MQ (2º Teste/1º Exame) 12/1/2015

I

1) Verdadeira $\quad D^\dagger = (L A L^\dagger)^\dagger = (L^\dagger)^\dagger A^\dagger L^\dagger = L A L^\dagger = D$

2) Falso $\quad \frac{d^2 u(x)}{dx^2} = -\frac{2mE}{\hbar^2} u(x)$. Da Figura $\frac{d^2u}{dx^2} < 0 \Rightarrow E>0$

3) Verdadeira $\quad$ Corresponde ao 2º estado par $\Rightarrow n=2 \Rightarrow E_2 = \hbar\omega \frac{5}{2}$

4) Falsa $\quad \langle 0|x^2|0\rangle \neq 0$ e $\langle 3|x^2|3\rangle \neq 0$ ambos >0

$$\langle \psi|x^2|\psi\rangle = \frac{1}{9}\langle 0|x^2|0\rangle + \frac{8}{9}\langle 3|x^2|3\rangle > 0$$

II

1) $P(E_1) = \frac{1}{3} \Rightarrow B^2 = \frac{1}{3} \Rightarrow B = \frac{1}{\sqrt{3}}$ (Real e positivo)

$$\sum_n |A_n|^2 = 1 \Rightarrow A^2 + B^2 = 1 \Rightarrow A^2 = \frac{2}{3} \Rightarrow A = \sqrt{\frac{2}{3}}$$

logo

$$\boxed{A = \sqrt{\frac{2}{3}}, \quad B = \sqrt{\frac{1}{3}}}$$

2) $\langle H\rangle = \sum_n E_n |A_n|^2 = E_0 \frac{2}{3} + E_1 \frac{1}{3}$

$$= \hbar\omega\left[\frac{2}{3}\times\frac{1}{2} + \frac{1}{3}\times\frac{3}{2}\right] = \frac{5}{6}\hbar\omega$$

3) Método 1 (operadores A e $A^\dagger$)

$$\langle x\rangle = \left(\sqrt{\frac{2}{3}}\langle 0| - \sqrt{\frac{1}{3}}\langle 1|\right) x \left(\sqrt{\frac{2}{3}}|0\rangle - \sqrt{\frac{1}{3}}|1\rangle\right)$$

$$= \frac{2}{3}\langle 0|x|0\rangle + \frac{1}{3}\langle 1|x|1\rangle - \frac{\sqrt{2}}{3}\langle 1|x|0\rangle - \frac{\sqrt{2}}{3}\langle 0|x|1\rangle$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}\ (A + A^+)$$

Então

$$\langle 0|x|0\rangle = \langle 1|x|1\rangle = 0$$

e

$$\langle 1|x|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\ \langle 1|(A+A^+)|0\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\ \underbrace{\langle 1|\ A^+|0\rangle}_{1} = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\langle 0|x|1\rangle = \sqrt{\frac{\hbar}{2m\omega}}\ \langle 0|(A+A^+)|1\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\ \underbrace{\langle 0|\ A\ |1\rangle}_{1} = \sqrt{\frac{\hbar}{2m\omega}}$$

Logo

$$\langle x\rangle = -\frac{2\sqrt{2}}{3}\ \sqrt{\frac{\hbar}{2m\omega}}$$

Método 2 (funções de onda)

$$\langle x\rangle = \int_{-\infty}^{+\infty} dx \left(\sqrt{\frac{2}{3}}u_0(x) - \sqrt{\frac{1}{3}}u_1(x)\right) x \left(\sqrt{\frac{2}{3}}u_0(x) - \sqrt{\frac{1}{3}}u_1(x)\right)$$

$$= \frac{2}{3}\underbrace{\int_{-\infty}^{\infty} dx\ u_0^2(x)\ x}_{=0\ (\text{ímpar})} + \frac{1}{3}\underbrace{\int_{-\infty}^{+\infty} dx\ u_1^2(x)\ x}_{=0\ (\text{ímpar})} - \frac{2\sqrt{2}}{3}\int_{-\infty}^{+\infty} dx\ u_0(x)\ u_1(x)\ x$$

