\, \widetilde{\Gamma}_{\sigma\pi\pi}(p; 0, -p; r) \tag{18.15}$$

where r is a cutoff parameter which should be set equal to zero at the end. We apply the R operation to relevant vertices and write

$$\begin{aligned}\Delta_\sigma^{-1}(p^2; r) &= p^2 Z - m_\sigma^2 + (1 - t^{\Gamma_1})\overline{[\Delta_\sigma^{-1}(p^2; r)]}, \\ \Delta_\pi^{-1}(p^2; r) &= p^2 - m_\pi^2 + (1 - t^{\Gamma_2})\overline{[\Delta_\pi^{-1}(p^2; r)]}, \\ \widetilde{\Gamma}_{\sigma\pi\pi}(p; g, k; r) &= -2\lambda u + (1 - t^{\Gamma_3})\overline{[\widetilde{\Gamma}_{\sigma\pi\pi}(p; g, k; r)]},\end{aligned} \tag{18.16}$$

where $D(\Gamma_1) = 2$, $D(\Gamma_2) = 2$, $D(\Gamma_3) = 0$, and the symbol $\overline{[\quad]}$ signifies the quantity constructed by the R operation as in (18.13), wherein the vertices P_a in eq. (18.8) take the values of the corresponding renormalization parts at subtraction points as determined up to the $(n - 1)$ loop approximation. In (18.16), the degrees of the subtraction polynomials are determined by the superficial degrees of divergence of the proper vertices in question. We have chosen the coefficient of p^2 in Δ_π^{-1} equal to one by convention, i.e., by renormalizing the π and σ fields appropriately. Likewise we have chosen the value of $\widetilde{\Gamma}_{\sigma\pi\pi}(0; 0, 0; r)$ to be $-2\lambda\, u$ by convention. Now substituting the expressions in (18.16) into (18.15) and identifying terms proportional to $(p^2)^0$ and (p^2), we obtain

$$Z = 1 + u \left\{ \frac{\mathrm{d}}{\mathrm{d}p^2} \widetilde{\Gamma}_{\sigma\pi\pi}(p; 0, -p; r) \right\}_{p^2=0}, \tag{18.17}$$

$$m_\sigma^2 = m_\pi^2 + 2\lambda u^2.$$

The BPH theorem then asserts that the quantities appearing in eq. (18.16) together with Z defined in (18.17) are cutoff independent, i.e., well-defined in the limit $r \to 0$; furthermore this procedure determines the values Z, m_σ^2 of the renormalization part Δ_σ^{-1} up to the n loop approximation. In fact, by a systematic exploitation of the Ward-Takahashi identities, it is possible, as Symanzik first showed, to determine the values of three- and four-point renormalization parts at subtraction points completely in terms of m^2, $-2\lambda u$ and $\tilde{\Gamma}_{\pi\pi\pi\pi}(0, 0, 0, 0) \equiv -6\lambda$.

The inductive procedure described above becomes complete when we realize that in the tree (zero loop) approximation the values of renormalization parts at subtraction points are those read off the Lagrangian (they, of course, satisfy the Ward-Takahashi identities). Thus the values of renormalization parts at subtraction points have the expansion

$$Z = 1 + z_1\lambda + z_2\lambda^2 + \dots,$$

$$\tilde{\Gamma}_{\sigma\sigma\sigma}(0, 0, 0) = -6\lambda u[1 + \alpha_1\lambda + \alpha_2\lambda^2 + \dots],$$

$$\tilde{\Gamma}_{\sigma\sigma\sigma\sigma}(0, 0, 0, 0) = -6\lambda[1 + \beta_1\lambda + \beta_2\lambda^2 + \dots],$$

$$\tilde{\Gamma}_{\sigma\sigma\pi\pi}(0, 0; 0, 0) = -2\lambda[1 + \gamma_1\lambda + \gamma_2\lambda^2 + \dots].$$

The symmetry breaking parameter c is given by eq. (17.23), or (17.24).

This discussion makes sense only if there is a regularization scheme which preserves the Ward-Takahashi identities, and this leads us to the subject of the next lecture.

The Symanzik procedure outlined above is equivalent to the renormalization procedure discussed in section 17. This statement is clearly true in the tree approximation. Let us recall that the Lagrangian is first written in terms of bare quantities as

$$\mathcal{L} = \tfrac{1}{2}[(\partial_\mu\sigma_0)^2 + (\partial_\mu\pi_0)^2] - \tfrac{1}{2}\mu_0^2(\sigma_0^2 + \pi_0^2) - \tfrac{1}{4}\lambda_0(\sigma_0^2 + \pi_0^2)^2 + c_0\sigma_0. \tag{18.18}$$

After making the renormalization transformations

$$\sigma_0 = Z_3^{1/2}(u + s), \qquad \pi_0 = Z_3^{1/2}\pi, \qquad c_0 = Z_3^{-1/2}c,$$
$$\lambda_0 = (\lambda + \delta\lambda)Z_3^{-2}, \qquad \mu_0^2 = Z_3[m_\pi^2 + \delta m_\pi^2 - u(\lambda + \delta\lambda)] \tag{18.19}$$

we can write the Lagrangian as

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu s)^2 - \tfrac{1}{2}(m_\pi^2 + 2\lambda u^2)s^2 + \tfrac{1}{2}(\partial_\mu\pi)^2 - \tfrac{1}{2}m_\pi^2\pi^2$$
$$-\lambda us(s^2 + \pi^2) - \tfrac{1}{4}\lambda(s^2 + \pi^2)^2 + \mathcal{L}_c, \tag{18.20}$$

where $\mathcal{L}_c$ is the sum of the renormalization counterterms:

$$\mathcal{L}_c = \tfrac{1}{2}(Z_3 - 1)[(\partial_\mu s)^2 + (\partial_\mu\pi)^2] - \tfrac{1}{2}\delta m_\pi^2\pi^2 - \tfrac{1}{2}(\delta m_\pi^2 + 2u^2\delta\lambda)s^2$$
$$-u\delta\lambda s(s^2 + \pi^2) - \tfrac{1}{4}\delta\lambda(s^2 + \pi^2)^2 + [c - u(m_\pi^2 + \delta m_\pi^2)]s. \tag{18.21}$$

Now, suppose that the Symanzik procedure is equivalent to the subtractions of infinities by the above counterterms up to the $(n - 1)$ loop approximation. We then have, in the n loop approximation,

$$\Delta_\pi^{-1}(p^2;r) = [1 + A(r) + (Z_3 - 1)]p^2 - (m_\pi^2 + B(r) + \delta m_\pi^2) + (1 - t^{\Gamma_1})\overline{[\Delta_\pi^{-1}(p^2;r)]},$$
$$\Delta_\sigma^{-1}(p^2;r) = [1 + C(r) + (Z_3 - 1)]p^2 - (m_\pi^2 + 2\lambda u + D(r) + \delta m_\pi^2 + 2u^2\delta\lambda) + (1 - t^{\Gamma_2})\overline{[\Delta_\sigma^{-1}(p^2;r)]},$$
$$\widetilde{\Gamma}_{\sigma\pi\pi} = -2\lambda u[1 + E(r)] - 2u\delta\lambda + (1 - t^{\Gamma_3})\overline{[\widetilde{\Gamma}_{\sigma\pi\pi}]} \tag{18.22}$$

where $A(r)$, ... $E(r)$ are infinite (i.e., r-dependent) quantities. We choose Z_3, δm_π^2 and $\delta\lambda$ such that

$$Z_3 = 1 - A(r), \qquad \delta m_\pi^2 = -B(r), \qquad \delta\lambda = \lambda E(r). \tag{18.23}$$

Then the Ward-Takahashi identity (18.15) tells us that

$$C(r) - A(r) = u\left\{\frac{\partial}{\partial p^2}\,\overline{[\widetilde{\Gamma}_{\sigma\pi\pi}(p;0,-p;r)]}\right\}_{p^2=0} \tag{18.24}$$

which is convergent as $r \to 0$, and

$$D(r) + \delta m_\pi^2 + 2u^2\delta\lambda = 0. \tag{18.25}$$

The combination of (18.22), (18.24) and (18.25) is clearly equivalent to eqs. (18.16) and (18.17).

Bibliography

For renormalization theory see

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4. N.N. Bogoliubov and D.V. Shirkov, Introduction to the Theory of Quantized Fields (Interscience Publishers, N.Y., 1959) Chapter IV, and references cited in p. 330 thereof.
5. K. Hepp, Comm. Math. Phys. 1 (1965) 95; Théorie de la Renormalisation (Springer, Berlin, 1969).
6. W. Zimmermann, in Lectures on Elementary Particles and Quantum Field Theory, eds. S. Deser, M. Grisaru and H. Pendleton (MIT Press, Cambridge, 1970) p. 395 et seq.

For the renormalization of the σ-model discussed here, refer to Symanzik's papers cited in the preceeding section.

19. The regularization scheme of 't Hooft and Veltman

Recently, 't Hooft and Veltman proposed a scheme for regularizing Feynman integrals which preserves various symmetries of the underlying Lagrangian. This method is applicable to the σ-model, electrodynamics, and non-Abelian gauge theories, and depends on the idea of analytic continuation of Feynman integrals in the number of space-time dimensions. The critical observations here are that the global or local symmetries of these theories are independent of space-time dimensions, and that Feynman integrals are convergent for sufficiently small, or complex N, where N is the "complex dimension" of space-time.

Let us first review the nature of ultraviolet divergence of a Feynman diagram. For this purpose, it is convenient to parametrize the propagators as

$$\Delta_F(p^2) = \frac{1}{\mathrm{i}} \int_0^\infty \mathrm{d}\alpha \, \exp\{\mathrm{i}\alpha(p^2 - m^2 + \mathrm{i}\epsilon)\}. \tag{19.1}$$

Making use of this representation, we can write a typical Feynman integral as

$$F_\Gamma \sim \left(\prod_{i=1}^{I} \int_0^\infty \mathrm{d}\alpha_i\right)\left(\prod_{j=1}^{L} \int \mathrm{d}^4 k_j\right)(k_{l_1})_\lambda (k_{l_2})_\mu \ldots (k_{l_n})_\nu \times \exp\{\mathrm{i}\sum_i \alpha_i(q_i^2 - m_i^2 + \mathrm{i}\epsilon)\} \tag{19.2}$$

where I is the number of internal propagators in Γ, L the number of loops, and $l_1, \ldots l_n$ may take any values from 1 to L. The momentum q_j carried by the jth propagator is a linear function of loop momenta k_i and external momenta p_m. The exponent on the right-hand side of eq. (19.2) can therefore be written as

$$\sum_{i=1}^{I} \alpha_i(q_i^2 - m_i^2 + \mathrm{i}\epsilon) = \tfrac{1}{2}\sum_{i,j} k_i A_{ij}(\alpha) k_j + \sum_{i,m} k_i B_{im}(\alpha) p_m - \sum_i^I \alpha_i(m_i^2 - \mathrm{i}\epsilon)$$

$$\equiv \tfrac{1}{2}\mathbf{k}^\mathrm{T} \cdot \mathbf{A} \cdot \mathbf{k} + \mathbf{k} \cdot \mathbf{B} \cdot \mathbf{p} - \sum_i \alpha_i(m_i^2 - \mathrm{i}\epsilon)$$

where $\mathbf{k}$ is a column matrix with entries which are four-vectors. The matrices $\mathbf{A}$ and $\mathbf{B}$ are homogeneous functions of first degree in α's, and $\mathbf{A}$ is symmetric. Upon translating the integration variables

$$\mathbf{k} \to \mathbf{k}' = \mathbf{k} + \mathbf{A}^{-1}\mathbf{B}\mathbf{p}$$

and diagonahalizing the matrix $\mathbf{A}$ by an orthogonal transformation on $\mathbf{k}'$, we can perform the loop integrations over k_j in eq. (19.2). The result is a sum over terms each of which has the form

$$F_\Gamma \sim T_{\lambda\mu\ldots\nu}\left(\prod_{i=1}^{I} \int_0^\infty \mathrm{d}\alpha_i\right)\frac{1}{\Pi_i [A_i(\alpha)]^{S_i}} \exp\left[-\mathrm{i}\left\{\tfrac{1}{2}\mathbf{p} \cdot \mathbf{C}(\alpha) \cdot \mathbf{p} + \sum_i \alpha_i(m_i^2 - \mathrm{i}\epsilon)\right\}\right] \tag{19.3}$$

where $T_{\lambda\mu\ldots\nu}$ is a tensor typically a product of $g_{\rho\sigma}$'s, $A_i(\alpha)$ is the ith eigenvalue of the matrix A, and s_i is a positive number which is determined by the tensorial structure of F_Γ. Note that $A_i(\alpha)$ is homogeneous of first degree in α's. The matrix $\mathbf{C}$ is

$$\mathbf{C} = \mathbf{B}^\mathrm{T}\mathbf{A}^{-1}\mathbf{B},$$

and is also a homogeneous function of first degree in α's. In this parametrization, the ultraviolet divergences of the integral appear as the singularities of the integrand on the right-hand side of eq. (19.3) arising from the vanishing of some factors $\Pi_i [A_i(\alpha)]^{s_i}$ as some or all α's approach to zero in certain orders, for example,

$$\alpha_{r_1} < \alpha_{r_2} < \ldots < \alpha_{r_I}$$

where $(r_1, r_2 \ldots r_I)$ is a permutation of $(1, 2, \ldots I)$. See, for instance, a more detailed and careful discussion of Hepp.

The 't Hooft-Veltman regularization consists in defining the integral F_Γ in n dimensions, $n > 4$ (one-time and $(N-1)$-space dimensions) while keeping external momenta and polarization vectors in the first four dimensions (i.e., in the physical space), performing the $n-4$ dimensional integrals in the space orthogonal to the physical space, and then continuing the result in n. (For single-loop graphs one may perform all n integrations together.) For sufficiently small n, or complex n, the subsequent four-dimensional integrations are convergent.

