# MQ (2º TESTE/ 1º EXAME) 13/1/2014



## I

1) Verdadeiro. $D^\dagger = (ABCA)^\dagger = A^\dagger C^\dagger B^\dagger A^\dagger = ACBA = ABCA = D$

2) Falso. Os estados ímpares não são alterados pela função delta porque a função de onda se anula na origem.

3) Falso. Os estados ligados são alternadamente pares e ímpares começando pelos pares. Assim deve haver dois estados pares e um ímpar, pois a função apresentada corresponde ao segundo estado (1 nodo).

4) Verdadeiro $\langle \psi | x | \psi \rangle \propto \langle \psi | A + A^\dagger | \psi \rangle$. A desce o estado de uma unidade e $A^\dagger$ sobe. Por isso

$$(A + A^\dagger)|\psi\rangle = \frac{1}{2}|1\rangle - \frac{\sqrt{3}}{2}\sqrt{4}\,|4\rangle - \frac{\sqrt{3}}{2}\sqrt{3}\,|2\rangle$$

e $\langle \psi | A + A^\dagger | \psi \rangle = 0$

## II

1) $\psi(x,0) = \sum_n A_n u_n(x)$ ; $\sum_n |A_n|^2 = 1$

$A_0 = C$ , $A_1 = -C$ , $A_n = 0 \quad n \geq 2$

$$1 = C^2 + C^2 \Rightarrow C = \frac{1}{\sqrt{2}} \quad (\text{real e positivo})$$

$$\text{Probabilidade } (E = E_1) = |A_1|^2 = C^2 = \frac{1}{2}$$

2) $\langle H \rangle = E_0 |C|^2 + E_1 |C|^2 = \frac{1}{2}\left(\frac{1}{2}\hbar\omega + \frac{3}{2}\hbar\omega\right) = \hbar\omega$

3) Método 1

$$\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\langle \psi | A + A^\dagger | \psi \rangle$$

onde $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Logo $(A+A^+)|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle - \sqrt{2}|2\rangle - |0\rangle)$

e

$$\langle\psi|(A+A^+)|\psi\rangle = \frac{1}{2}(\langle 0| - \langle 1|)(|1\rangle - \sqrt{2}|2\rangle - |0\rangle)$$

$$= \frac{1}{2}(-1-1) = -1$$

Portanto

$$\langle x\rangle = -\sqrt{\frac{\hbar}{2m\omega}}$$

<u>Método 2</u>

$$\langle x\rangle = \int_{-\infty}^{+\infty} dx \, \frac{1}{\sqrt{2}}(u_0(x) - u_1(x))\, x\, \frac{1}{\sqrt{2}}(u_0(x) - u_1(x))$$

$$= \frac{1}{2}\underbrace{\int_{-\infty}^{\infty} dx \, u_0^2(x)\, x}_{=0} + \frac{1}{2}\underbrace{\int_{-\infty}^{+\infty} dx \, u_1^2(x)\, x}_{=0}$$

função ímpar

$$- \int_{-\infty}^{+\infty} dx \, u_0(x) u_1(x)\, x$$

$$= -\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{1}{\sqrt{2}}\, 2 \left(\frac{m\omega}{\hbar}\right)^{1/2} \int_{-\infty}^{+\infty} dx \, x^2 e^{-\frac{m\omega}{\hbar}x^2}$$

$$= -\frac{\sqrt{2}}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)\left(\frac{\hbar}{m\omega}\right)^{3/2} \underbrace{\int_{-\infty}^{+\infty} dy \, y^2 e^{-y^2}}_{\frac{\sqrt{\pi}}{2}}$$

$$= -\sqrt{\frac{\hbar}{2m\omega}}$$

Método 3 

Do método 2 temos

$$\langle x \rangle = -\int_{-\infty}^{+\infty} dx\, u_0(x)\, u_1(x)\, x$$

Mas $\quad u_0(x)\, x = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-y^2/2}\, x$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(\frac{\hbar}{m\omega}\right)^{1/2} \frac{1}{2}\frac{\sqrt{2}}{\sqrt{2}}\, 2y\, e^{-y^2/2}$$

$$= \left(\frac{\hbar}{2m\omega}\right)^{1/2} \underbrace{\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2}}\, 2y\, e^{-y^2/2}}_{u_1(x)}$$

$$= \left(\frac{\hbar}{2m\omega}\right)^{1/2} u_1(x)$$

Então

$$\langle x \rangle = -\left(\frac{\hbar}{2m\omega}\right)^{1/2} \underbrace{\int_{-\infty}^{+\infty} dx\, u_1(x)\, u_1(x)}_{=1} \quad \text{(normalização)}$$