$$= -\frac{2\sqrt{2}}{3}\int_{-\infty}^{+\infty} dx\ u_0(x)\ u_1(x)\ x$$

Mas

$$\int_{-\infty}^{\infty} dx \; \underbrace{u_0(x)\, u_1(x)\, x}_{\text{par}} = 2 \int_0^{\infty} dx\, u_0(x)\, u_1(x)\, x$$

$$= 2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{1}{\sqrt{2}} \int_0^{\infty} dx \; x \; 2\left(\frac{m\omega}{\hbar}\right)^{1/2} x \; e^{-\frac{m\omega}{\hbar}x^2}$$

Fazendo a mudança de variável

$$y^2 = \frac{m\omega}{\hbar}x^2 \Rightarrow x = \sqrt{\frac{\hbar}{m\omega}}\, y$$

$$= \frac{\cancel{4}}{\sqrt{2}} \frac{1}{\sqrt{\pi}} \left(\frac{m\omega}{\hbar}\right)\left(\frac{\hbar}{m\omega}\right)^{3/2} \underbrace{\int_0^{\infty} dy\; y^2 e^{-y^2}}_{\frac{1}{2}\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\times\frac{1}{2}\sqrt{\pi}}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}$$

e portanto

$$\langle x \rangle = -\frac{2\sqrt{2}}{3}\sqrt{\frac{\hbar}{2m\omega}}$$

4). $\psi_2(x,t) = \sqrt{\frac{1}{3}}\, u_0(x)\, e^{-\frac{i}{\hbar}E_0 t} + \sqrt{\frac{2}{3}}\, u_4(x)\, e^{-\frac{i}{\hbar}E_4 t}$

$$E_0 = \hbar\omega\frac{1}{2} \; ; \quad E_4 = \hbar\omega\frac{9}{2}$$

$$\psi_2(x,t) = \sqrt{\frac{1}{3}}\, u_0(x)\, e^{-i\frac{\omega}{2}t} + \sqrt{\frac{2}{3}}\, u_4(x)\, e^{-i\frac{9\omega}{2}t}$$

$$= e^{-i\frac{\omega}{2}t}\left[\sqrt{\frac{1}{3}}\,u_0(x) + \sqrt{\frac{2}{3}}\,u_4(x)\,e^{-i4\omega t}\right]$$

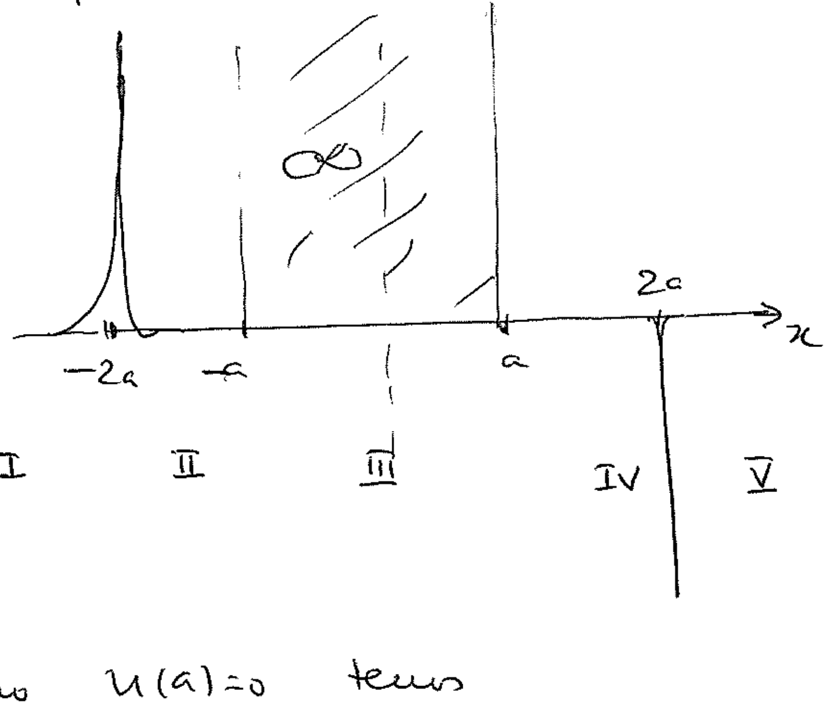
$$|\psi_2(x,t)|^2 = \left|\sqrt{\frac{1}{3}}\,u_0(x) + \sqrt{\frac{2}{3}}\,u_4(x)\,e^{-i4\omega t}\right|^2$$