To see how it works, consider the integral

$$F_\Gamma(n) \sim \left(\prod_i \int \mathrm{d}\alpha_i\right)\left(\prod_{j=1}^{L} \int \mathrm{d}^n k_j\right)\prod (k_a \cdot k_b)\prod(k_c \cdot p_m)\prod(k_d \cdot e_l)\exp\left\{\mathrm{i}\sum_i \alpha_i(q_i^2 - m_i^2 + \mathrm{i}\epsilon)\right\} \tag{19.4}$$

where, now, the k_j are n-dimensional vectors. As before we can express the q_i as linear functions of the k_i and the external momenta p_i, where the p_i have only first four component nonvanishing. From now, we shall denote an n-dimensional vector by $(\hat{k}, K)$, where $\hat{k}$ is the projection of k onto the physical space-time and $K = k - \hat{k}$. Thus, $p = (\hat{p}, 0)$. Equation (19.4) may be written as a sum of terms of the form

$$\begin{aligned} F_\Gamma(n) \sim &\left(\prod_{i=1}^{I} \int \mathrm{d}\alpha_i\right)\left(\prod_{j=1}^{L} \int \mathrm{d}^4\hat{k}_j\right)\left(\prod_{j=1}^{L} \int \mathrm{d}^{n-4}K_j\right)\left(\prod_{a,b} K_a \cdot K_b\right) \\ &\times \left(\prod_{c,m} \hat{k}_c \cdot \hat{p}_m\right)\left(\prod_{d,l} \hat{k}_d \cdot \hat{e}_l\right)\left(\prod_{e,f} \hat{k}_e \cdot \hat{k}_f\right) \exp\left[\mathrm{i}\left\{\hat{\mathrm{k}}^{\mathrm{T}} \cdot \mathrm{A}\hat{\mathrm{k}} + \hat{\mathrm{k}} \cdot \mathrm{Bp} - \mathrm{K}^{\mathrm{T}} \cdot \mathrm{AK} - \mathrm{i}\sum_i \alpha_i(m_i^2 - \mathrm{i}\epsilon)\right\}\right] \end{aligned} \tag{19.5}$$

The integrals over K_j can be performed immediately, using the formulas

$$\int \mathrm{d}^{n-4}K\, K_{\alpha_1} K_{\alpha_2} \ldots K_{\alpha_{2r}} \exp(-\mathrm{i}AK^2) = \frac{\pi^{n/2}}{2^r r!} \sum_{\sigma \in S_{2r}} \delta_{\sigma(\alpha_1),\sigma(\alpha_2)}\delta_{\sigma(\alpha_3),\sigma(\alpha_4)} \ldots \delta_{\sigma(\alpha_{2r-1}),\sigma(\alpha_{2r})} (\mathrm{i}A)^{-n/2+2-r}$$

where the summation is over the elements σ of the symmetric group on $2r$ objects $(\alpha_1, \alpha_2, \ldots \alpha_{2r})$, and

$$\delta_{\alpha\beta}\delta_{\beta\alpha} = n - 4.$$

Thus F_Γ of eq. (19.5) will have the form

$$\begin{aligned} F_\Gamma(n) \sim &\left(\prod_{i=1} \int \mathrm{d}\alpha_i\right) \frac{f(n)}{\Pi_i [A_i(\alpha)]^{n/2-2+r_i}} \left(\prod_{j=1} \int \mathrm{d}^4\hat{k}_j\right) \prod_{e,f} (\hat{k}_e \cdot \hat{k}_f) \prod_{c,m} (\hat{k}_e \cdot \hat{P}_m) \\ &\times \prod_{d,l} (\hat{k}_d \cdot \hat{e}_l) \exp\left\{\mathrm{i}\prod_i \alpha_i(\hat{q}_i^2 - m_i^2 + \mathrm{i}\epsilon)\right\} \end{aligned}$$

where $f(n)$ is a polynomial in n and r_i is a nonnegative integer depending on the structure of $\Pi K_a \cdot K_b$ in eq. (19.5). For sufficiently small $n < 4$, the singularities of the integrand as some or all α's go to zero disappear.

The reasons this regularization preserves the Ward-Takahashi identities of the kind discussed in the preceeding sections are, firstly, that the vector manipulations such as

$$k^\mu(2p + k)_\mu = [(p + k)^2 - m^2] - (p^2 - m^2)$$

or partial fractioning of a product of two propagators, which are necessary to verify these identities "by hand", are valid in any dimensions, and, secondly, that the shifts of integration variables, dangerous when integrals are divergent, are justified for small enough, or complex n, since the integral in question is convergent.

The divergence in the original integral is manifested in the poles of $F_\Gamma(n)$ at $n = 4$. These poles are removed by the R-operation, so that $J_\Gamma(n)$ as defined by the R-operation is finite and well-defined as $n \to 4$. Actually, to our knowledge the proof of this has not appeared in the literature, except for the original discussion of 't Hooft and Veltman. Hepp's proof, for example, does not really apply here, since the analytical discussion of Hepp is not tailored for this kind of regularization. However, the argument of 't Hooft and Veltman is sufficiently convincing and we have no reason to believe why a suitable modification of Hepp's proof, for example, of the BPHZ theorem should not go through with the dimensional regularization.

The above discussion is fine for theories with bosons only. When there are fermions in the theory, a complication may arise. This has to do with the occurrence of the so-called Adler-Bell-Jackiw anomalies, which we discussed briefly in section 5. The subject of anomalies in Ward-Takahashi identities has been discussed thoroughly in two excellent lectures by Adler, and by Jackiw, and we shall not go into any further details here. In short, the Adler-Bell-Jackiw anomalies may occur when the verification of certain Ward-Takahashi identities depends on the algebra of Dirac gamma matrices with γ_5, such as $\gamma_\mu\gamma_5 + \gamma_5\gamma_\mu = 0$. Typically, this happens when a proper vertex involving an odd number of axial vector currents cannot be regularized in a way that preserves all the Ward-Takahashi identities on such a vertex, and as a consequence some of the Ward-Takahashi identities have to be broken. The occurrence of these anomalies is not a matter of not being clever enough to devise a proper regularization scheme: for certain models such a scheme is impossible to devise. The dimensional regularization does not help in such a case, due to the fact that γ_5 and the completely antisymmetric tensor density $\epsilon_{\lambda\mu\nu\rho}$ are unique to four dimensions and do not allow a logically consistent generalization to n dimensions. When there are anomalies in a spontaneously broken gauge theory, the unitarity of the S-matrix is in jeopardy since, as we shall see in the forthcoming sections, the unitarity of the S-matrix, i.e., cancellation of spurious singularities introduced by a particular choice of gauge is inferred from the Ward-Takahashi identities. Gross and Jackiw have shown that, in an Abelian gauge theory, the occurrence of anomalies runs afoul of the dual requirements of unitarity and renormalizability of the theory.

Thus, a satisfactory theory should be free of anomalies. Fortunately, it is possible to construct models which are anomaly-free, by a judicious choice of fermion fields to be included in the model. There are two "lemmas" which make the above assertion possible. One is that the anomalies are not "renormalized", which in particular means that the absence of anomalies in lowest order insures their absence to all orders. This was shown by Adler and Bardeen in the context of an SU(3) version of the σ-model, and by Bardeen in a more general context which encompasses non-Abelian gauge theories. The second is the observation that all anomalies are related; in particular, if the simplest anomaly involving the vertex of three currents is absent in a model, so are all other anomalies. This can be inferred from an explicit construction of all anomalies by Bardeen, or from a more general and elegant argument of Wess and Zumino.

To make sure that a non-Abelian gauge theory is anomaly-free, therefore, it suffices to check that one-fermion-loop contribution to the three-gauge-boson-vertex is free of anomaly. Let

$$\bar{\psi}\gamma^\mu \Gamma_a \psi$$

be the coupling term of the gauge boson $A_\mu{}^a$ to the fermions. Here ψ is a column matrix of all fermion fields in the theory and Γ_a is a matrix whose elements may depend on γ_5. Now the one-fermion-loop contribution to the cubic coupling of the gauge bosons is

$$\Gamma^{abc}_{\lambda\mu\nu}(p, q, r) \sim \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \operatorname{Tr} \left\{ \gamma_\lambda \Gamma_a \frac{1}{\gamma\cdot(k+p)-M} \gamma_\mu \Gamma_b \frac{1}{\gamma\cdot(k-r)-M} \gamma_\nu \Gamma_c \frac{1}{\gamma\cdot k-M} + \begin{pmatrix} b & & c \\ \mu & \leftrightarrow & \nu \\ q & & r \end{pmatrix} \right\} \tag{19.6}$$

where M is the mass matrix of the fermions, and $p+q+r=0$. As can be deduced from the discussion of Gross and Jackiw, for example, the vertex of eq. (19.6) is anomaly-free if the part of this vertex proportional to $\epsilon_{\lambda\mu\nu\rho}p^\rho$ or $\epsilon_{\lambda\mu\nu\rho}q^\rho$ is convergent. This calls for

$$\operatorname{Tr}\gamma_5\,\Gamma_a\{\Gamma_b, \Gamma_c\}_+ = 0. \tag{19.7}$$

Equation (19.7) is a sufficient condition for the absence of anomalies in a gauge theory.

Georgi and Glashow have discussed various ramifications of this condition. Physically, eq. (19.7) implies that the anomaly caused by one kind of fermions is cancelled by that caused by another. In some models, this cancellation may be arranged among leptons and among hadrons, separately; in some other models this cancellation takes place between leptons and hadrons. In any case anomaly-free theories tend to contain more leptons and hadrons (quarks) than the phenomenology warrants at this time.

The rather restrictive constraints which the consideration of the absence of anomalies imposes on model building may in fact be a blessing in disguise. The possibility of a certain correspondence between leptonic and hadronic building blocks or of new-quantum numbers and new dimensions in hadron spectroscopy is intriguing and perhaps exciting.

Let us conclude with a simple example of dimensional regularization: the vacuum polarization in scalar electrodynamics. The Lagrangian is

$$\mathcal{L} = (\partial^\mu\phi^* - \mathrm{i}eA^\mu\Phi^*)(\partial_\mu\phi + \mathrm{i}eA_\mu\phi) - \tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - V(\phi)$$

and the relevant vertices are shown in fig. 19.1. There are two diagrams which contribute to the vacuum polarization, shown in fig. 19.2. The sum of these contributions is

$$I = e^2 \frac{\mathrm{d}^n k}{(2\pi)^n} \int \frac{1}{(k+p)^2-\mu^2}\,\frac{1}{k^2-\mu^2}\,[(2k+p)_\mu(2k+p)_\nu - 2((k+p)^2-\mu^2)g_{\mu\nu}]\,. \tag{19.8}$$

We use the exponential parametrization of the propagators to obtain

$$I = \frac{e^2}{(\mathrm{i})^2}\int_0^\infty \mathrm{d}\alpha \int_0^\infty \mathrm{d}\beta \int \frac{\mathrm{d}^n k}{(2\pi)^n} \exp\{\mathrm{i}[\alpha(k+p)^2 + \beta k^2 - (\alpha+\beta)(\mu^2 - \mathrm{i}\epsilon)]\}\,[(2k+p)_\mu(2k+p)_\nu - 2((k+p)^2-\mu^2) \tag{19.9}$$

The exponent is proportional to

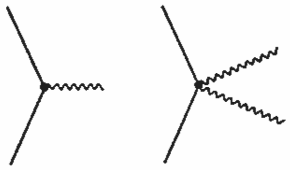

Fig. 9.1. Photon-scalar meson vertices in charged scalar electrodynamics.

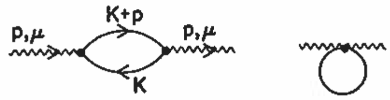

Fig. 9.2. Second order vacuum polarization diagrams in charged scalar electrodynamics.

$$(\alpha+\beta)k^2 + 2k\cdot p\alpha + \alpha p^2 - (\alpha+\beta)(\mu^2-\mathrm{i}\epsilon) = (\alpha+\beta)\left(k+\frac{\alpha}{\alpha+\beta}p\right)^2 + \frac{\alpha\beta}{\alpha+\beta}p^2-(\alpha+\beta)(\mu^2-\mathrm{i}\epsilon),$$

so we may write

$$I = -e^2\int_0^\infty \mathrm{d}\alpha\int_0^\infty \mathrm{d}\beta\int\frac{\mathrm{d}^n k}{(2\pi)^n}\exp\left\{\mathrm{i}(\alpha+\beta)k^2+\mathrm{i}\left[\frac{\alpha\beta}{\alpha+\beta}p^2-(\alpha+\beta)(\mu^2-\mathrm{i}\epsilon)\right]\right\}$$

$$\times\left\{4k_\mu k_\nu + \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 p_\mu p_\nu - g_{\mu\nu}\left[2(k^2-\mu^2)+\frac{\alpha^2+\beta^2}{(\alpha+\beta)^2}p^2\right]\right\}$$

$$= e^2(p_\mu p_\nu - p^2 g_{\mu\nu})\int_0^\infty \mathrm{d}\alpha\int_0^\infty \mathrm{d}\beta\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2\int\frac{\mathrm{d}^n k}{(2\pi)^n}\exp\left\{\mathrm{i}(\alpha+\beta)k^2+\mathrm{i}\left[\frac{\alpha\beta}{\alpha+\beta}p^2-(\alpha+\beta)(\mu^2-\mathrm{i}\epsilon)\right]\right\}$$

$$-e^2\int_0^\infty \mathrm{d}\alpha\int_0^\infty \mathrm{d}\beta\,\frac{\mathrm{d}^n k}{(2\pi)^n}\exp\left\{\mathrm{i}(\alpha+\beta)k^2+\mathrm{i}\left[\frac{\alpha\beta}{\alpha+\beta}p^2-(\alpha+\beta)(\mu^2-\mathrm{i}\epsilon)\right]\right\}$$

$$\times\left\{4k_\mu k_\nu - 2g_{\mu\nu}(k^2-\mu^2) - g_{\mu\nu}\frac{\alpha\beta}{(\alpha+\beta)^2}p^2\right\}. \tag{19.10}$$

The first term is explicitly gauge invariant and only logarithmically divergent, so that a subtraction will make it convergent. It is the second term that requires a careful handling. We need the formulas

$$\int\frac{\mathrm{d}^n k}{(2\pi)^n}\exp(\mathrm{i}\lambda k^2) = \frac{\exp(\mathrm{i}\pi n/4)}{(2\sqrt{\pi\lambda})^n}$$

$$\int\frac{\mathrm{d}^n k}{(2\pi)^n}k^2\exp(\mathrm{i}\lambda k^2) = \frac{1}{\mathrm{i}\lambda}\left(-\frac{n}{2}\right)\frac{\exp(\mathrm{i}\pi n/4)}{(2\sqrt{\pi\lambda})^n}$$

$$\int\frac{\mathrm{d}^n k}{(2\pi)^n}k_\mu k_\nu\exp(\mathrm{i}\lambda k^2) = \frac{1}{n}g_{\mu\nu}\frac{\mathrm{d}^n k}{(2\pi)^n}k^2\exp(\mathrm{i}\lambda k^2) = g_{\mu\nu}\frac{1}{\mathrm{i}\lambda}(-\tfrac{1}{2})\frac{\exp(\mathrm{i}\pi n/4)}{(2\sqrt{\pi\lambda})^n}, \tag{19.11}$$

so that the second term, I_2, is

$$I_2 = -e^2 g_{\mu\nu} \frac{\exp(i\pi n/4)}{(2\sqrt{\pi})^n} \int_0^\infty d\alpha \int_0^\infty d\beta \frac{1}{(\alpha+\beta)^{n/2}} \exp\left\{ i\left[\frac{\alpha\beta}{\alpha+\beta} p^2 - (\alpha+\beta)(\mu^2 - i\epsilon)\right]\right\} \tag{19.12}$$

$$\times \frac{2}{\alpha+\beta}\left\{ i\left(1-\frac{n}{2}\right) - \left[\frac{\alpha\beta}{\alpha+\beta}p^2 - (\alpha+\beta)\mu^2\right]\right\}$$

$$= -2ie^2 g_{\mu\nu} \frac{\exp(i\pi n/4)}{(2\sqrt{\pi})^n} \int d\alpha\, d\beta\, \delta(1-\alpha-\beta) \int_0^\infty \frac{d\lambda}{\lambda^{n/2-1}} \exp\{i\lambda(\alpha\beta p^2 - \mu^2 + i\epsilon)\} \left[\frac{1-n/2}{\lambda} + i(\alpha\beta p^2 - \mu^2)\right]$$

For sufficiently small n, $n < 2$, the λ-integration is convergent, and

$$\int_0^\infty \frac{d\lambda}{\lambda^{n/2-1}} \exp\{i\lambda(A+i\epsilon)\}\left(\frac{1-n/2}{\lambda} + iA\right) = \int_0^\infty d\lambda \frac{d}{d\lambda}\{\lambda^{1-n/2}\exp[i\lambda(A+i\epsilon)]\} = 0. \tag{19.13}$$

So the dimensional regularization gives the gauge invariant result,

$$I_2 = 0.$$

Bibliography

The dimensional regularization, in the form discussed here, is due to

1. G. 't Hooft and M.T. Veltman, Nucl. Phys. 44B (1973) 189.

See also

2. C.G. Bollini, J.-J. Giambiagi and A. Gonzales Dominguez, Nuovo Cimento 31 (1964) 550.
3. G. Cicuta and E. Montaldi, Lettere al Nuovo Cimento 4 (1972) 329.

