Mecânica Quântica Teste 1B (12/11/2009) 1

I

1) Verdadeira : $B^\dagger = -i(A-A^\dagger)^\dagger = -i(A^\dagger - A) = i(A-A^\dagger)$
$= B$

2) Falsa : $[H,p] \neq 0$ pois $[V(x), p] \neq 0$

3) Verdadeira : Uma função delta sem mais nada tem sempre uma solução para o estado ligado.

4) Falsa : $p^2 = \alpha A^2 + \beta (A^\dagger)^2 + \gamma A A^\dagger + \delta A^\dagger A$

$\langle n-1|(A^\dagger)^2|n+1\rangle = \langle n-1|AA^\dagger|n+1\rangle = 0$

$\langle n-1|A^\dagger A|n+1\rangle = 0$, mas

$\langle n-1|(A)^2|n+1\rangle = \langle n-1|A(\sqrt{n+1}|n\rangle)$

$= \sqrt{n+1}\langle n-1|A|n\rangle = \sqrt{n+1}\sqrt{n}\langle n-1|n-1\rangle$

$= \sqrt{(n+1)n} \neq 0$

II

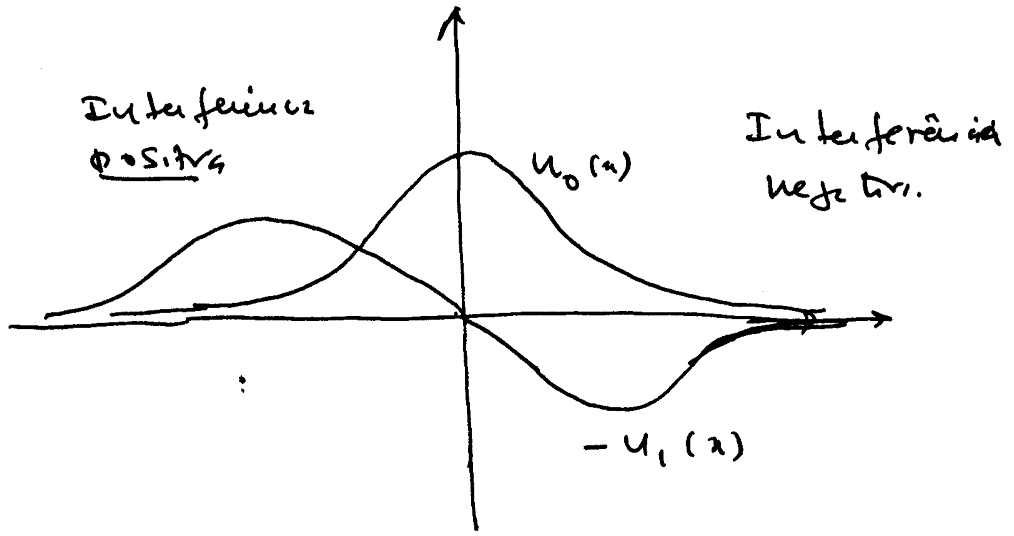
1) $\psi(x,0) = \sum_n A_n u_n(x) \Rightarrow A_0 = A$; $A_1 = -A$; $A_n = 0$ $n \geq 2$

$\sum_n |A_n|^2 \Rightarrow \quad 2|A|^2 = 1 \Rightarrow |A| = \frac{1}{\sqrt{2}}$

$P(E=E_1) = |A_1|^2 = \frac{1}{2}$

2) $\langle E\rangle = |A_0|^2 \frac{\hbar\omega}{2} + |A_1|^2 \frac{3\hbar\omega}{2} = \frac{1}{2}\hbar\omega\left(\frac{1}{2}+\frac{3}{2}\right) = \hbar\omega$

3) Método Gráfico



logo $\quad \mathbb{P}(]-\infty,0]) > \mathbb{P}([0,+\infty[)$

Método analítico

$$\mathbb{P}([0,+\infty[) = \frac{1}{2}\int_0^\infty dx\left[u_0^2(x)+u_1^2(x)-2u_0(x)u_1(x)\right]$$

$$\boxed{\begin{array}{l} u_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-y^2/2} \\ u_1 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{2y}{\sqrt{2}}\, e^{-y^2/2} \\ y = \sqrt{\frac{m\omega}{\hbar}}\, x \end{array}}$$