$$|\psi_2(x,0)|^2 = |\psi_2(x,T)|^2 \Rightarrow 4\omega T = 2\pi$$

$$\Rightarrow \boxed{T = \frac{\pi}{2\omega}}$$

III

1) Só pode haver estados com $E<0$ para $x>a$.



$$\frac{d^2u}{dx^2} - \alpha^2 u = 0$$

$$\alpha^2 = \frac{2m}{\hbar}|E|$$

como $u(a)=0$ temos

$$\begin{cases} u_{IV}(x) = A\sinh(\alpha(x-a)) \\ u_{V}(x) = B\,e^{-\alpha x} \end{cases}$$

- Continuidade em $2a$

$$A \sinh(\alpha a) = B e^{-\alpha a}$$

- Descontinuidade em $2a$

$$-B\alpha e^{-\alpha a} - A\alpha \cosh \alpha a = -\frac{\beta}{a} A \sinh(\alpha a)$$

ou seja

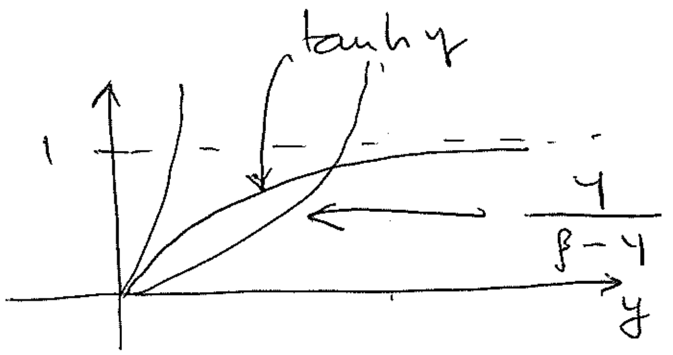
$$A\alpha a \sinh(\alpha a) + A\alpha a \cosh(\alpha a) = \beta A \sinh(\alpha a)$$

$$\tanh y\,(\beta - y) = y \qquad \boxed{y = \alpha a}$$

ou ainda

$$\boxed{\tanh y = \frac{y}{\beta - y}}$$

2)



A condição para haver intersecção não nula é

$$\left(\frac{y}{\beta - y}\right)'_{y=0} < (\tanh y)'_{y=0} = 1$$

$$\left(\frac{y}{\beta - y}\right)'_{y=0} = \left.\frac{\beta - y + y}{(\beta - y)^2}\right|_{y=0} = \frac{1}{\beta} \Rightarrow \frac{1}{\beta} < 1 \Rightarrow \boxed{\beta > 1}$$

3) Existe só um pois há só uma intersecção

4) $$\begin{cases} u_I(x) = e^{ikx} + R e^{-ikx} \\ u_{II}(x) = A \sin(k(x+a)) \end{cases}$$

$$\frac{d^2u}{dx^2} + k^2 u = 0$$

$$u(-a) = 0$$

- Continuidade em $x = -2a$

$$e^{-2ika} + R e^{i2ka} = -A \sin(ka)$$

- Descontinuidade em $x = -2a$

$$A k \cos ka - ik\left(e^{-ik2a} - R e^{2ika}\right) = \frac{\beta}{a}\left(e^{-2ika} + R e^{2ik}\right.$$

$$-k \cot(ka)\left(e^{-2ika} + R e^{2ika}\right) - ik\left(e^{-2ika} - R e^{2ika}\right) =$$

$$= \frac{\beta}{a}\left(e^{-2ika} + R e^{2ika}\right)$$

$$R e^{2ika} \left(\beta + ka \cot ka - ika\right) =$$

$$= e^{-2ika}\left(-\beta - ka \cot ka - ika\right)$$

$$R = e^{-4ika} \frac{-\beta - ka \cot ka - ika}{\beta + ka \cot ka - ika} =$$

$$= e^{-4ika} e^{i\pi} \frac{\beta + ka \cot ka + ika}{\beta + ka \cot ka - ika} = e^{i\delta}$$