A closely related regularization method – analytic regularization – is discussed in

4. E.R. Speer, Generalized Feynman Amplitudes (Princeton University Press, Princeton, 1969).

Excellent reviews on the Adler-Bell-Jackiw anomalies are:

5. S.L. Adler, in Lectures on Elementary Particles and Quantum Field theory, eds. S. Deser, M. Grisaru and H. Pendelton, (MIT Press, Cambridge, 1970).
6. R. Jackiw, in Lectures on Current Algebra and its Applications (Princeton University Press, Princeton, N.J. 1970).

For a complete list of anomalies vertices, involving only currents (not pions), see

7. W.A. Bardeen, Phys. Rev. 184 (1969) 1848.
8. J. Wess and B. Zumino, Phys. Letters 37B (1971) 95.

The following papers discuss the problem of anomalies in gauge theories

9. D.J. Gross and R. Jackiw, Phys. Rev. D6 (1972) 477.
10. H. Georgi and S.L. Glashow, Phys. Rev. D6 (1972) 429.
11. C. Bouchiat, J. Iliopoulos and P. Meyer, Phys. Letters 38 B (1972) 519.
12. W.A. Bardeen, in Proc. XVI Intern. Conf. on High Energy Physics (National Accelerator, 1972) Vol. 2, p. 295.

The last reference gives a concise algorithm for dimensional regularization valid for scalar loops. Our prescription agrees with it for this case.

20. Feynman rules and renormalization of spontaneously broken gauge theories: Landau gauge

The reader who has followed the developments so far should have no difficulty in comprehending the recent literature on various renormalizable formulations of spontaneously broken gauge symmetries. In the following sections, we shall try to convey the general ideas underlying the discussions of Lee and Zinn-Justin on this subject, without getting involved too much in mathematical manipulations.

For concreteness let us consider an O(3) gauge theory in which the triplet gauge bosons are interacting with a triplet of real scalar fields ϕ_0. As we explained in section 14, the generating functional W_L of Green's functions in the Landau gauge is written as [the subscript "0" refers to unrenormalized quantities]

$$W_L[J_\mu, J] = \int [dA_{0\mu}][dc_0][dc_0^\dagger] \exp\left[i\left\{S + S_c + \int d^4x\left[-\frac{1}{2\alpha_0}(\partial^\mu A_{0\mu}(x))^2 + J_{0\mu}(x)\cdot A_0^\mu(x) + J_0(x)\cdot\phi_0(x)\right]\right\}\right] \tag{20.1}$$

where, we recall, c_0 is a triplet of fictitious complex scalar fields of the wrong statistics and the limit $\alpha_0 \to 0$ is understood. The action, S, and S_c, are given by

$$S = \int d^4x[-\tfrac{1}{4}(\partial_\mu A_{0\nu} - \partial_\nu A_{0\mu} + g_0 A_{0\mu}\times A_{0\nu})^2 + \tfrac{1}{2}(\partial_\mu\phi_0 + g_0 A_{0\mu}\times\phi_0)^2 - \tfrac{1}{2}\mu_0^2\phi_0^2 - \tfrac{1}{4}\lambda_0(\phi_0^2)^2], \tag{20.2}$$

$$S_c = \int d^4x\{-\partial^\mu c_0^\dagger(x)\cdot\partial_\mu c_0(x) - g_0\partial^\mu c_0^\dagger(x)\cdot A_{0\mu}(x)\times c_0(x)\}. \tag{20.3}$$

If $\mu^2 > 0$, the theory is the usual one of massless gauge bosons interacting with a multiplet of scalar fields of mass μ. Let us consider the renormalization of the theory in this case. All the three-point and four-point vertices are logarithmically divergent (i.e., the superficial degrees of divergence $D = 0$), and all the self-energy parts, for $A_{0\mu}$, ϕ_0 and c_0, have $D = 2$, i.e., are quadratically divergent, according to the power counting procedure discussed in section 18.

The Ward-Takahashi identity for W_L of eq. (20.1) is obtained by considering the effects on W_L of the transformation

$$A_{0\mu}(x) \to A_{0\mu}(x) - \omega(x)\times A_{0\mu}(x) + \frac{1}{g_0}\partial_\mu\omega(x)$$

$$\phi_0(x) \to \phi_0(x) - \omega(x)\times\phi_0(x) \tag{20.4}$$

which leaves S invariant, after eliminating the c_0- and $c_0^\dagger$-fields. Since we are going to derive the Ward-Takahashi identity for W for a more general class of gauge conditions in a later section, we shall forego writing it down here. Now, when momentum-space Green's functions are dimensionally regularized, they satisfy the Ward-Takahashi automatically. When we generate renormalization counterterms in the manner described below (or scale fields and parameters in the way specified below), the renormalized Green's functions are finite as $n \to 4$, satisfy the renormalized form of the Ward-Takahashi identities [see eq. (2.8) of Lee and Zinn-Justin II]. It is necessary to ensure that the renormalized Green's functions satisfy the renormalized Ward-Takahashi identities, because the latter will be used to show that the renormalized S-matrix is free of spurious singularities.

A simple way of generating all the necessary renormalization counterterms in the effective action is to perform the following scale transformations on the quantities appearing in eq. (20.1):

$$A_{0\mu} = Z_3^{1/2}A_\mu, \qquad J_{0\mu} = Z_3^{-1/2}J_\mu, \qquad \phi_0 = Z_2^{1/2}\phi,$$

$$J_0 = Z_2^{-1/2} J,$$

$$c_0 = \tilde{Z}_3^{1/2} c, \qquad g_0 = g Z_1/Z_3^{3/2} = g \tilde{Z}_1/\tilde{Z}_3 Z_3^{1/2},$$

$$(\mu_0)^2 = \mu^2 + \delta\mu^2/Z_2, \qquad \lambda_0 = \lambda Z_4/Z_2^2$$

and

$$\alpha_0 = Z_3 \alpha,$$

where the superscript "0" signifies the quantities appearing in eq. (20.1). In terms of the new (renormalized) quantities, the generating functional W_{I} has the same form as eq. (20.1), except that S and S_c acquire additional pieces ΔS and ΔS_c, where

$$\begin{aligned}\Delta S = \int d^4x \{ &-\tfrac{1}{4}(Z_3 - 1)(\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu)^2 + \tfrac{1}{2} g (Z_1 - 1) \mathbf{A}_\mu \times \mathbf{A}_\nu \cdot (\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu)^2 \\ &- \tfrac{1}{4} g^2 (Z_1^2/Z_3 - 1)(\mathbf{A}_\mu \times \mathbf{A}_\nu)^2 + \tfrac{1}{2}(Z_2 - 1)[(\partial_\mu \boldsymbol{\phi})^2 - \mu^2 \boldsymbol{\phi}^2] - g[Z_1(Z_2/Z_3) - 1] \mathbf{A}_\mu \cdot (\boldsymbol{\phi} \times \partial^\mu \boldsymbol{\phi}) \\ &+ \tfrac{1}{2} g^2 [(Z_1^2/Z_3)(Z_2/Z_3) - 1](\mathbf{A}_\mu \times \boldsymbol{\phi})^2 - \tfrac{1}{2}\delta\mu^2 \boldsymbol{\phi}^2 - \tfrac{1}{4}\lambda(Z_4 - 1)(\boldsymbol{\phi}^2)^2 \}\end{aligned} \tag{20.6}$$

and

$$\Delta S_c = \int d^4x \{ -(\tilde{Z}_3 - 1)\partial^\mu \mathbf{c}^\dagger \cdot \partial_\mu \mathbf{c} - g(\tilde{Z}_1 - 1)\partial^\mu \mathbf{c}^\dagger \cdot \mathbf{A}_\mu \times \mathbf{c} \}. \tag{20.7}$$

When properly regulated, the self energy-parts of $\mathbf{A}_\mu$ and $\mathbf{c}$ are only logarithmically divergent, and if we choose Z_3, $\tilde{Z}_3$, Z_2 and $\delta\mu^2$ to make the propagators for $\mathbf{A}_\mu$, $\mathbf{c}$ and $\boldsymbol{\phi}$ finite; Z_1, $\tilde{Z}_1$ and Z_4 to make the $A_\mu \phi^2$-, $A_\mu c^\dagger c$- and ϕ^4 proper vertices finite, then the counterterms exhibited in eqs. (20.6) and (20.7) render finite all renormalization parts of the theory. In particular, the renormalization constants Z_1, Z_3, $\tilde{Z}_1$ and $\tilde{Z}_3$ can be chosen so that

$$Z_1/Z_3 = \tilde{Z}_1/\tilde{Z}_3. \tag{20.8}$$

This is first shown by A. Slavnov and J.C. Taylor. Also, if we choose the renormalization counterterms in the above manner, then the counterterms for the A_μ^4- and $A_\mu^2\phi^2$-vertices shown in eqs. (20.6) and (20.7) remove divergences from these vertices.

The proof for this is considerably complicated by the fact that we should not perform subtractions from renormalization parts at the points where all external momenta vanish, since at these points infrared divergences of the renormalization parts are uncontrollable. For this reason, the BPHZ R-operation has to be performed at some points where all external momenta p_m are Euclidean, $p_m^2 < 0$. In any case, the gauge invariant renormalizability of Green's functions in the Landau gauge, i.e., the possibility of renormalizing Green's functions in terms of the scaling as in eq. (20.5) as indicated above, were shown in paper I and paper II, section 2 of Lee and Zinn-Justin*.

Let us now consider the case $\mu^2 < 0$. For this case, let us mimic the developments of the σ-model we presented in section 17. From the generating functions $Z[\mathbf{J}_\mu, \mathbf{J}]$ of the connected Green's functions

*It should be borne in mind that the gauge transformation for the *renormalized* gauge fields is

$$\mathbf{A}_\mu \to \mathbf{A}_\mu - \boldsymbol{\omega} \times \mathbf{A}_\mu + \frac{1}{g}\left(\frac{Z_3}{Z_1}\right)\partial_\mu \boldsymbol{\omega} = \mathbf{A}_\mu - \boldsymbol{\omega} \times \mathbf{A}_\mu + \frac{1}{g}\left(\frac{\tilde{Z}_3}{\tilde{Z}_1}\right)\partial_\mu \boldsymbol{\omega}.$$

$$W(J_\mu, J) = \exp\{iZ[J_\mu, J]\}, \tag{20.9}$$

we define the generating functional of proper vertices

$$\Gamma[\mathcal{A}_\mu, \Phi] = Z[J_\mu, J] - \int d^4x [\mathcal{A}_\mu(x) \cdot J^\mu(x) + J(x) \cdot \Phi(x)] \tag{20.10}$$

where

$$\mathcal{A}_\mu(x) = \delta Z/\delta J^\mu(x) \tag{20.11}$$

and

$$\Phi(x) = \delta Z/\delta J(x). \tag{20.12}$$

The Maxwell equations dual to eqs. (20.11) and (20.12) are

$$-J_\mu(x) = \delta\Gamma/\delta\mathcal{A}_\mu(x) \tag{20.13}$$

$$-J(x) = \delta\Gamma/\delta\Phi(x). \tag{20.14}$$

The expansion of Γ around $\Phi = 0$ and $\mathcal{A}_\mu = 0$ generates proper vertices of the symmetric theory, $\mu^2 > 0$, and conversely, the knowledge of the renormalized proper vertices amounts to knowing the renormalized form of Γ. Now we consider the equation

$$-\boldsymbol{\gamma} = \delta\Gamma/\delta\Phi(x)\big|_{\Phi=u,\,\mathcal{A}_\mu=0} \tag{20.15}$$

which determines the vacuum expectation value of the scalar fields, when the system is subjected to a constant external source $\boldsymbol{\gamma}$:

$$u = u(\boldsymbol{\gamma}). \tag{20.16}$$

The direction of $\boldsymbol{\gamma}$ may be defined as the z-direction in the isospin space. The isospin invariance of Γ implies that $\mathbf{u}$ is along the z-direction. Just as for eq. (17.22) of section 17, we obtain from eq. (20.15)

$$-u\,\Delta_\chi^{-1}(0) = \gamma \tag{20.17}$$

where Δ_χ is the momentum-space propagator of those components of the scalar field which are perpendicular to γ. Denoting

$$-\Delta_\chi^{-1}(0) = m_\chi^2,$$

we see that eq. (20.17) may be written as

$$u\,m_\chi^2 = \gamma. \tag{20.18}$$

So the spontaneous breakdown of the gauge symmetry entails

$$m_\chi^2 = 0 \tag{20.19}$$

and

$$v \equiv u(\boldsymbol{\gamma} = 0) \neq 0 \tag{20.20}$$

where v is the spontaneous vacuum expectation value of the scalar field. We can adjust $\delta\mu^2$ so that* $m_\chi^2 \to 0$: the expansion coefficients of Γ about $\mathcal{A}_\mu = 0, \Phi = \mathrm{v}$ are the proper vertices of the spontaneously broken gauge theory. The generating functions $Z[\mathrm{J}_\mu, \mathrm{J}]$ satisfies the same Ward-Takahashi identity in the limit $m_\chi^2 \to 0$, for its response to the gauge transformation (20.4) is independent of the value of μ^2.

In the spontaneously broken symmetry case, it is convenient to write the scalar field ϕ_0 as

$$\phi_0 = (\mathrm{v}_0 + \psi_0) + \chi_0 \tag{20.21}$$

so that $\mathrm{v}_0 \cdot \chi_0 = 0$, $\psi_0 \cdot \chi_0 = 0$. The action (20.2) can be written as

$$\begin{aligned} S[\mathrm{A}_{0\mu}, \psi_0, \chi_0] = \int \mathrm{d}^4 x \{ & -\tfrac{1}{4}(\partial_\mu \mathrm{A}_{0\nu} - \partial_\nu \mathrm{A}_{0\mu} + g_0 \mathrm{A}_{0\mu} \times \mathrm{A}_{0\nu})^2 + \tfrac{1}{2} g_0^2 (\mathrm{v}_0 \times \mathrm{A}_{0\mu})^2 \\ & + \tfrac{1}{2}[(\partial_\mu \psi_0)^2 - (2\lambda_0 v_0^2)\psi^2] + \tfrac{1}{2}(\partial_\mu \chi_0)^2 - g_0 \mathrm{v}_0 \cdot (\mathrm{A}_{0\mu} \times \partial^\mu \chi) - g_0 \psi_0 \cdot (\mathrm{A}_{0\mu} \times \partial^\mu \chi_0) \\ & - g_0 \chi_0 \cdot (\mathrm{A}_{0\mu} \times \partial^\mu \psi_0) - g_0 \chi_0 \cdot (\mathrm{A}_{0\mu} \times \partial^\mu \chi_0) + \tfrac{1}{2} g_0^2 [(\psi_0 \times \mathrm{A}_{0\mu})^2 \\ & + 2(\mathrm{v}_0 \times \mathrm{A}_0^\mu) \cdot (\chi_0 \times \mathrm{A}_{0\mu}) + 2(\mathrm{v}_0 \times \mathrm{A}_0^\mu) \cdot (\psi_0 \times \mathrm{A}_{0\mu}) + 2(\psi_0 \times \mathrm{A}_0^\mu) \cdot (\chi_0 \times \mathrm{A}_{0\mu}) \\ & + (\chi_0 \times \mathrm{A}_{0\mu})^2] - \lambda_0 v_0 \psi_0 (\psi_0^2 + \chi_0^2) - \tfrac{1}{4}\lambda_0 (\psi_0^2 + \chi_0^2)^2 - \tfrac{1}{2}\Delta\mu^2 (\psi_0^2 + \chi_0^2) - v_0 \Delta\mu^2 \psi_0 \} \end{aligned} \tag{20.22}$$

where we have written

$$\Delta\mu^2 = \mu_0^2 + \lambda_0 v_0^2 .$$

Since the vacuum expectation value of ϕ_0 is v_0, ψ_0 must not have one. This leads to the condition that

$$v_0[\Delta\mu^2 + S] = 0 \tag{20.23}$$

where $v_0 \Delta\mu^2$ is the contribution of the last term on the right-hand side of eq. (20.22) to the vacuum expectation value of ψ, and $v_0 S$ is the higher order contribution to the process $\psi \to$ vacuum. Actually it can be shown that $(\Delta\mu^2 + S) = -[\Delta_\chi^{-1}(0)]_{\text{unrenormalized}}$ so that eq. (20.23) is nothing but (20.18) in the limit $\gamma = 0$, and tells us that $\Delta\mu^2$ should be chosen to make $m_\chi^2 \equiv -\Delta_\chi^{-1}(0)$ vanish. We can perform the renormalization transformation of eq. (20.5) in $\mathrm{A}_{0\mu}$, $\phi_0 = \mathrm{v}_0 + \psi_0 + \chi_0$, g_0 and λ_0 in eq. (20.22) [note that v_0, ψ_0 and χ_0 must all transform like ϕ_0] to generate necessary renormalization counterterms. The discussion of the preceeding paragraph then implies that choosing renormalization constants Z_1, Z_2, Z_3, Z_4, $\widetilde{Z}_1$ and $\widetilde{Z}_2$ to be the same as in a symmetric theory will eliminate divergences completely from the spontaneously broken symmetry version of the theory.