e

$$\langle x \rangle = -\sqrt{\frac{\hbar}{2m\omega}}$$

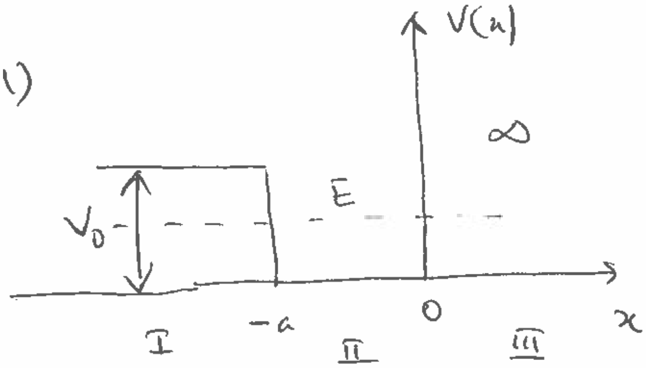
4) $\quad \psi(x,t) = \frac{1}{\sqrt{2}} u_0(x)\, e^{-i\frac{\omega t}{2}} - \frac{1}{\sqrt{2}} u_1(x)\, e^{-i\frac{3}{2}\omega t}$

$$\frac{\omega T}{2} = 2\pi \Rightarrow T = \frac{4\pi}{\omega}.$$ Notar que

$$\frac{3}{2}\omega T = 6\pi$$

III

1)




$$u_{III}(x) = 0$$

$$\frac{d^2u}{dx^2} - \frac{2m}{\hbar^2}(V-E)u = 0$$

I

$$\frac{d^2u}{dx^2} - \frac{2m}{\hbar^2}(V_0-E)u = 0 \qquad \alpha^2 = \frac{2m}{\hbar^2}(V_0-E) > 0$$

$$\frac{d^2u}{dx^2} - \alpha^2 u = 0$$

$$u_I(x) = A e^{+\alpha x}$$

II

$$\frac{d^2u}{dx^2} + \frac{2m}{\hbar^2}Eu = 0$$

$$k^2 = \frac{2m}{\hbar^2}E$$

$$\frac{d^2u}{dx^2} + k^2 u = 0$$

$$u_{II}(x) = B\sin kx \qquad \text{car} \qquad u_{II}(0) = 0$$

En $x = -a$

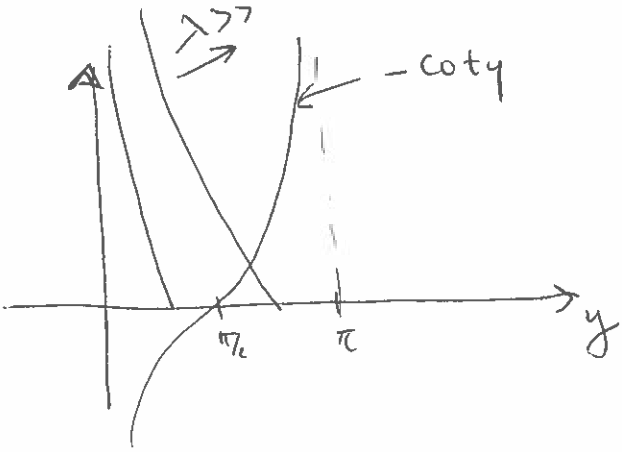
$$A e^{-a\alpha} = -B\sin ka$$

$$\alpha A e^{-\alpha a} = kB\cos ka$$

$$-\cot(ka) = \frac{\alpha}{k} = \frac{\alpha a}{ka} = \frac{\sqrt{\lambda - y^2}}{y}$$

$$\Rightarrow \boxed{-\cot y = \frac{\sqrt{\lambda - y^2}}{y}}$$

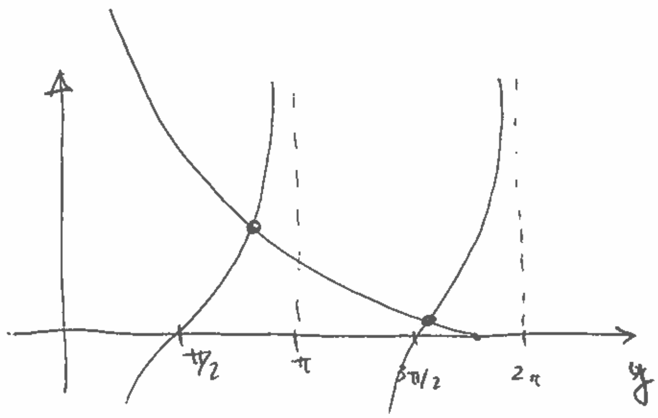
2)




Valor mínimo é quando $\lambda = \left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{4}$. Portanto

$$\frac{\pi^2}{4} = \frac{2mV_0a^2}{\hbar^2} \Rightarrow V_0 = \frac{\pi^2\hbar^2}{8ma^2}$$