Cru

$$\delta = -4ka + \pi + 2\arctan\left(\frac{ka}{\beta + ka\cot ka}\right)$$

5) $\beta \to 0 \Rightarrow R = -e^{-i4ka}\dfrac{ka\cot ka + ika}{ka\cot ka - ika}$

$$= -e^{-i4ka}\frac{\cos ka + i\sin ka}{\cos ka - i\sin ka}$$

$$= -e^{-4ika}\frac{e^{ika}}{e^{-ika}} = -e^{-2ika}$$

$$\boxed{R_{\beta\to 0} = -e^{-2ika}}$$

No limite $\beta \to 0$ a função delta não existe e a região I e II coincidem. Então

$$u_I(-a) = 0 = e^{-ika} + R\,e^{ika}$$

$$\Rightarrow \boxed{R = -e^{-2ika}}$$

IV

1) <u>Falso</u>: $\int d\Omega\, Y_{00} = 4\pi\, Y_{00} = \sqrt{4\pi}$

2) <u>Falso</u>: $|10\rangle = \sqrt{\frac{3}{10}}\,|21\rangle|1-1\rangle - \sqrt{\frac{2}{5}}\,|20\rangle|10\rangle + \sqrt{\frac{3}{10}}\,|2-1\rangle|1$

3) <u>Verdadeira</u>: $\sin 2\varphi = \dfrac{e^{2i\varphi} - e^{-2i\varphi}}{2i}$

$$\Rightarrow \psi \propto (Y_{22} - Y_{2,-2})$$

$$\Rightarrow P(L_z = 2\hbar) = P(L_z = -2\hbar) = \frac{1}{2}$$

4) <u>Verdadeira</u> $\theta = 90°$, $\phi = 180°$ corresponde ao sentido negativo do eixo dos x $\Rightarrow |\uparrow\vec{n}\rangle = |\downarrow S_x\rangle$

V

1) Da normalização

$$\frac{1}{4} + a^2 + b^2 = 1 \Rightarrow a^2 + b^2 = \frac{3}{4}$$

$$\langle H\rangle = \frac{1}{4}E_1 + \frac{3}{4}E_2 = E_1\left(\frac{1}{4} + \frac{3}{16}\right)$$

$$= \frac{7}{16}E_1 = \frac{7}{16}(-13.6\ \text{eV})$$

2) $\langle L_z\rangle = \frac{1}{4}\times 0 + a^2\times 0 + b^2 \times \hbar = \frac{1}{2}\hbar$

$$b^2 = \frac{1}{2} \Rightarrow b = \sqrt{\frac{1}{2}}$$

$$a^2 + b^2 = \frac{3}{4} \Rightarrow a^2 = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$$

$$\boxed{a = \frac{1}{2} \; ; \; b = \frac{1}{\sqrt{2}}}$$

VI

1) A equação de Schrödinger é

$$H_0 \psi_m = E_m \psi_m$$

$$H_0 \psi_m = -\frac{\hbar^2}{2I} \frac{d^2}{d\varphi^2} \left(\frac{1}{\sqrt{2\pi}} e^{im\varphi} \right)$$

$$= \frac{\hbar^2 m^2}{2I} \frac{1}{\sqrt{2\pi}} e^{im\varphi}$$

$$= \frac{\hbar^2 m^2}{2I} \psi_m \Rightarrow \boxed{E_m = \frac{\hbar^2 m^2}{2I}}$$