The Feynman rules for this theory are obtained if we write

$$S[\mathrm{A}_{0\mu}, \phi_0] + S_c[\mathrm{A}_{0\mu}, \mathrm{c}_0, \mathrm{c}_0^\dagger] = S_0[\mathrm{A}_\mu, \psi, \chi, \mathrm{c}, \mathrm{c}^\dagger] + S_\mathrm{I}[\mathrm{A}_\mu, \psi, \chi, \mathrm{c}, \mathrm{c}^\dagger] \tag{20.24}$$

with

$$S_0 = \int \mathrm{d}^4 x \{ -\tfrac{1}{4}(\partial_\mu \mathrm{A}_\nu - \partial_\mu \mathrm{A}_\nu)^2 + \tfrac{1}{2} g^2 (\mathrm{v} \times \mathrm{A}_\mu)^2 + \tfrac{1}{2}(\partial_\mu \chi)^2 + \tfrac{1}{2}[(\partial_\mu \psi)^2 - (2\lambda \mathrm{v}^2)\psi^2] - g\mathrm{v} \cdot (\mathrm{A}_\mu \times \partial^\mu \chi) - \partial^\mu \mathrm{c}^\dagger \cdot \partial_\mu \mathrm{c} \} \tag{20.25}$$

where all quantities in the definition of S_0 refer to the renormalized ones, and S_I is defined as the rest, including renormalization counterterms. The generating functional $W_\mathrm{L}[\mathrm{J}_\mu, \mathrm{J}]$ may be written as

* A change in $\delta\mu^2$ affects Z's only by finite multiplicative factors, see for example K. Symanzik, Comm. Math. Phys. 23 (1972) 46.

$$W_{\mathrm{L}}[J_\mu, J] = \exp\{\mathrm{i}v\int \mathrm{d}^4x J_1(x)\}\exp\{\mathrm{i}\, S_1[\delta/\mathrm{i}\delta J^\mu, \delta/\mathrm{i}\delta J_3, \delta/\mathrm{i}\delta J_{1,2}, \delta/\delta K^\dagger, \delta/\delta K]\, W_{\mathrm{Lo}}[J^\mu, J, K, K^\dagger]\}\big|_{K=K^\dagger=0} \qquad (20.26)$$

where

$$W_{\mathrm{Lo}} = \int [\mathrm{d}A_\mu][\mathrm{d}\psi][\mathrm{d}\chi][\mathrm{d}c^\dagger][\mathrm{d}c]\exp\Big[\mathrm{i}\Big\{S_0 + \int \mathrm{d}^4x\Big[-\frac{1}{2\alpha}(\partial^\mu A_\mu(x))^2 + J_\mu(x)\cdot A^\mu(x) + J_3(x)\psi(x) + J(x)\cdot\chi(x) + K^\dagger(x)\cdot c(x) + c^\dagger(x)\cdot K(x)\Big]\Big\}\Big] \ , \qquad (20.27)$$

K and $K^\dagger$ being anticommuting c-numbers. The propagators of the theory are easily obtained from eq. (20.27), and perturbation theory is based on the formula (20.26) and on the idea of loop-wise expansion, as explained in section 16. The propagators are, as $\alpha \to 0$,

$A^{1,2}$: $-\mathrm{i}(g_{\mu\nu} - k_\mu k_\nu/(k^2 + \mathrm{i}\epsilon))/(k^2 - \mu^2 + \mathrm{i}\epsilon)$, $\qquad \mu = gv$,

$\chi^{1,2}$: $\mathrm{i}/(k^2 + \mathrm{i}\epsilon)$

A^3_μ : $-\mathrm{i}(g_{\mu\nu} - k_\mu k_\nu/k^2)/(k^2 + \mathrm{i}\epsilon)$

ψ : $\mathrm{i}/\{k^2 - (2\lambda v^2) + \mathrm{i}\epsilon\}$

$c^{1,2,3}$: $\mathrm{i}/(k^2 + \mathrm{i}\epsilon)$.

This model may be considered as the Georgi-Glashow model discussed in Part I, without fermions. A^3_μ is the photon, ψ is the physical neutral Higgs boson, $(A^1_\mu \pm \mathrm{i}A^2)/\sqrt{2}$ are the $W^\pm$ boson fields. The W boson propagator in this gauge may be written as

$$-\mathrm{i}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2 + \mathrm{i}\epsilon}\right)\frac{\mathrm{i}}{k^2 - \mu^2 + \mathrm{i}\epsilon} = -\mathrm{i}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{\mu^2}\right)\frac{1}{k^2 - \mu^2 + \mathrm{i}\epsilon} - \mathrm{i}\frac{k_\mu k_\nu}{\mu^2}\frac{1}{k^2 + \mathrm{i}\epsilon} \ . \qquad (20.28)$$

The first term on the right-hand side is the canonical propagator for a massive vector boson with three degrees of polarization freedom. The second term corresponds to a massless scalar boson which couples to the source of the vector meson gradiently. The trouble is that this scalar particle is associated with a negative probability. So Green's functions of this theory are full of "ghosts".

What happens in an S-matrix element, which is obtained from the Green's function by removing external lines, setting external momenta on the mass-shell, and contracting tensor indices with appropriate physical polarization vectors, is that the poles at $k^2 = 0$ associated with the projection operator $(g_{\mu\nu} - k_\mu k_\nu/k^2)$ in the vector boson propagators, and of the propagators for the unphysical Higgs scalars $\chi^{1,2}$ and for the scalars $c^{1,2,3}$ of the wrong statistics cancel, so that none of the massless scalar particles in the theory are physical. The physical particles are the photon (A^3_μ) which is massless and has two polarizations, a neutral massive scalar meson (ψ) and a pair of massive charged vector bosons with three polarizations. This precisely is what is predicted by the Higgs-Kibble theorem discussed in Part I. The reader is invited to verify this fact for a simple process like $W^+ + \psi \to W^+ + \psi$ in lowest order. In this case there are four diagrams which contri-

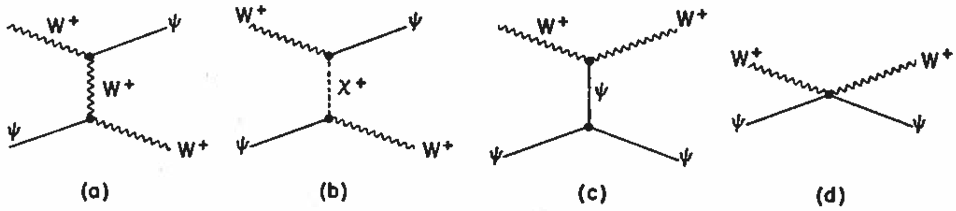

Fig. 20.1. Lowest order graphs for $\psi - W^+$ elastic scattering.

bute in lowest order (see fig. 20.1). When all external particles are physical, the pole in the t-channel at $t = 0$ is absent.

The proof of the cancellation of spurious poles at $k^2 = 0$ in the S-matrix proceeds from the Ward-Takahashi identities satisfied by *renormalized* Green's functions. These relations are used to show that when the imaginary part of an S-matrix element is computed by the unitarity relation, the contributions from massless particles associated with the three kinds of the $k^2 = 0$ poles add up to zero. The proof is extremely tedious and was worked out explicitly and in detail for intermediate states containing one, two and three such unphysical quanta in paper II of Lee and Zinn-Justin.

The discussion to be presented in the forthcoming sections obviates the necessity of proving the cancellation of spurious singularities in this manner.

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The quantization of spontaneously broken gauge theories in a renormalizable gauge was first suggested, and carried out by
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This section is based largely on
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which were referred to as I and II respectively in the text.

The Taylor-Slavnov identity was derived in
3. J.C. Taylor, Nucl. Phys. B33 (1971) 436.
4. A. Slavnov, Theor. and Math. Phys. 10 (1972) 153 (in Russian); 10 (1972) 99 (English translation).

In this section, as well as in ref. [2], renormalization is discussed primarily in the context of an O(3) gauge theory. A discussion applicable to any gauge theory will be presented in
5. B.W. Lee, to be published.
based in the Ward-Takahashi identities satisfied by irreducible vertices, derived in
6. B.W. Lee, Phys. Letters, to be published.

21. The R_ξ-gauges

In this section we will discuss a formulation of spontaneously broken gauge theories in a class of gauges in which the proof of the unitarity of the S-matrix is fairly simple. But first, let us describe spontaneous broken gauge theories in a general way, without making commitments as to the group involved and the representation of scalar fields.

Let ϕ_i ($i = 1, 2, ...K$) be a set of scalar fields transforming, in general, reducibly under G of dimension N:

$$\phi \to (1 + \mathrm{i}\, u^\alpha L_\alpha)\phi, \qquad \alpha = 1, 2, ...N \tag{21.1}$$

where u^α are infinitesimal parameters of the group, and L_α's are representations of the generators of G. We include the coupling constants of the gauge theory in L so that the structure constants, $C_{\alpha\beta\gamma}$, defined by

$$[L_\alpha, L_\beta] = \mathrm{i}\, C_{\alpha\beta\gamma} L_\gamma \tag{21.2}$$

depend on the coupling constants. We choose ϕ to be real so that L can be made imaginary antisymmetric, and $C_{\alpha\beta\gamma}$ real and completely antisymmetric. With this convention, the gauge-invariant renormalizable Lagrangian is written as

$$\mathcal{L} = -\tfrac{1}{4} F^\alpha_{\mu\nu} F^{\alpha\mu\nu} + \tfrac{1}{2}(\partial^\mu \phi^\mathrm{T} + \mathrm{i}\, \phi^\mathrm{T} L_\alpha A^{\alpha\mu})(\partial_\mu \phi - \mathrm{i} L_\beta A^\beta_\mu \phi) - V(\phi) \tag{21.3}$$

where $V(\phi)$ is an invariant quartic polynomial in ϕ, and

$$F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + C_{\alpha\beta\gamma} A^\beta_\mu A^\gamma_\nu . \tag{21.4}$$

Let v be the vacuum expectation value of ϕ in the Landau gauge, and define ϕ' by

$$\phi = v + \phi' . \tag{21.5}$$

[As Appelquist et al. (see bibliography) have stressed, the vacuum expectation value of a scalar field v depends in general on the gauge. In final results, we can always trade v for the mass of a surviving Higgs scalar meson, for example, which is gauge invariant, as a fundamental parameter of the theory.] In terms of ϕ', the Lagrangian (21.3) can be written as

$$\mathcal{L} = -\tfrac{1}{4} F^\alpha_{\mu\nu} F^{\alpha\mu\nu} + \tfrac{1}{2}(v, L_\alpha L_\beta v) A^\alpha_\mu A^{\mu\beta} + \tfrac{1}{2}(\partial^\mu \phi'^\mathrm{T} + \mathrm{i}\phi'^\mathrm{T} A^{\alpha\mu} L_\alpha)(\partial_\mu \phi' - \mathrm{i} A^\alpha_\mu L_\alpha \phi')$$
$$+ \mathrm{i}(v, L_\alpha \partial_\mu \phi') A^{\alpha\mu} + (v, L_\alpha L_\beta \phi') A^\alpha_\mu A^{\beta\mu} - V(\phi' + v). \tag{21.6}$$

The gauge invariant potential can be written as

$$V(\phi) = V(v) + \phi'_i \frac{\partial V(v)}{\partial v_i} + \frac{1}{2!} \phi'_i \phi'_j \frac{\partial^2 V(v)}{\partial v_i \partial v_j} + \frac{1}{3!} \phi'_i \phi'_j \phi'_k \frac{\partial^3 V(v)}{\partial v_i \partial v_j \partial v_k} + \frac{1}{4!} \phi'_i \phi'_j \phi'_k \phi'_l \frac{\partial^4 V(v)}{\partial v_i \partial v_j \partial v_k \partial v_l} . \tag{21.7}$$

Let us recall the discussion of section 2. The vector boson mass matrix

$$(\mu^2)_{\alpha\beta} \equiv (v, L_\alpha L_\beta v), \qquad \mu^2 = (\mu^2)^\mathrm{T} \tag{21.8}$$

has rank $N - M$ where M is the dimension of the little group of v. We can decompose the representation space of ϕ by the projection operators P and $(1 - P)$:

$L_\alpha \phi = 0$ if $\phi \in$ little group of v.

$$\delta_{ij} = P_{ij} + (1 - P)_{ij} \qquad P_{ij} = \sum_{\alpha,\beta} (L_\alpha v)_i \left(\frac{1}{\mu^2}\right)_{\alpha\beta} (v^\mathrm{T} L_\beta)_j ; \tag{21.9}$$

P is the projection operator onto the space of the Goldstone bosons which is $N - M$ dimensional. Note that the sum over α and β in (21.9) actually extends over the $N - M$ generators which are not of the little group of v. We may decompose the quadratic terms of ϕ' of (21.7) into two parts:

$$\partial^2 V(v)/\partial v_i \partial v_j = (M^2)_{ij} + P_{ik} \partial^2 V(v)/\partial v_k \partial v_j ,$$
$$(M^2)_{ij} = (1 - P)_{ik} \partial^2 V(v)/\partial v_k \partial v_j . \tag{21.10}$$

We can always adjust the quadratic term of $V(\phi)$, $\frac{1}{2}\phi^\mathrm{T} A \phi$, $[A, L_\alpha] = 0$, so that M^2 is a positive semi-definite matrix.

To lowest order, v is determined by the condition

$$\frac{\partial V(v)}{\partial v_i} = 0, \qquad v \neq 0, \tag{21.11}$$

from which it follows that [see eq. (2.18)]

$$P_{ik}\,\partial^2 V(v)/\partial v_k \partial v_j = 0. \tag{21.12}$$

In higher orders in the Landau gauge, v is given by

$$P_{ik}\,[\Delta^{-1}(0)]_{kj} = 0 \tag{21.13}$$

which is the generalization of eq. (20.17) (with $\gamma = 0$). In (21.13) $\Delta(0)$ is the scalar meson propagator matrix at zero momentum, which is a function of v.