3) $V_0 = \frac{14\hbar^2}{ma^2} \Rightarrow \lambda = \frac{2ma^2}{\hbar^2}\frac{14\hbar^2}{ma^2} = 28$




$\left(\frac{3\pi}{2}\right)^2 < 28 < (2\pi)^2$

Dois estados ligados

4) Quando $V_0 \to \infty$ o Estado fundamental corresponde a $y = \pi$

Logo

$$\sqrt{\frac{2ma^2}{\hbar^2}E_1} = \pi$$

donde

$$\boxed{E_1 = \frac{\pi^2\hbar^2}{2ma^2}}$$

5) $E > V_0$. Agora

$$u_I(x) = e^{iqx} + R\,e^{-iqx} \qquad q^2 = \frac{2m}{\hbar^2}(E - V_0)$$

$$u_{II}(x) = A \sin kx \qquad \text{as } u_{II}(0) = 0$$

<u>On $x = -a$</u>

$$iq \frac{e^{-iqa} - R e^{iqa}}{e^{-iqa} + R e^{iqa}} = -k \cot ka$$

$$R e^{2iqa} \left(-iq + k \cot ka\right) = -k \cot ka - iq$$

$$R = e^{-2iqa} \frac{-k \cot ka - iq}{k \cot ka - iq}$$

$$= e^{-2iqa} \, e^{i\pi} \, e^{2i\beta} \qquad \beta = \operatorname{atan}\left(\frac{q}{k \cot ka}\right)$$

$$= e^{i(-2qa + \pi + 2\beta)}$$

$$\boxed{\delta = \pi - 2qa + 2\beta}$$

IV

1) Verdadeiro $\quad \int d\Omega\, Y_{120} = \sqrt{4\pi} \int d\Omega\, Y_{00}^* Y_{20} = 0$

2) Verdadeiro. ver Tabela de Clebch-Gordon

3) Falso Probabilidade $\boxed{\text{Zero}}$ pois o estado está num estado que é uma combinação de $Y_{21}$ e $Y_{2,-1}$ com $l=2 \Rightarrow L^2 = \hbar^2 2(2+1) = 6\hbar^2$.

4) Verdadeiro. Estado próprio de $S_y$ com valor próprio $+\frac{\hbar}{2}$

V

1) $\sum_{n,l,m} |A_{n,l,m}|^2 = 1 \Rightarrow \left(\frac{1}{2}\right)^2 + a^2 + b^2 = 1 \Rightarrow a^2 + b^2 = \frac{3}{4}$

$$\langle H \rangle = \left(\frac{1}{2}\right)^2 E_1 + (a^2+b^2) E_2 = \frac{1}{4} E_1 + \frac{3}{4} E_2$$

$$= \frac{1}{4} E_1 + \frac{3}{4}\frac{1}{4} E_1 = \frac{7}{16} E_1 = -\frac{7}{16}\frac{1}{2}(\mu c^2)\alpha^2$$

2) $\langle L_z \rangle = \frac{1}{4} \times 0 + a^2 \times \hbar - b^2 \hbar = (a^2 - b^2)\hbar = -\frac{1}{4}\hbar$

logo

$$\begin{cases} a^2 + b^2 = \frac{3}{4} \\ a^2 - b^2 = -\frac{1}{4} \end{cases} \Rightarrow a^2 = \frac{1}{4}, \quad b^2 = \frac{1}{2}$$

e

$$\boxed{a = \frac{1}{2}, \quad b = \frac{1}{\sqrt{2}}}$$

1) $$H_0 = \mu_B B_0 \sigma_x = \mu_B B_0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Valores próprios

$$(-\lambda)^2 - 1 = 0 \quad \Rightarrow \lambda = \pm 1$$

Logo $$E_1 = -\mu_B B_0 , \quad E_2 = +\mu_B B_0$$

Vectores próprios

$|1\rangle$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = - \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \Rightarrow \begin{array}{l} \beta = -\alpha \\ \alpha^2 + \beta^2 = 1 \end{array}$$

logo

$$|1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$|2\rangle$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \Rightarrow \begin{array}{l} \beta = \alpha \\ \alpha^2 + \beta^2 = 1 \end{array}$$

logo

$$|2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

2) $$H = \mu_B B_0 \sigma_x + \eta \mu_B B_0 \sigma_z = \mu_B B_0 \begin{bmatrix} \eta & 1 \\ 1 & -\eta \end{bmatrix}$$

Valores Próprios

$$(\eta - \lambda)(-\eta - \lambda) = 1$$

$$\lambda^2 - \eta^2 - 1 = 0 \Rightarrow \lambda = \pm\sqrt{1+\eta^2}$$

Logo

$$E_1 = -\mu_B B_0 \sqrt{1+\eta^2} \quad ; \quad E_2 = +\mu_B B_0 \sqrt{1+\eta^2}$$