2) $H_1 = 4V_0 \sin\varphi \cos\varphi = 2V_0 \sin 2\varphi = \frac{V_0}{i}\left(e^{2i\varphi} - e^{-2i\varphi}\right)$

logo $\boxed{H_1 = -iV_0\left(e^{2i\varphi} - e^{-2i\varphi}\right)}$

o estado fundamental tem $m=0$ logo $\psi_0 = \frac{1}{\sqrt{2\pi}}$

$$\Delta E_0^{(1)} = \langle \psi_0 | H_1 | \psi_0 \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, (-iV_0)\left(e^{2i\varphi} - e^{-2i\varphi}\right)$$

$$= 0$$

3) O 1º estado excitado é degenerado pois

$$E_1 = E_{-1} = \frac{\hbar^2}{2I}$$

Temos portanto de encontrar a matriz H_{ij} no subespaço degenerado. Obtemos

$$\langle \psi_1 | H_1 | \psi_1 \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, e^{-i\varphi} e^{i\varphi} H_1$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, (-iV_0) \left(e^{2i\varphi} - e^{-2i\varphi} \right)$$

$$= 0$$

$$\langle \psi_{-1} | H_1 | \psi_{-1} \rangle = 0$$

$$\langle \psi_1 | H_1 | \psi_{-1} \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, e^{-i\varphi} e^{-i\varphi} \, H_1$$

$$= \frac{1}{2\pi} (-iV_0) \int_0^{2\pi} d\varphi \, e^{-2i\varphi} \left(e^{2i\varphi} - e^{-2i\varphi} \right)$$

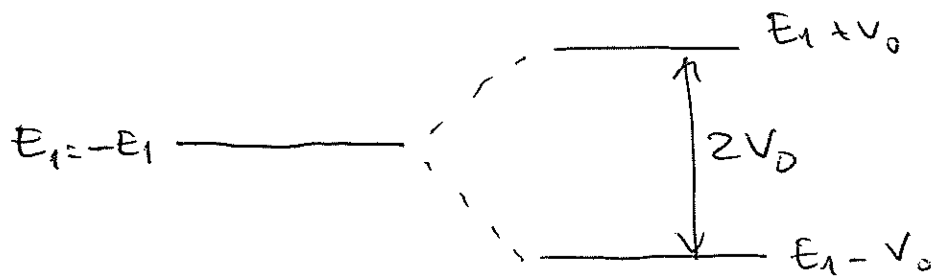
$$= (-iV_0)$$

$$\langle \psi_{-1} | H_1 | \psi_1 \rangle = \langle \psi_1 | H_1 | \psi_{-1} \rangle^* = (iV_0)$$

Logo

$$H_{ij} = \begin{bmatrix} 0 & -iV_0 \\ iV_0 & 0 \end{bmatrix} \Rightarrow \text{Valores próprios } \pm V_0$$

$$(\lambda^2 - V_0^2 = 0)$$



4) Da análise da questão anterior a perturbação só liga estados com $\Delta m = \pm 2$. Então

$$\Delta E_0^{(2)} = \sum_{k \neq 0} \frac{|\langle \psi_0 | H_1 | \psi_k \rangle|^2}{E_0^{(0)} - E_k^{(0)}}$$

$$= \frac{|\langle \psi_0 | H_1 | \psi_{-2} \rangle|^2}{E_0^{(0)} - E_{-2}^{(0)}} + \frac{|\langle \psi_0 | H_1 | \psi_2 \rangle|^2}{E_0^{(0)} - E_2^{(0)}}$$

Com $E_0^{(0)} = 0$; $E_2^{(0)} = E_{-2}^{(0)} = \frac{4\hbar^2}{2I} = \frac{2\hbar^2}{I}$