We shall write the Lagrangian $\mathcal{L}$ as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\mathrm{I}$$

where

$$\mathcal{L}_0 = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^\mathrm{T}(\partial^\mu A^\nu - \partial^\nu A^\mu) + \tfrac{1}{2} A_\lambda^\mathrm{T} \mu^2 A^\lambda + \tfrac{1}{2}(\partial^\mu \phi')^\mathrm{T}(\partial_\mu \phi') - \tfrac{1}{2}\phi'^\mathrm{T} M^2 \phi' + \mathrm{i} A_\mu^\alpha (v, L_\alpha \partial^\mu \phi') \tag{21.14}$$

and $\mathcal{L}_\mathrm{I}$ is the rest [the sum of interaction terms and the counter terms $\phi_i' \partial V(v)/\partial v_i + \tfrac{1}{2}\phi_i P_{ik} \phi_j \partial^2 V(v)/\partial v_k \partial v_j$]. In eq. (21.14), we have expressed A_μ^α as a column matrix. The Lagrangian (21.14) is a perfectly straightforward free Lagrangian for M massless vector bosons, $N - M$ massive vector bosons and a multiplet of scalar mesons, but for the last term, which couples the longitudinal components of the massive vector bosons to some of the scalar mesons.

The development in section 14 suggests that we consider a wider class of gauge conditions than the Landau gauge. Let us further generalize the considerations there and consider gauge conditions of the form

$$F_\alpha(A_\mu, \phi) - a_\alpha = 0 \tag{21.15}$$

where a_α is in general an arbitrary function of space-time, so that

$$F_\alpha(A_\mu^g, \phi^g) - a_\alpha = 0 \tag{21.16}$$

has a unique solution for g, given A_μ and ϕ. Following the discussion of section 14, we define the determinant Δ_F by

$$\Delta_\mathrm{F}^{-1}[A_\mu, \phi] = \int \prod_x \mathrm{d}g(x) \prod_{x,\alpha} \delta[F_\alpha(A_\mu^g, \phi^g) - a_\alpha]. \tag{21.17}$$

The counterparts of eqs. (14.12) and (14.13) are

$$\Delta_\mathrm{F}[A_\mu, \phi] = \det M_\mathrm{F} \tag{21.18}$$

where

$$[M_\mathrm{F}(x, y)]_{\alpha\beta} = \delta F_\alpha(A_\mu^g(x), \phi^g(x))/\delta u_\beta(y)\big|_{u=0} \tag{21.19}$$

and for g in the neighborhood of the identity,

$$F_\alpha(A^g_\mu(x), \phi^g(x)) = F_\alpha(A_\mu(x), \phi(x)) + \int \mathrm{d}^4y \sum_\beta [M_\mathrm{F}(x, y)]_{\alpha\beta} u_\beta(y) + \mathrm{O}(u^2).$$

The generating functional of Green's functions in this gauge is

$$W_{\mathrm{F,a}}[J^\alpha_\mu, J_i] = \int [\mathrm{d}A^\alpha_\mu][\mathrm{d}\phi_i]\,\Delta_\mathrm{F}[A_\mu, \phi] \prod_{x,\alpha} \delta[F_\alpha(A_\mu, \phi) - a_\alpha]\exp\{\mathrm{i}\int \mathrm{d}^4x[\mathcal{L}(x) + J^\mathrm{T}_\mu A^\mu + J^\mathrm{T}\phi]\}. \quad (21.20)$$

Repeating the argument in section 16, where we showed that the renormalized S-matrix is the same in the Coulomb and Landau gauges, we can show that the renormalized S-matrix is independent of the arbitrary function a_α in the gauge condition (21.15). One may therefore integrate over $a_\alpha(x)$ (with an arbitrary weight factor) the right-hand side of eq. (21.20) without changing the renormalized S-matrix

$$W_\mathrm{F} \equiv \int \prod_{x,\alpha} \mathrm{d}a_\alpha(x) \exp\left\{-\frac{\mathrm{i}}{2}\int a^2_\alpha(x)\,\mathrm{d}^4x\right\} W_{\mathrm{F,a}} \quad (21.21)$$

where we have inserted a Gaussian factor arbitrarily. We obtain finally

$$W_\mathrm{F} \sim \int [\mathrm{d}A^\alpha_\mu][\mathrm{d}\phi_i] \det M_\mathrm{F} \exp[\mathrm{i}\int \mathrm{d}^4x\{\mathcal{L}(x) - \tfrac{1}{2}F^\mathrm{T}F + J^\mathrm{T}_\mu A^\mu + J^\mathrm{T}\phi\}]. \quad (21.22)$$

Equation (21.22) is the basis of a formulation based on the general gauge condition (21.15).

The idea is to choose F_α in such a way that the part of $\mathcal{L}(x) - \frac{1}{2}F^\mathrm{T}F$ which is quadratic in A_μ and ϕ' is nonsingular. We choose F_α to be, with a real, nonnegative ξ,

$$F_\alpha = \sqrt{\xi}\left[\partial^\mu A^\alpha_\mu - \frac{\mathrm{i}}{\xi}(v, L^\alpha\phi')\right]. \quad (21.23)$$

Then the sum of $\mathcal{L}_0$ in eq. (21.14) and $-\frac{1}{2}F^\mathrm{T}F$ is

$$\mathcal{L}_0 - \tfrac{1}{2}F^\mathrm{T}F = -\tfrac{1}{2}A^\mathrm{T}_\mu[-g^{\mu\nu}\partial^2 + \partial^\mu\partial^\nu(1-\xi) - g^{\mu\nu}\mu^2]A_\nu + \tfrac{1}{2}\phi'^\mathrm{T}\left[-\partial^2 - M^2 - \frac{1}{\xi}\sum_\alpha L_\alpha v)(v^\mathrm{T}L_\alpha\right]\phi' \quad (21.24)$$

where we have dropped terms which are four-divergences. The cross-term in the square of F_α has cancelled the term which couples the longitudinal components of the massive vector bosons to some of the scalar bosons.

The propagators in this gauge are obtained from the formula

$$W_{\mathrm{F}0} \sim \int [\mathrm{d}A^\alpha_\mu][\mathrm{d}\phi_i] \exp[\mathrm{i}\int \mathrm{d}^4x\{\mathcal{L}_0(x) - \tfrac{1}{2}F^\mathrm{T}F + J^\mathrm{T}_\mu A^\mu + J^\mathrm{T}\phi\}]$$

$$\sim \exp\left(-\frac{\mathrm{i}}{2}\right)\int \mathrm{d}^4x\,\mathrm{d}^4y\,[J^\mathrm{T}_\mu(x)\Delta^{\mu\nu}_\mathrm{F}(x-y)J_\nu + \mathrm{i}J^\mathrm{T}(x)\Delta_\mathrm{F}(x-y)J(y)]. \quad (21.25)$$

[We apologize for the proliferation of the symbol Δ_F: in eqs. (21.17)–(21.20), it is a Jacobian of field variables; here it stands for the Feynman propagator.] The propagators satisfy

$$[(\partial^2 + \mu^2 - \mathrm{i}\epsilon)g^{\mu\nu} - \partial^\mu\partial^\nu(1-\xi)]\,\Delta_{\mathrm{F}\nu\lambda}(x-y;\xi) = -g^{\mu\lambda}\delta^4(x-y),$$

$$\left[-\partial^2 - M^2 - \frac{1}{\xi}\sum_\alpha L_\alpha v)\,(v^{\mathrm{T}}L_\alpha + \mathrm{i}\epsilon\right]\Delta_{\mathrm{F}}(x-y;\xi) = \delta^4(x-y) \tag{21.26}$$

which incorporate the boundary condition given by the Euclidicity postulate. The momentum space propagators are given by

$$\Delta_{\mathrm{F}}^{\mu\nu}(k;\xi) = -\left[g^{\mu\nu} - \left(1-\frac{1}{\xi}\right)k_\mu k_\nu\,\frac{1}{k^2-\mu^2/\xi+\mathrm{i}\epsilon}\right]\frac{1}{k^2-\mu^2+\mathrm{i}\epsilon} \tag{21.27}$$

and

$$\Delta_{\mathrm{F}}(k^2;\xi) = \frac{1}{k^2-M^2-(1/\xi)\,\Sigma_\alpha L_\alpha v)\,(v^{\mathrm{T}}L_\alpha+\mathrm{i}\epsilon} = P\,\frac{1}{k^2-\mu^2/\xi+\mathrm{i}\epsilon} + (1-P)\,\frac{\mathrm{i}}{k^2-M^2+\mathrm{i}\epsilon} \tag{21.28}$$

where the projection operator P is defined in eq. (21.12). (The derivation of the second line of eq. (21.28) from the first is left as a challenge to the dedicated reader.)

Now let us consider $\det M_{\mathrm{F}}$. For infinitesimal u_α we have

$$\phi^g = \phi - \mathrm{i}\,u^\alpha L_\alpha\phi, \tag{21.29}$$

$$[A_\mu^g]^\alpha = A_\mu^\alpha - \mathrm{i}\,u^\beta C_{\alpha\beta\gamma}A_\mu^\nu - \partial_\mu u^\alpha, \tag{21.30}$$

so that

$$F_\alpha(A_\mu^g,\phi^g) - F_\alpha(A_\mu,\phi) = \partial^2 u^\alpha - \mathrm{i}\,C_{\alpha\beta\gamma}\partial^\mu(A_\mu^\nu u^\beta) + \frac{\mathrm{i}}{\xi}(v,\,L^\alpha L^\beta\phi)\mathrm{i}\,u^\beta + \mathrm{O}(u^2). \tag{21.31}$$

Making use of eqs. (21.18), (21.19) and discussions of section 14, we can write

$$\det M_{\mathrm{F}} = \det\{\delta F_\alpha(A_\mu^g(x)\phi^g(x))/\delta u_\beta(y)\}\,|_{g=0} \tag{21.32}$$

$$= \int[\mathrm{d}c_\alpha]\,[\mathrm{d}c_\alpha^\dagger]\exp\left[\mathrm{i}\int\mathrm{d}^4x\left\{\partial^\mu c_\alpha^\dagger\partial_\mu c_\alpha - c_\alpha^\dagger\left(\frac{1}{\xi}\mu^2\right)_{\alpha\beta}c_\beta + \mathrm{i}c_{\alpha\beta\gamma}\partial^\mu c_\alpha^\dagger A_\mu^\gamma c_\beta - \frac{1}{\xi}c_\alpha^\dagger(v^{\mathrm{T}}L^\alpha L^\beta\phi')c_\beta\right\}\right]$$

where c and $c^\dagger$ are N-component complex fields of anticommuting c-numbers. The propagator for c is

$$\Delta^c(k^2,\xi) = \mathrm{i}/(k^2-\mu^2/\xi+\mathrm{i}\epsilon). \tag{21.33}$$

The propagators (21.27), (21.28) and (21.33) become those of the Landau gauge as $\xi\to\infty$. In fact in this limit, the Feynman rules for the ξ-gauge are identical to those of the Landau gauge. Incidentally, the vector propagator of the form of eq. (21.27) is precisely that devised by T.D. Lee and C.N. Yang over a decade ago in their attempt to construct a regularizable theory of weak interactions, which they called the ξ-limiting procedure. In this gauge, the spurious $k^2 = 0$ singularities in the vector and scalar propagators, and the propagators for the fictitious scalars of the wrong statistics in the Landau gauge, have been displaced to $k^2 = \mu^2/\xi$. Furthermore, we shall show in the next lecture that *the renormalized S-matrix is independent of the parameter* ξ. This can only mean that *the poles at* $k^2 = \mu^2/\xi$ *disappear in the S-matrix completely.*

Note that the vector boson propagator of eq. (21.27) behaves as $1/k^2$ for large k^2, and all the interactions of the theory are of the renormalizable type, so that the superficial degree of divergence of any proper diagram in this gauge is at most two. One is at liberty to consider the limit $\xi \to 0$, *after* Feynman integrations are performed. To take this limit in the integrand is dangerous, since the integral may not then be well-defined (with the dimensional regularization, however, only-loop diagrams for S-matrix elements seem to be controllable even in this gauge). In any case, we may just take this limit in the propagators to see the particle content of the theory. In the limit $\xi \to 0$, there are M massless, and $N - M$ massive vector bosons, $K - N - M$ massive scalar bosons and no other spurious particles. Thus in this limit we are in the unitary gauge discussed in section 3. In this limit

$$\det M_{\mathrm{F}} \sim \int [\mathrm{d}c_\alpha]\,[\mathrm{d}c_\alpha^\dagger] \exp\left[\mathrm{i}\int \mathrm{d}^4x\, c_\alpha^\dagger(x)\left\{-\frac{1}{\xi}\mu^2_{\alpha\beta} - \frac{1}{\xi}(v^{\mathrm{T}}L^\alpha L^\beta \phi'(x))\right\} c_\beta(x)\right]$$

$$\sim \exp\left[\delta^4(0)\int \mathrm{d}^4x \sum_\alpha [\ln(1 + J(x)]_{\alpha\alpha}\right]$$

where

$$[J(x)]_{\alpha\beta} = (1/\mu^2)_{\alpha\gamma}(v,\, L^\gamma L^\beta \phi'(x))$$

a result originally due to Weinberg.

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22. Proof that the renormalized S-matrix is independent of ξ

In this section we will derive the Ward-Takahashi identity for W_{F} and show that the renormalized S-matrix is independent of ξ. In particular, it will then follow that the poles in propagators at μ^2/ξ are spurious. We shall then comment on the practical way of performing renormalization in this scheme: In the following discussion all fields refer to unrenormalized ones, and all Feynman integrals are dimensionally regularized.