3) 1ª ordem

$$E_1^{(1)} = \langle 1 | H_1 | 1 \rangle = \frac{1}{2}\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mu_B B_0 \eta$$

$$= \frac{\mu_B B_0 \eta}{2} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

$$E_2^{(1)} = \langle 2 | H_1 | 2 \rangle = \frac{1}{2}\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mu_B B_0 \eta$$

$$= \frac{\mu_B B_0 \eta}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

Conclusão: As correcções de 1ª ordem são nulas

2ª ordem

$$E_1^{(2)} = \frac{|\langle 1 | H_1 | 2 \rangle|^2}{E_1 - E_2} \quad ; \quad E_2^{(2)} = \frac{|\langle 2 | H_1 | 1 \rangle|^2}{E_2 - E_1}$$

$$|\langle 1|H_1|2\rangle| = |\langle 2|H_1|1\rangle|$$

$$\langle 1|H_1|2\rangle = \mu_B B_0 \frac{1}{2} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \eta & \\ & -\eta \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{\mu_B B_0 \eta}{2} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \mu_B B_0 \eta$$

e

$$E_1^{(2)} = \frac{(\mu_B B_0)^2 \eta^2}{-2\mu_B B_0} = -\frac{1}{2} \mu_B B_0 \eta^2$$

$$E_2^{(2)} = \frac{(\mu_B B_0)^2 \eta^2}{+2\mu_B B_0} = +\frac{1}{2} \mu_B B_0 \eta^2$$

4) Expandindo $\sqrt{1+\eta^2} \simeq 1 + \frac{1}{2}\eta^2$ vem

$$E_1 = -\mu_B B_0 \sqrt{1+\eta^2} \simeq -\mu_B B_0 - \frac{1}{2} \mu_B B_0 \eta^2$$

$$E_2 = +\mu_B B_0 \sqrt{1+\eta^2} \simeq \mu_B B_0 + \frac{1}{2} \mu_B B_0 \eta^2$$

1) Temos

$$\vec{J} - \vec{J}_1 = \vec{J}_2 \qquad e \qquad \vec{J} - \vec{J}_2 = \vec{J}_1$$

Quadrando

$$J^2 + J_1^2 - 2\vec{J}\cdot\vec{J}_1 = J_2^2 \qquad e \qquad J^2 + J_2^2 - 2\vec{J}\cdot\vec{J}_2 = J_1^2$$

Logo

$$\vec{J}\cdot\vec{J}_1 = \frac{1}{2}\left(J^2 + J_1^2 - J_2^2\right)$$

$$\vec{J}\cdot\vec{J}_2 = \frac{1}{2}\left(J^2 + J_2^2 - J_1^2\right)$$

2) $$\langle M_i \rangle = \frac{1}{\hbar^2 j(j+1)} \langle (\vec{M}\cdot\vec{J})\, J_i \rangle$$

Mas

$$\vec{M}\cdot\vec{J} = \gamma_1\, \vec{J}\cdot\vec{J}_1 + \gamma_2\, \vec{J}\cdot\vec{J}_2$$

$$= \frac{\gamma_1}{2}\left(J^2 + J_1^2 - J_2^2\right) + \frac{\gamma_2}{2}\left(J^2 + J_2^2 - J_1^2\right)$$

$$= \frac{\gamma_1 + \gamma_2}{2}\hbar^2 j(j+1) + \frac{\gamma_1 - \gamma_2}{2}\hbar^2\left[j_1(j_1+1) - j_2(j_2+1)\right]$$

ou seja

$$\langle M_i \rangle = \left[\frac{\gamma_1+\gamma_2}{2} + \frac{\gamma_1-\gamma_2}{2}\,\frac{\left[j_1(j_1+1) - j_2(j_2+1)\right]}{j(j+1)}\right] \langle J_i \rangle$$

Agora para $i = x$

$$\langle J_x \rangle = \langle j\, m_j | \tfrac{1}{2}(J_+ + J_-) | j\, m_j \rangle = 0$$

$$\langle J_y \rangle = \langle j\, m_j | \tfrac{1}{2i}(J_+ - J_-) | j\, m_j \rangle = 0$$

e portanto

$$\langle M_x \rangle = \langle M_y \rangle = 0$$

3) usando os resultados de 2) e

$$\langle J_z \rangle = \langle j\, m_j | J_z | j\, m_j \rangle = \hbar m_j$$

obtemos

$$\langle M_z \rangle = \hbar m_j \left[ \frac{\gamma_1 + \gamma_2}{2} + \frac{\gamma_1 - \gamma_2}{2} \, \frac{j_1(j_1+1) - j_2(j_2+1)}{j(j+1)} \right]$$