Logo

$$\Delta E_0^{(2)} = -\frac{I}{2\hbar^2} \left[|\langle \psi_0 | H_1 | \psi_{-2} \rangle|^2 + |\langle \psi_0 | H_1 | \psi_2 \rangle|^2 \right]$$

$$\langle \psi_0 | H_1 | \psi_2 \rangle = \frac{1}{2\pi} (-iV_0) \int_0^{2\pi} d\varphi \, e^{2i\varphi} \left(e^{2i\varphi} - e^{-2i\varphi} \right)$$

$$= (-iV_0)(-1) = iV_0$$

$$\langle \psi_0 | H_1 | \psi_{-2} \rangle = \frac{1}{2\pi} (-iV_0) \int_0^{2\pi} d\varphi \, e^{-2i\varphi} \left(e^{2i\varphi} - e^{-2i\varphi} \right)$$

$$= -iV_0$$

Portanto:

$$\Delta E_0^{(2)} = -\frac{I}{2\hbar^2}\left[V_0^2 + V_0^2\right] = -\frac{I V_0^2}{\hbar^2}$$

VN

(ver Nota no final)

1) $|\psi(t)\rangle = \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}$ com $a(0) = b(0) = \frac{1}{\sqrt{2}}$

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle = \hbar\omega_0 \sigma_z |\psi(t)\rangle$$

$$= \hbar\omega_0 \begin{bmatrix} \cos\omega t & 0 \\ 0 & -\cos\omega t \end{bmatrix}\begin{bmatrix} a(t) \\ b(t) \end{bmatrix}$$

Obtemos:

$$\begin{cases} \dfrac{da}{dt} = -i\omega_0 \cos(\omega t)\, a(t) \\ \dfrac{db}{dt} = i\omega_0 \cos(\omega t)\, b(t) \end{cases}$$

A Solução é

$$\begin{cases} a(t) = a(0)\, e^{-i\frac{\omega_0}{\omega}\sin(\omega t)} \\ b(t) = b(0)\, e^{i\frac{\omega_0}{\omega}\sin(\omega t)} \end{cases}$$

e
$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\frac{\omega_0}{\omega}\sin(\omega t)} \\ e^{i\frac{\omega_0}{\omega}\sin(\omega t)} \end{bmatrix}$$

2)
$$|\psi(t)\rangle = \alpha(t)|\uparrow S_x\rangle + \beta(t)|\downarrow S_x\rangle$$

$$\beta(t) = \langle \downarrow S_x | \psi(t)\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\frac{\omega_0}{\omega}\sin\omega t} \\ e^{i\frac{\omega_0}{\omega}\sin\omega t} \end{bmatrix}$$

$$= -\frac{1}{2}\left(e^{i\frac{\omega_0}{\omega}\sin(\omega t)} - e^{-i\frac{\omega_0}{\omega}\sin(\omega t)} \right)$$

$$= -i \sin\left[\frac{\omega_0}{\omega}\sin(\omega t)\right]$$

$$\mathbb{P}(S_x = -\hbar/2) = |\beta(t)|^2 = \sin^2\left[\frac{\omega_0}{\omega}\sin(\omega t)\right]$$

3) Como $-1 < \sin(\omega t) < 1$, para ter $\mathbb{P}(S_x = -\hbar/2) = 1$ para algum instante só ocorre se

$$\frac{\omega_0}{\omega} \geq \pi/2 \Rightarrow \omega_0 \geqslant \frac{\pi}{2}\omega$$

ou

$$\frac{\mu_B B_0}{\hbar} \geqslant \frac{\pi}{2}\omega \Rightarrow \boxed{B_0 \geqslant \frac{\pi}{2}\frac{\hbar\omega}{\mu_B}}$$

Nota Sobre o Problema VII

Ao tentar resolver este problema muitos alunos procederam do modo seguinte:

1) encontraram os valores próprios da matriz H
2) Aplicaram a evolução temporal para estados estacionários.

No entanto este procedimento está incorrecto. Quando separamos a equação de Schrödinger na equação independente do tempo e na evolução temporal dos estados estacionários, usámos o facto do Hamiltoniano não depender do tempo (constante do movimento). Ora neste problema $H = H(t)$ e é portanto necessário voltar atrás e resolver a equação de Schrödinger dependente do tempo, isto é:

$$\boxed{i\hbar \frac{d\psi}{dt} = H\psi}$$