The Ward-Takahashi identity for W_{F} is so complicated that unless we use a compact notation it is almost impossible to print it. We shall let ϕ_a be the set of all fields including the gauge fields, so that a runs over $\alpha = 1, 2, \ldots N$ and $i = 1, 2, 3, \ldots, K$, in the notation of the previous section. As before, α labels the generators of the group G. We let the indices a and α stand for the space-time variables and tensor indices as well as the internal symmetry indices, and summation and integration over repeated indices will always be understood in this section. With this convention, the infinitesimal transformation law of the field ϕ_a may be written as

$$\phi_a^g = \phi_a + [\Gamma^\alpha_{ab}\phi_b + \Lambda^\alpha_a]u_\alpha + \mathrm{O}(u^2) \tag{22.1}$$

where Γ^α_{ab} is a reducible representation of the generator labeled by α, so that, for example,

$$\Gamma^\alpha_{bc} = \mathrm{i}\, C_{b\alpha c}\delta^4(x_b - x_c)$$

if b and c refer to one of the α's, and

$$\Lambda^\alpha_b = -\delta_{\alpha b}\frac{\partial}{\partial x_b}\,\delta^4(x_b - x_\alpha)$$

so that

$$\Lambda^\alpha_\beta u_\alpha = \partial_{\mu(\beta)}u_\beta(x_\beta).$$

According to the discussion of the preceeding section, the generating functional of Green's functions in the general gauge F can be written as

$$W_F[J] \sim \int [\mathrm{d}\phi]\,\Delta_F[\phi]\exp[\mathrm{i}\{S[\phi] - \tfrac{1}{2}F^2_\alpha + J_a\phi_a\}] \tag{22.2}$$

where

$$\Delta_F[\phi] = \det M_F,$$

$$(M_F)_{\alpha\beta} = \delta F_\alpha(\phi^g)/\delta u_\beta\big|_{u=0} = \frac{\delta F_\alpha}{\delta\phi_a}(\Gamma^\beta_{ab}\phi_b + \Lambda^\beta_a), \tag{22.3}$$

so that

$$F_\alpha(\phi^g) = F_\alpha(\phi) + (M_F)_{\alpha\beta}u_\beta + \mathrm{O}(u^2). \tag{22.4}$$

The Ward-Takahashi identity for $W_F[J]$ is obtained if we consider the change of integration variables ϕ to ϕ^g where

$$\phi_a \to \phi_a^g \equiv \phi_a' = \phi_a + (\Gamma^\beta_{ab}\phi_b + \Lambda^\beta_a)u_\beta \tag{22.5}$$

where we shall restrict u_β such that

$$(M_F)_{\alpha\beta}u_\beta = \lambda_\alpha \tag{22.6}$$

λ_α being some constant independent of ϕ. Since M_F depends on ϕ, the allowed u's depend on ϕ:

$$u_\alpha = [M_F^{-1}(\phi)]_{\alpha\beta}\lambda_\beta. \tag{22.7}$$

The reason for this restriction is that the change in F_α is then simple. From eq. (22.4) we see that

$$F_\alpha(\phi') = F_\alpha(\phi) + \lambda_\alpha + \mathrm{O}(\lambda^2). \tag{22.8}$$

The functional metric $[\mathrm{d}\phi]$ and $\Delta_F[\phi]$ defined by eq. (22.2) are *not* invariant under the transformation (22.5) when u is restricted by eq. (22.6). The reason is that the transformation is no longer linear in ϕ. [A simple example is afforded by $\mathrm{d}x\mathrm{d}y$ which is invariant under the transformation $x \to x \cos\theta - y \sin\theta$, $y \to y\cos\theta + x \sin\theta$. The metric is *not* invariant if θ depends on x and y.] However there is an important lemma, derived by Fradkin and Tyutin, and Slavnov for the Landau gauge, and generalized to any gauge by Lee and Zinn-Justin in the Appendix of paper IV, which states that the product $\Delta_F[\phi]\,[\mathrm{d}\phi]$ is invariant under such nonlinear gauge transformations:

$$[\mathrm{d}\phi]\,\Delta_F[\phi] = [\mathrm{d}\phi']\,\Delta_F[\phi'], \tag{22.9}$$

for ϕ' given by eqs. (22.5) and (22.6).

The proof is interesting but somewhat lengthy, so we refer the reader to Appendix of Lee – Zinn-Justin IV.

Since a change of integration variables does not change the value of the integral, the change in W_F with respect to λ must be zero when we change the variables by eqs. (22.5) and (22.6). Writing ϕ for ϕ', we have

$$0 = \int [\mathrm{d}\phi]\,\Delta_F[\phi] \exp[\mathrm{i}\{S[\phi] - \tfrac{1}{2}F_\alpha^2(\phi) + J_a\phi_a\}]\, \frac{\delta}{\delta\lambda_\alpha}\{S[\phi] - \tfrac{1}{2}F_\alpha^2(\phi) + J_a\phi_a\}$$

or

$$\int [\mathrm{d}\phi]\,\Delta_F[\phi] \exp[\mathrm{i}\{S[\phi] - \tfrac{1}{2}F_\alpha^2(\phi) + J_a\phi_a\}]\,\{-F_\alpha(\phi) + J_a(\Gamma^\beta_{ab}\phi_b + \Lambda^\beta_a)[M_F^{-1}(\phi)]_{\beta\alpha}\} = 0. \tag{22.10}$$

Equation (22.10) can be converted into a functional differential equation satisfied by $W_F[J]$:

$$\left\{-F_\alpha\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right) + J_a\left(\Gamma^\beta_{ab}\left\{\frac{1}{\mathrm{i}}\frac{\delta}{\delta J_b}\right\} + \Lambda^\beta_a\right)\left[M_F^{-1}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)\right]_{\beta\alpha}\right\} W_F[J] = 0. \tag{22.11}$$

This is the Ward-Takahashi identity for $W_F[J]$.

To make use of eq. (22.11), we must know what

$$\left[M_F^{-1}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)\right]_{\beta\alpha} W_F[J]$$

is. Consider

$$W_{\alpha\beta}[J] = \int [\mathrm{d}\phi]\,[\mathrm{d}c_\alpha]\,[\mathrm{d}c_\alpha^\dagger]\, c_\alpha c_\beta^\dagger \exp[\mathrm{i}\{S[\phi] - \tfrac{1}{2}F_\alpha^2(\phi) + c_\alpha^\dagger (M_F)_{\alpha\beta} c_\beta + J_a\phi_a\}] \tag{22.12}$$

where $c_\alpha^\dagger$ and c_α are anticommuting fields, so that

$$\Delta_F[\phi] = \det M_F = \int [\mathrm{d}c_\alpha]\,[\mathrm{d}c_\alpha^\dagger] \exp\{\mathrm{i}\, c_\alpha^\dagger (M_F)_{\alpha\beta} c_\beta\}. \tag{22.13}$$

The functional $W_{\alpha\beta}[J]$ is the Green's function for the fictitious scalar fields of the wrong statistics in the presence of external sources J, and satisfied the equation

$$\left[M_F\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)\right]_{\alpha\beta} W_{\beta\gamma}[J] = \int [\mathrm{d}\phi]\,[\mathrm{d}c]\,[\mathrm{d}c^\dagger]\,(M_F)_{\alpha\beta} c_\beta c_\gamma^\dagger \exp[\mathrm{i}\{S[\phi] - \tfrac{1}{2}F_\alpha^2(\phi) + c_\alpha^\dagger (M_F)_{\alpha\beta} c_\beta + J_a\phi_a\}]$$

$$= \int [\mathrm{d}\phi]\,[\mathrm{d}c]\,[\mathrm{d}c^\dagger]\, c_\gamma^\dagger \left(-\frac{\partial}{\partial c_\alpha^\dagger}\right) \exp[\mathrm{i}\{S[\phi] - \tfrac{1}{2}F_\alpha^2(\phi) + c_\alpha^\dagger (M_F)_{\alpha\beta} c_\beta + J_a\phi_a\}],$$

i.e.,

$$\left[M_F\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)\right]_{\alpha\beta} W_{\beta\gamma}[J] = \delta_{\alpha\gamma} W_F[J]. \tag{22.14}$$

We see that

$$\left[M_F^{-1}\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right)\right]_{\beta\alpha} W_F[J] = W_{\beta\alpha}[J]. \tag{22.15}$$

Next, let us consider what happens to $W_F[J]$ when we vary the gauge condition F_α by ΔF_α:

$$W_{F+\Delta F}[J] - W_F[J] = \int [\mathrm{d}\phi]\, \Delta_F \exp(\mathrm{i}\{S[\phi] - \tfrac{1}{2}F_\alpha^2(\phi) + J_a\phi_a\})\left\{-\mathrm{i}F_\alpha \Delta F_\alpha + \frac{\delta \Delta F_\alpha}{\delta\phi_a}(\Gamma^\beta_{ab}\phi_b + \Lambda^\beta_a)[M_F^{-1}(\phi)]_{\beta\alpha}\right\} \tag{22.16}$$

where we have used the fact that, from eq. (22.3),

$$\Delta_{F+\Delta F} = \det\left\{\left(\frac{\delta F_\alpha}{\delta\phi_a} + \frac{\delta \Delta F_\alpha}{\delta\phi_a}\right)(\Gamma^\beta_{ab}\phi_b + \Lambda^\beta_a)\right\}$$

$$= \det\left\{M_F + \frac{\delta\Delta F_\alpha}{\delta\phi_a}(\Gamma^\beta_{ab}\phi_b + \Lambda^\beta_a)\right\} \approx \Delta_F \frac{\delta\Delta F_\alpha}{\delta\phi_a}(\Gamma^\beta_{ab}\phi_b + \Lambda^\beta_a)(M_F^{-1})_{\beta\alpha}$$

to first order in ΔF_α. We will operate $\mathrm{i}\,\Delta F_\alpha(\delta/\mathrm{i}\,\delta J)$ on eq. (22.10). Noting that

$$\mathrm{i}\,\Delta F_\alpha\left(\frac{1}{\mathrm{i}}\frac{\delta}{\delta J}\right) J_a \exp(\mathrm{i}\,J_a\phi_a) = \exp(\mathrm{i}\,J_a\phi_a)\{J_a\Delta F_\alpha(\phi) + \delta\Delta F_\alpha(\phi)/\delta\phi_a\},$$

we obtain

$$0 = \int [\mathrm{d}\phi]\, \Delta_F[\phi] \exp(\mathrm{i}\{S[\phi] - \tfrac{1}{2}F_\alpha^2(\phi) + J_a\phi_a\})\left\{-\mathrm{i}F_\alpha\Delta F_\alpha + \left[J_a\Delta F_\alpha(\phi) + \frac{\delta\Delta F_\alpha(\phi)}{\delta\phi_a}\right](\Gamma^\beta_{ab}\phi_b + \Lambda^\beta_a)(M_F^{-1})_{\beta\alpha}\right\}. \tag{22.17}$$

Combining eqs. (22.16) and (22.17), we finally obtain

$$W_{F+\Delta F}[J] - W_F[J] = \int [\mathrm{d}\phi]\, \Delta_F[\phi] \exp(\mathrm{i}\{S[\phi] - \tfrac{1}{2}F_\alpha^2(\phi) + J_a\phi_a\}) J_a \Delta F_\alpha(\phi)(\Gamma^\beta_{ab}\phi_b + \Lambda^\beta_a)(M_F^{-1})_{\beta\alpha}$$

or, valid to first order in ΔF,

$$W_{F+\Delta F}[J] = \int [\mathrm{d}\phi]\, \Delta_F[\phi] \exp(\mathrm{i}\{S[\phi] - \tfrac{1}{2}F_\alpha^2(\phi) + J_a[\phi_a + (\Gamma^\alpha_{ab}\phi_b + \Lambda^\alpha_a)(M_F^{-1})_{\alpha\beta}\Delta F_\beta(\phi)]\}) \tag{22.18}$$

i.e., the effect of changing the gauge from F to $F + \Delta F$ is merely to add extra vertices between the source and fields:

$$J_a\phi_a \xrightarrow{F \to F + \Delta F} J_a\{\phi_a + (\Gamma^\alpha_{ab}\phi_b + \Lambda^\alpha_a)[M_F^{-1}(\phi)]_{\alpha\beta}\Delta F_\beta(\phi)\}.$$

What happens to the S-matrix under such circumstances has been discussed thoroughly in section 15, in conjunction with the change in the S-matrix as one passes from the Coulomb gauge to the Landau gauge. We shall adopt the conclusion there to this case, that, if two $W[J]$'s differ only in the external source term as in (22.18), the result is only a redefinition of the renormalization of the unrenormalized S-matrix. The change in an S-matrix element S may be expressed as

$$S_{F+\Delta F} = \prod_l (Z_{F+\Delta F}/Z_F)_l^{1/2} S_F$$

where $(Z_F)_l$ is the wave function renormalization for the lth external line and the product Π extends over all external lines. Thus, the quantity

$$S = S_F / \prod_l (Z_F)_l^{1/2} \tag{22.19}$$

is independent of F. Let us recall that we have been dealing with dimensionally regularized (with $n = 4-\epsilon$) quantities. In eq. (22.19) $(Z_F)_l$ is to be calculated from the two-point function $[G_F^{(2)}(\epsilon)]_l$ for the particle of type l. In general S_F and Z_F depend both on F and ϵ, and are finite as long as $\epsilon \neq 0,2$, etc.

When we specialize to the ξ-gauge, the quantity in (22.19) is

$$S(\epsilon) = \lim_{p_l^2 \to m_l^2} \prod_{l=1}^{n} \frac{(p_l^2 - m_l^2)}{[Z(\xi,\epsilon)]_l^{1/2}} G^{(n)}(\xi,\epsilon),$$

$$\lim_{p_l^2 \to m_l^2} [G^{(2)}(\xi,\epsilon)] = \frac{[Z(\xi,\epsilon)]_l}{p_l^2 - m_l^2} \tag{22.20}$$

where $G^{(n)}(\xi,\epsilon)$ is the regularized n-point Green's function in the ξ-gauge. The discussion of the preceeding paragraph implies that $S(\epsilon)$ is independent of ξ. Therefore it is devoid of spurious singularities in k^2 depending on ξ for all $\epsilon \neq 0,2$, as can be seen by taking the limit $\xi \to 0$ in (22.20).

To obtain the physical S-matrix, we must take the limit $\epsilon \to 0$ after renormalization. One way of accomplishing this is renormalize the Green's function $G^{(n)}(\xi,\epsilon)$ for arbitrary ξ. In paper IV, Lee and Zinn-Justin discuss the renormalization scheme for arbitrary ξ based on the Ward-Takahashi identities for a particular model. The same has not been worked out for a general class of models.* However this is not necessary. Since $S(\epsilon)$ is independent of ξ one may take the case $\xi = \infty$ (the Landau gauge) and obtain

$$S(\epsilon) = \lim_{p_l^2 \to m_l^2} \prod_{l=1}^{n} \frac{(p_l^2 - m_l^2)}{[Z(\infty,\epsilon)]_l^{1/2}} G^{(n)}(\infty,\epsilon).$$

This is precisely the regularized S-matrix element in the Landau gauge, and we now learn that this quantity is devoid of spurious singularities in k^2. We can now rescale the coupling constants and other parameters according to the discussion of section 20, i.e.,

$$g^{\text{o}} = g_{\text{r}} \frac{Z_1(\epsilon)}{[Z_3(\epsilon)]^{3/2}} = g_{\text{r}} \frac{\tilde{Z}_1(\epsilon)}{[Z_3(\epsilon)]^{1/2}\tilde{Z}_3(\epsilon)}, \text{ etc.,}$$

* *Note added in proof:* This has now been done; this will be reported in a forthcoming paper by B.W.L.

and take the limit $\epsilon \to 0$, while keeping g_r and other renormalized parameters fixed. The result is a finite S-matrix element, which is devoid of spurious singularities.

Bibliography

This section is based on

1. B.W. Lee and J. Zinn-Justin, Spontaneously Broken Gauge Symmetries IV-General Gauge Formulation, Phys. Rev. D7 (1973) 1049.

Presumably, the discussion here is equivalent to a combinatoric discussion of

2. G. 't Hooft and M.T. Veltman, Nucl. Phys. B50 (1972) 318.

23. Anomalous magnetic moment of the muon in the Georgi-Glashow model

As a practical application of these ideas, we calculate the first weak correction to the anomalous magnetic moment of the muon in the Georgi-Glashow model, using the R_ξ gauge. In general, the parity-conserving terms of the μ^- electromagnetic vertex have the form

$$V_\mu = \bar{u}(p+q/2)\left[\gamma_\mu F_1(q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2m} F_2(q^2)\right]u(p-q/2) \tag{23.1}$$

where $p\pm q/2$ is the final (initial) muon momentum, q is the momentum transferred to the photon, and m is the muon mass. Using the Gordon decomposition

$$2m\gamma_\mu = 2p_\mu + i\sigma_{\mu\nu}q^\nu$$

eq. (23.1) can be rewritten

$$V_\mu = \bar{u}(p+q/2)\left[\gamma_\mu[F_1(q^2)+F_2(q^2)] - \frac{p_\mu}{m}F_2(q^2)\right]u(p-q/2) \tag{23.2}$$

or

$$V_\mu = \bar{u}(p+q/2)\left[\frac{p_\mu}{m}F_1(q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2m}[F_1(q^2)+F_2(q^2)]\right]u(p-q/2). \tag{23.3}$$

In any of these forms, $F_1(q^2)$ is the electric form factor, and $F_2(q^2)$ is the magnetic form factor. In particular. $F_1(0)$ is always renormalized to be 1, and $F_2(0)$ is the anomalous magnetic moment.

In ordinary electrodynamics, V_μ is just a matrix element of the neutral, gauge-invariant, electro-

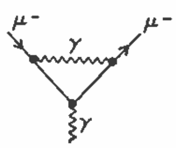

Fig. 23.1. Photon exchange contribution to the meson anomalous magnetic moment.

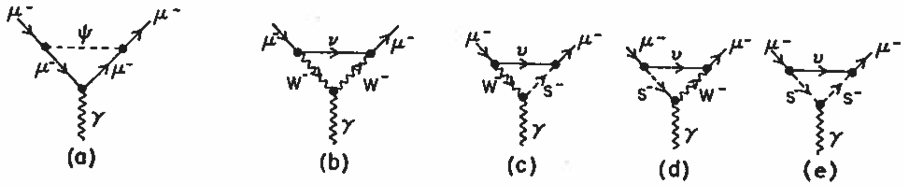

Fig. 23.2. Neutrino and scalar meson exchange contributions to the muon anomalous magnetic moment.

magnetic current, and therefore the form factors are independent of the gauge. In the more general case we have been considering, this current is not invariant under general transformations of the non-Abelian gauge group, and the form factors will be gauge-dependent. In particular, for the R_ξ gauge, $F_1(q^2)$ and $F_2(q^2)$ will depend upon ξ.

However, the electric charge and the anomalous magnetic moment, $F_1(0)$ and $F_2(0)$ respectively, are physically measurable quantities; they are related to residues of the photon pole in the S-matrix for $\mu^+\mu^-$ elastic scattering, for example. Therefore, since the S-matrix is gauge independent, we expect them to be independent of ξ. This is the fact we wish to demonstrate by an explicit calculation.

In all models, there is a contribution to $F_2(0)$ from the photon-exchange graph, fig. 23.1, which was calculated long ago to be $\alpha/2\pi$. The remaining graphs are formally of order α also, but they are all proportional to α/μ^2 (μ is the W mass, as in the previous sections) and so are numerically of the order of the Fermi constant G, and may be thought of as weak corrections to $F_2(0)$. In the Weinberg-Salam model, all these corrections are equal to Gm^2 times constants of the order one, so are very small indeed. In the Georgi-Glashow model, graphs with heavy-lepton exchange contribute terms of the order GM_0m (M_0 is the mass of the neutral heavy muon) and may be more interesting experimentally. We outline the calculation in a simple approximation.

Heavy muons M^0 and M^+, of mass M_0 and M_+, can be added to the Georgi-Glashow model discussed in section 9 in exact analogy to E^0 and E^+. In addition to the photon exchange graph (fig. 23.1), the graphs of figs. 23.2 and 23.3 all contribute to $F_2(0)$. In the U-gauge, graphs containing the charged scalars $s^\pm$ are absent, but the remaining ones are not all unambiguously convergent. In the R_ξ gauge, we must include them, but this gauge has the advantage that the contribution of each graph to $F_2(0)$ is convergent. The propagators for the scalars and the W mesons are given in eqs. (21.27) and (21.28).

The neutral scalar exchange graph, fig. 23.2a, contributes a term proportional to $Gm^2(m^2/m_\psi^2)$, which we take to be negligible, even though the model does not strictly require m_ψ to be very large. The neutrino exchange graphs, figs. 23.2b–e, are all of the order $G\mu^2$ while the M^0 exchange contributions of fig. 23.3 are proportional to GmM_0 and are therefore the largest.

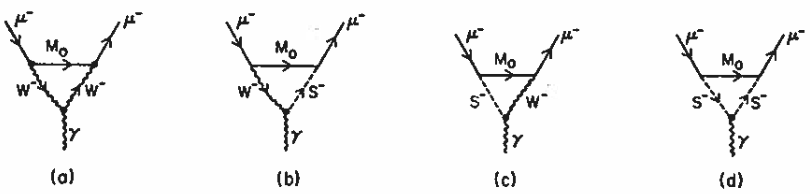

Fig. 23.3. Heavy muon exchange contribution to the muon anomalous magnetic moment.

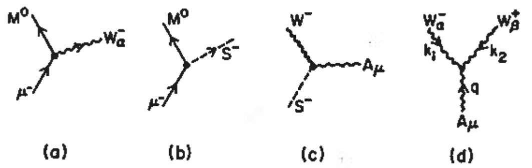

Fig. 23.4. Vertices needed for calculating the graphs in fig. 23.3.

We calculate these M° exchange graphs only. Since M_0 is arbitrary, the term proportional to M_0 must in any case be independent of ξ. We make the approximation

$$m^2 \ll M_0^2 \ll \mu^2 \tag{23.4}$$

and for convenience restrict ξ so that

$$M_0^2 \ll \mu^2/\xi \tag{23.5}$$

as well. Our result for each graph separately will therefore not be valid in the Landau gauge $\xi \to \infty$, which may be treated as a separate case.

The vertices which occur in the graphs in fig. 23.3 can be obtained from the coupling terms we wrote down in section 9.

From the muon analog (19.12), the μWM° vertex [fig. 23.4a] is

$$e\overline{M^\circ}[\cos\beta\gamma^\alpha\mu_L^- + \gamma^\alpha\mu_R^-]. \tag{23.6}$$

The sM°μ vertex, fig. 23.4b, is obtained from the muon analog of eq. (19.22):

$$G_1[\overline{M_L^\circ}s^+\mu_R^- \cos\beta + \mu_L^- s^- M_R^\circ] + \text{H.C.} + G_2 \sin\beta\, \overline{M_L^\circ}s^+\mu_R^- + \text{H.C.} \tag{23.7}$$

where

$$G_1 = \frac{e}{\mu}[m - M_0\cos\beta], \qquad G_2 = \frac{-e}{\mu}M_0\sin\beta.$$

According to eq. (19.20), the $W_\alpha^- A_\mu s^-$ vertex of fig. 23.4c is just

$$-e\,\mu\, g_{\alpha\mu}. \tag{23.8}$$

The W–W–A vertex, fig. 23.4d, can be read off the trilinear term in $-\mathbf{F}_{\mu\nu}\cdot\mathbf{F}^{\mu\nu}$, as in fig. 14.1. The result is

$$e\,\Gamma_{\alpha\beta\mu} \equiv e[g_{\alpha\beta}(k_1 - k_2)_\mu + g_{\mu\alpha}(q - k_1)_\beta + g_{\beta\mu}(k_2 - q)_\alpha)]. \tag{23.9}$$

A simplifying feature of the calculation is that we need only the term in V_μ linear in q to obtain $F_2(0)$. Therefore, terms of higher order may be dropped, but the terms proportional to q must be kept until V_μ is expressed in one of the forms (23.1–3). We use the following notation: V_μ^a, V_μ^b, V_μ^c and V_μ^d are the contributions to V_μ from the graphs in figs. 23. a–d, respectively, and F_2^a, F_2^b, F_2^c and F_2^d are defined similarly.

We turn to fig. 23.3a:

$$V^{a}_{\mu} = \frac{-ie^2}{(2\pi)^4}\int d^4k\, \bar{u}(p+q/2)N^{\alpha\gamma}u(p-q/2)\left[-g^{\beta}_{\alpha} + \frac{(1-1/\xi)(k+q/2)_{\alpha}(k+q/2)^{\beta}}{(k+q/2)^2-\mu^2/\xi+i\epsilon}\right]$$

$$\times \left[-g^{\delta}_{\gamma} + \frac{(1-1/\xi)(k-q/2)_{\gamma}(k-q/2)^{\delta}}{(k-q/2)^2-\mu^2/\xi+i\epsilon}\right]\Gamma_{\delta\beta\mu}[(p-k)^2-M_0^2+i\epsilon]^{-1}$$

$$\times\ [(k-q/2)^2-\mu^2+i\epsilon]^{-1}[(k+q/2)^2-\mu^2+i\epsilon]^{-1} \tag{23.10}$$

where $\Gamma_{\delta\beta\mu}$ is defined in (23.9), with $k_1 = k-q/2$, $k_2 = -(k_1+q/2)$. From (23.6) and (23.9)

$$\bar{u}(p+q/2)N^{\alpha\gamma}u(p-q/2) = \bar{u}(p+q/2)\left[\cos\beta\,\gamma^{\alpha}\,\frac{(1-\gamma_5)}{2} + \gamma^{\alpha}\,\frac{(1+\gamma_5)}{2}\right] \tag{23.11}$$

$$\times\ [\gamma\cdot(p-k)+M_0]\left[\cos\beta\,\gamma^{\gamma}\,\frac{(1-\gamma_5)}{2} + \gamma^{\gamma}\,\frac{(1+\gamma_5)}{2}\right]u(p-q/2)$$

$$= \bar{u}(p+q/2)[M_0\cos\beta\,\gamma^{\alpha}\gamma^{\gamma} + \tfrac{1}{2}(1+\cos^2\beta)\gamma^{\alpha}\gamma\cdot(p-k)\gamma^{\gamma} + \text{terms in } \gamma_5]u(p-q/2).$$

We ignore the parity-violating form factors proportional to γ_5. The first term in (23.11) is the one proportional to M_0, so we neglect the term in $1+\cos^2\beta$ as well. Thus, we replace (23.11) by

$$N^{\alpha\gamma} = M_0\cos\beta\,\gamma^{\alpha}\gamma^{\gamma} = M_0\cos\beta[g^{\alpha\gamma} - i\sigma^{\alpha\gamma}]. \tag{23.12}$$

First, we compute the term in (23.10) proportional to $g^{\beta}_{\alpha}g^{\delta}_{\gamma}$. It is

$$-\frac{i}{(2\pi)^4}e^2\int d^4k\, u(p+q/2)N^{\alpha\gamma}u(p-q/2)[2k_{\mu}g_{\alpha\gamma} + (\tfrac{3}{2}q-k)_{\alpha}g_{\mu\gamma}$$

$$-(k+\tfrac{3}{2}q)_{\gamma}g_{\alpha\mu}][(p-k)^2-M_0^2-i\epsilon]^{-1}[(k-q/2)^2-\mu^2+i\epsilon]^{-1}[(k+q/2)^2+\mu^2+i\epsilon]^{-1}. \tag{23.13}$$

We parametrize the denominator in the standard way:

$$[(p-k)^2-M_0^2+i\epsilon]^{-1}[(k-q/2)^2-\mu^2+i\epsilon]^{-1}[(k+q/2)+\mu^2+i\epsilon]^{-1}$$

$$= 2\int_0^1 dx\int_0^{1-x} dy\,[k^2-2k\cdot(\alpha p-\beta q)+\alpha m^2+\tfrac{1}{4}(1-\alpha)q^2-(1-\alpha)\mu^2-\alpha M_0^2+i\epsilon]^{-3} \tag{23.14}$$

where

$$\alpha = 1-x-y, \qquad \beta = \tfrac{1}{2}(x-y) \tag{23.15}$$

and change the integration variable from k to l:

$$k = l+\alpha p-\beta q. \tag{23.16}$$

Then, using (23.12), the expression (23.13) becomes (we omit writing the spinors)

$$-\frac{3e^2}{4\pi^4}M_0\cos\beta\int d^4l\,dx\,dy\,\frac{[l_{\mu}+(\alpha-1)p_{\mu}-\beta q_{\mu}+m\gamma_{\mu}]}{[l^2-(1-\alpha)\mu^2+i\epsilon]^3} \tag{23.17}$$

where, using (23.4), we have neglected terms in the denominator proportional to m^2, q^2, and M_0^2. The l_μ and βq_μ terms vanish after integration, leaving

$$\frac{3}{4\pi}\frac{\alpha M_0 \cos\beta}{\mu^2}(p_\mu + m\gamma_\mu).$$

From (23.2), we conclude that this term contributes

$$-\frac{3}{4\pi}\frac{\alpha m M_0 \cos\beta}{\mu^2} \tag{23.18}$$

to $F_2^a(0)$.

Next we compute the "crossed" terms in eq. (23.10), i.e., those proportional to $g_\alpha^\beta(1 - 1/\xi)$ and $g_\gamma^\delta(1 - 1/\xi)$.

We use the identities

$$k_{1\delta}\Gamma_\mu^{\delta\beta} = k_2^\beta k_{2\mu} - g_\mu^\beta k_2^2 \tag{23.19a}$$

$$-k_{2\beta}\Gamma_\mu^{\delta\beta} = k_1^\delta k_{1\mu} - g_\mu^\delta k_1^2 \tag{23.19b}$$

which hold up to terms quadratic in q. Using (23.19), the sum of the crossed terms in V_μ^a can be written

$$\frac{-\mathrm{i}e^2}{(2\pi)^4} M_0 \cos\beta(1 - 1/\xi)\bar{u}(p+q/2)\gamma^\alpha\gamma^\gamma u(p-q/2)(I_{\alpha\gamma}^{(1)} + I_{\alpha\gamma}^{(2)}) \tag{23.20}$$

where

$$I_{\alpha\gamma}^{(1)} = \int \mathrm{d}^4k\, g_{\alpha\beta}k_{1\gamma}(k_2^\beta k_2^\mu - g_\mu^\beta k_2^2)(k_1^2 - u^2/\xi + \mathrm{i}\epsilon)(k_1^2 - \mu^2 + \mathrm{i}\epsilon)^{-1}(k_2^2 - \mu^2 + \mathrm{i}\epsilon)^{-1}[(k-p)^2 - M_0^2 + \mathrm{i}\epsilon]$$

$$= 6\int \frac{\mathrm{d}^4k\, \mathrm{d}w\mathrm{d}x\mathrm{d}y\mathrm{d}z\, \delta(w + x + y + z - 1)g_{\alpha\beta}k_{1\gamma}(k_2^\beta k_{2\mu} - g_\mu^\beta k_2^2)}{[w(k_1^2 - \mu^2/\xi) + x(k_1^2 - \mu^2) + y(k_2^2 - \mu^2) + z[(k-p)^2 - M_0^2] + \mathrm{i}\epsilon]^4}$$

and $I_{\alpha\gamma}^{(2)}$ is defined analogously.

We define l by

$$k = l + zp + \tfrac{1}{2}\lambda q \tag{23.21}$$

where

$$\lambda = x - y + w$$

and make the same approximations in the denominator as before to obtain

$$I_{\alpha\gamma}^{(1)} = 6\int \frac{\mathrm{d}^4l\, \mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}w\, \delta(x+y+z+w-1)}{[l^2 - (x+y+w/\xi)\mu^2 + \mathrm{i}\epsilon]^4} \cdot T_{\alpha\gamma\mu}^{(1)} \tag{23.22}$$

where

$$T_{\alpha\gamma\mu}^{(1)} = [l+zp+\tfrac{1}{2}(\lambda-1)q]_\gamma \cdot \{[l+zp+\tfrac{1}{2}(\lambda+1)q]_\alpha[l+zp+\tfrac{1}{2}(\lambda+1)q]_\mu - [l+zp+\tfrac{1}{2}(\lambda+1)q]^2 g_{\alpha\mu}\}. \tag{23.23}$$

Linear and cubic terms in l in (23.23) vanish upon symmetric integration. The term independent of l is proportional to m^2/μ^2, so we neglect it. Thus, effectively, we may write

$$T^{(1)}_{\alpha\gamma\mu} = \tfrac{1}{4} l^2 \{g_{\alpha\gamma}[zp_\mu + \tfrac{1}{2}(\lambda+1)q_\mu] + g_{\gamma\mu}[zp_\alpha + \tfrac{1}{2}(\lambda+1)q_\alpha] - g_{\alpha\mu}[5zp_\gamma + \tfrac{1}{2}(5\lambda+1)q_\gamma]\} \quad (23.24)$$

$I^{(2)}$ can be worked out analogously. The total contribution of (23.20) to V^a_μ is

$$-\frac{6ie^2}{(2\pi)^4} M_0 \cos\beta\,(1-1/\xi)\bar{u}\,\gamma^\alpha\gamma^\gamma u \int \frac{d^4l\, dxdydzdw\;\delta(x+y+z+w-1)}{[l^2-(x+y+w/\xi)\mu^2+i\epsilon]}\,\frac{l^2}{4}\,T_{\alpha\gamma\mu} \quad (23.25)$$

where, with $\mu = x - y - w$,

$$T_{\alpha\gamma\mu} = q_{\alpha\mu}[\tfrac{1}{2}(\mu-5\lambda)q_\gamma - 4zp_\gamma] + g_{\alpha\mu}[\tfrac{1}{2}(\lambda-5\mu)q_\alpha - 4zp_\alpha] + g_{\alpha\gamma}[2zp_\mu + \tfrac{1}{2}(\lambda+\mu)q_\mu]. \quad (23.26)$$

Inserting (23.26) into (23.25) and contracting $T_{\alpha\gamma\mu}$ with $\gamma^\alpha\gamma^\gamma$, we obtain for the contribution (23.20) to V^a_μ

$$\frac{-9\,e^2 M_0\cos\beta}{(2\pi)^4}(1-1/\xi)\bar{u}\,\sigma_{\mu\nu}q^\nu u \int \frac{d^4l\, dxdydxdw\;\delta(x+y+z+w-1)l^2 w}{[l^2-(x+y+w/\xi)\mu^2+i\epsilon]^4} \quad (23.27)$$

and the contribution to $F^a_2(0)$ is

$$-\frac{3\alpha}{2\pi}\frac{m\,M_0\cos\beta}{\mu^2}(1-1/\xi)\int dxdydzdw\;\delta(x+y+z+w-1)\left[\frac{w}{x+y+w/\xi}\right]. \quad (23.28)$$

The integral in (23.28) is

$$\xi\int_0^1 t\left[(1-t) - \xi\, t \log\left(1+\frac{1-t}{\xi t}\right)\right]dt$$

and (23.28) becomes

$$-\frac{\alpha}{\pi}\frac{m\,M_0}{\mu^2}\left[\frac{1}{4} - \frac{1}{2}\frac{\xi}{(\xi-1)} + \frac{1}{2}\frac{\xi}{(\xi-1)^2}\log\xi\right]. \quad (23.29)$$

The remaining term in V^a_μ [eq. (23.10)] is

$$\frac{-ie^2}{(2\pi)^4} M_0\cos\beta\,\bar{u}(p+q/2)\gamma^\alpha\gamma^\gamma u(p+q/2)(1-1/\xi)^2$$

$$\times\int d^4k\,\frac{k_{2\alpha}k_2^\beta k_{1\gamma}k_1^\delta\,\Gamma_{\delta\beta\mu}}{[(k-p)^2-M_0^2+i\epsilon]^{-1}[k_1^2-\mu^2+i\epsilon]^{-1}[k_1^2-\mu^2/\xi+i\epsilon]^{-1}[k_2^2-\mu^2+i\epsilon]^{-1}[k_2^2-\mu^2/\xi+i\epsilon]^{-1}}. \quad (23.30)$$

From either eqs. (23.19), it follows that

$$k_2^\beta k_1^\delta T_{\delta\beta\mu} = 0 \quad (23.31)$$

so there is no contribution from (23.30). Therefore, from (23.18) and (23.29),

$$F^a_2(0) = \frac{-\alpha}{\pi}\frac{m\,M_0}{\mu^2}\left[1 - \frac{1}{2}\frac{\xi}{(\xi-1)} + \frac{1}{2}\frac{\xi}{(\xi-1)^2}\log\xi\right]. \quad (23.32)$$

Next we turn to the graphs of figs. 23.3b and 23.3c. The relevant vertices have been worked out in (23.6), (23.7), and (23.8). For instance,

$$V_\mu^{\mathrm{b}} = \frac{-\mathrm{i}e^2}{(2\pi)^4}\int \mathrm{d}^4k\ \bar{u}(p+q/2)N_{\mathrm{b}}^\alpha u(p+q/2)\left[-g_{\alpha\mu} + \frac{(1-1/\xi)(k-q/2)_\alpha(k-q/2)_\mu}{(k-q/2)^2-\mu^2/\xi+\mathrm{i}\epsilon}\right]$$
$$\times\ [(k-p)^2-M_0^2+\mathrm{i}\epsilon]^{-1}[(k+q/2)^2-\mu^2/\xi+\mathrm{i}\epsilon]^{-1}[(k-q/2)^2-\mu^2+\mathrm{i}\epsilon]^{-1} \tag{23.33}$$

where

$$N_{\mathrm{b}}^\alpha = \left[(m\cos\beta - M_0)\,\frac{(1-\gamma_5)}{2} + (m - M_0\cos\beta)\,\frac{(1+\gamma_5)}{2}\right][M_0+\gamma\cdot(p-k)]\left[\cos\beta\,\gamma^\alpha\frac{(1-\gamma_5)}{2}+\gamma^\alpha\frac{(1+\gamma_5)}{2}\right]$$
$$= M_0\cos\beta\,\gamma\cdot(k-p)\gamma^\alpha - M_0^2(1+\cos^2\beta)\gamma^\alpha + \text{parity violating terms} + \text{terms of order } m/M_0. \tag{23.34}$$

The second term, proportional to M_0^2, contributes terms of the order m/μ times those we are keeping to $F_2^{\mathrm{b}}(0)$, so we shall neglect it.

Similarly, we take V_μ^{c} to be

$$V_\mu^{\mathrm{c}} = \frac{-\mathrm{i}e^2}{(2\pi)^4}M_0\cos\beta\int \mathrm{d}^4k\ u(p+q/2)\gamma^\alpha\gamma\cdot(k-p)u(p-k/2)$$
$$\times\ \frac{[-g_{\alpha\mu}+(1-1/\xi)(k+q/2)_\alpha(k+q/2)_\mu/\{(k+q/2)^2-\mu^2/\xi+\mathrm{i}\epsilon\}}{[(k-p)^2-M_0^2-\mathrm{i}\epsilon]\,[k+q/2)^2-\mu^2+\mathrm{i}\epsilon]\,[(k-q/2)^2-\mu^2/\xi+\mathrm{i}\epsilon]}\,. \tag{23.35}$$

First we calculate the terms in (23.33) and (23.35) proportional to $g_{\alpha\mu}$. This can be done by introducing Feynman parameters into (23.33) just as we did in eq. (23.14), and interchanging them in (23.34) to obtain a common denominator. The calculation is straightforward, with the result that the contribution of these terms to $F_2^{\mathrm{b}}(0) + F_2^{\mathrm{c}}(0)$ is

$$\frac{\alpha}{\pi}\frac{mM_0\cos\beta}{\mu^2}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\ \frac{X}{(x/\xi+y)} = \frac{\alpha}{\pi}\frac{mM_0\cos\beta}{\mu^2}\int_0^1 x\log\left[1+\frac{1-x}{\xi x}\right]\mathrm{d}x$$
$$= -\frac{\alpha}{\pi}\frac{mM_0\cos\beta}{\mu^2}\left[\frac{1}{2}\frac{\xi}{(\xi-1)} - \frac{1}{2}\frac{\xi}{(\xi-1)^2}\log\xi\right]. \tag{23.36}$$

There remain the terms proportional to $(1-1/\xi)$ in (23.33) and (23.35). We can parametrize them just as we did the "crossed" terms in fig. 23.3a to obtain

$$\frac{-6\mathrm{i}e^2}{(2\pi)^4}(1-1/\xi)M_0\cos\beta\int \mathrm{d}^4k\ \mathrm{d}w\mathrm{d}x\mathrm{d}y\mathrm{d}z\ \delta(w+x+y+z-1)$$
$$\times\ \bar{u}(p+q/2)\left\{\frac{\gamma\cdot(k-p)\gamma\cdot k_1}{[w(k_1^2-\mu^2)+x(k_1^2-\mu^2/\xi)+y(k_2^2-\mu^2/\xi)+z[(k-p)^2-M_0^2]+\mathrm{i}\epsilon]^4}\right.$$
$$\left.+\ \frac{\gamma\cdot k_2\gamma\cdot(k-p)}{[w(k_2^2-\mu^2)+y(k_2^2-\mu^2/\xi)+y(k_1^2-\mu^2/\xi)+z[(k-p)^2-M_0^2]+\mathrm{i}\epsilon]^4}\right\}u(p-q/2). \tag{23.27}$$

Define

$$k = l + zp + \lambda q/2 \qquad (\lambda = x - y + w)$$

in the first term, and

$$k = l + zp + \mu q/2 \qquad (u = x - y - w)$$

in the second. In the approximations (23.4) and (23.5), the denominators in both terms are the same. Up to terms of order m^2/μ^2, (23.37) becomes, after doing the l-integration,

$$\frac{-e^2}{32\pi^2}\frac{M_0\cos\beta}{\mu^2}(1-1/\xi)\int\frac{\mathrm{d}w\mathrm{d}x\mathrm{d}y\mathrm{d}x\;\delta(x+y+z+w-1)[(12z-4)p_\mu+3(x-y)q_\mu]}{[w+(x+y)/\xi]} \tag{23.38}$$

The expression (23.38) is identically zero for all ξ. We conclude that

$$F_2^{\mathrm{b}}(0)+F_2^{\mathrm{c}}(0) = \frac{-\alpha}{2\pi}\frac{mM_0\cos\beta}{\mu^2}\left[\frac{\xi}{(\xi-1)}-\frac{1}{2}\frac{\xi}{(\xi-1)^2}\log\xi\right]. \tag{23.39}$$

Finally, we compute the contribution of fig. 23.3d. The $A_\mu s^+s^-$ vertex is the usual charged scalar meson electromagnetic vertex, and the other two are given by (23.7). Thus

$$\Gamma_\mu^{\mathrm{d}} = \frac{-\mathrm{i}}{(2\pi)^4}\frac{e^2}{\mu^2}\int \mathrm{d}^4k\;\bar{u}(p+q/2)N\,u(p-q/2)2k_\mu[k_1^2-\mu^2/\xi+\mathrm{i}\epsilon]^{-1}[k_2^2-\mu^2/\xi+\mathrm{i}\epsilon]^{-1}[(p-k)^2-M_0^2+\mathrm{i}\epsilon]^{-1} \tag{23.40}$$

where

$$N = \left[(m\cos\beta-M_0)\frac{(1+\gamma_5)}{2}+(m-M_0\cos\beta)\frac{(1-\gamma_5)}{2}\right][\gamma\cdot(p-k)+M_0]$$
$$\times\left[(m\cos\beta-M_0)\frac{(1-\gamma_5)}{2}+(m-M_0\cos\beta)\frac{(1+\gamma_5)}{2}\right]. \tag{23.41}$$

Therefore

$$\Gamma_\mu^{\mathrm{d}} = \frac{-2\mathrm{i}e^2}{(2)^4}\frac{M_0^3\cos\beta}{\mu^2}\int\mathrm{d}^4k\;k_\mu[k_1^2-\mu^2/\xi+\mathrm{i}\epsilon]^{-1}[k_2^2-\mu^2/\xi+\mathrm{i}\epsilon]^{-1}[(p-k)^2-M_0^2+\mathrm{i}\epsilon]^{-1} \tag{23.42}$$

$$+ \text{ terms smaller by at least } m/M_0 .$$

Provided $M_0^2 \ll \mu^2/\xi$, the integral in (23.42) is of order μ^{-2}, so that, in our approximation,

$$\Gamma_\mu^{\mathrm{d}} = F_2^{\mathrm{d}}(0) = 0. \tag{23.43}$$

From (23.32), (23.39), and (23.43), we conclude that the leading term in $F_2(0)$ is

$$F_2(0) = F_2^{\mathrm{a}}(0)+F_2^{\mathrm{b}}(0)+F_2^{\mathrm{c}}(0)+F_2^{\mathrm{d}}(0) = \frac{-\alpha}{\pi}\frac{mM_0}{\mu^2} \tag{23.44}$$

which is independent of ξ as expected.

This result is independent of our limitation on ξ. In the Landau gauge limit, $\xi \to \infty$, the integral in (23.42) is evidently of order M_0^{-2} instead of μ^{-2}, and in that case

$$F_2^{\mathrm{d}}(0) = \frac{-\alpha}{4\pi} \frac{m M_0}{\mu^2} \qquad (\xi \to \infty). \tag{23.45}$$

Careful treatment of the graphs in figs. 23.3b and 23.3c shows that $F_2^{\mathrm{b}}(0) + F_2^{\mathrm{c}}(0)$ also has the value (23.45) in that limit. The calculation leading to the expression (23.22) for $F_2^{\mathrm{a}}(0)$ is correct in this limit, so that

$$F_2^{\mathrm{a}}(0) = \frac{-\alpha}{2\pi} \frac{m M_0}{\mu^2} \qquad (\xi \to \infty). \tag{23.46}$$

The sum is still given by (23.44), independently of the approximation (23.5).

What is the experimental situation? Using eqs. (9.14) and (9.27) we can rewrite (23.44) as

$$F_2(0) = -GmM_+/2\pi^2\sqrt{2}\sin^2\beta \tag{23.47}$$

where M_+ is the mass of the charged heavy muon.

Electromagnetic corrections to $F_2(0)$ have been calculated up to sixth order in quantum electrodynamics, with the result

$$F_2^{\mathrm{QED}}(0) = (116582 \pm 1) \times 10^{-8}. \tag{23.48}$$

Hadron corrections to the photon propagator have been estimated to add $(6.5 \pm 0.5) \times 10^{-8}$ to this value, so that, neglecting weak corrections, the theoretical prediction is

$$F_2^{\mathrm{Th}}(0) = (116589 \pm 2) \times 10^{-8}. \tag{23.49}$$

The most recent available experimental figure is

$$F_2^{\mathrm{exp}}(0) = (116618 \pm 32) \times 10^{-8} \tag{23.50}$$

so that the theoretical value is well within the present experimental error without adding any weak corrections.

The weak correction calculated in eq. (23.47) has the value

$$F_2^{\mathrm{W}}(0) = -4.5 \times 10^{-9}(M_+/m)/\sin^2\beta. \tag{23.51}$$

We know that $M_+/m > M_{\mathrm{K}}/m = 4.7$, so that $|F_2^{\mathrm{W}}(0)|$ in the heavy muon model is at least 2.14×10^{-8}. Let us take two standard deviations as a reasonable lower limit for the true experimental value for $F_2(0)$, so that $F_2^{\mathrm{Th}}(0) + F_2^{\mathrm{W}}(0) > 116542 \times 10^{-8}$. Then $F_2^{\mathrm{W}}(0)$ must be less than 47×10^{-8}, and $(M_+/m)/\sin^2\beta$ must be less than 100. Thus

$$M_+ < 10\ \mathrm{GeV}. \tag{23.52}$$

Clearly, a more accurate measurement of $F_2(0)$ could put a much lower upper bound on the mass of the heavy muon.

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Is it to be concluded that to be on intimate terms with nature has a soothing influence, while the passion to penetrate the mystery lying behind appearances provokes expenditure of nervous energy which ultimately wears out the body and soul?

Germain Bazin