~~Agora~~

$$\int_0^\infty dx\, u_0^2(x) = \int_0^\infty dx\, u_1^2(x) = \frac{1}{2}$$

e

$$\int_0^\infty dx\, u_0 u_1 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{1}{\sqrt{2}}\sqrt{\frac{\hbar}{m\omega}}\int_0^\infty dy\, 2y\, e^{-y^2}$$

$$= \left(\frac{1}{2\pi}\right)^{1/2} 2\int_0^\infty dy\, y\, e^{-y^2} = \frac{1}{\sqrt{2\pi}}\, 2\, \frac{1}{2}\Gamma(1) = \frac{1}{\sqrt{2\pi}}$$

logo

$$\mathbb{P}([0,+\infty[) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} < \frac{1}{2}.$$

4) $$\psi(x,t) = \frac{1}{\sqrt{2}} u_0 e^{-i\frac{\omega}{2}t} - \frac{1}{\sqrt{2}} u_1 e^{-3i\frac{\omega}{2}t}$$

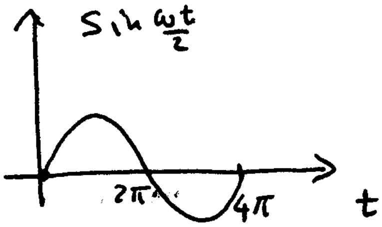
$$= \frac{1}{\sqrt{2}} e^{-i\frac{\omega}{2}t} \left(u_0(x) - u_1(x) e^{-i\omega t} \right)$$

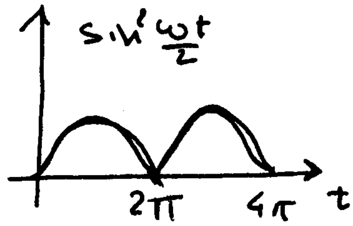
$$|\psi(x,t)|^2 = \frac{1}{2} \left| u_0(x) - u_1(x) e^{-i\omega t} \right|^2$$

$$= \frac{1}{2} \left[\left(u_0(x) - u_1(x) \cos\omega t \right)^2 + u_1^2(x) \sin^2\omega t \right]$$

$$= \frac{1}{2} \left[\left(u_0(x) - u_1(x) \right)^2 + 2u_0u_1 (1 - \cos\omega t) \right]$$

$$= \frac{1}{2} \left[\left(u_0(x) - u_1(x) \right)^2 + 4u_0u_1 \sin^2\left(\frac{\omega t}{2}\right) \right]$$

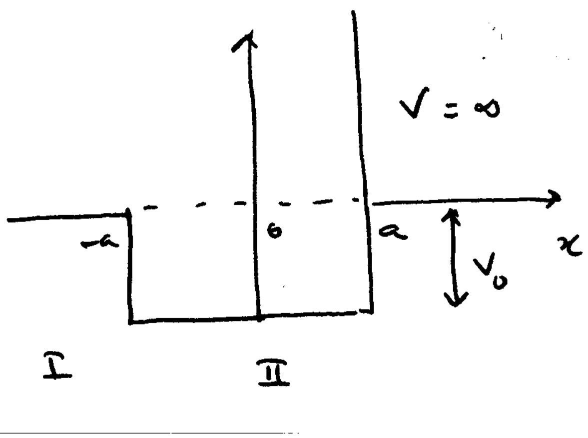




période $\omega T = 2\pi \Rightarrow T = \frac{2\pi}{\omega}$

III

1)



En I : $\frac{d^2u}{dx^2} - \alpha^2 u = 0$

$$\alpha^2 = \frac{2m|E|}{\hbar^2}$$

Em II) $\frac{d^2u}{dx^2} + q^2 u = 0$; $q^2 = \frac{2m}{\hbar^2}(V_0 - |E|)$

Logo

$$\begin{cases} u_I(x) = A\, e^{\alpha x} & x < -a \\ u_{II}(x) = B \sin(q(x-a)) & -a < x < a \quad \text{pois } u(a) = 0 \end{cases}$$

Em $x = -a$

$$\alpha = q \cot(-2qa)$$

ou

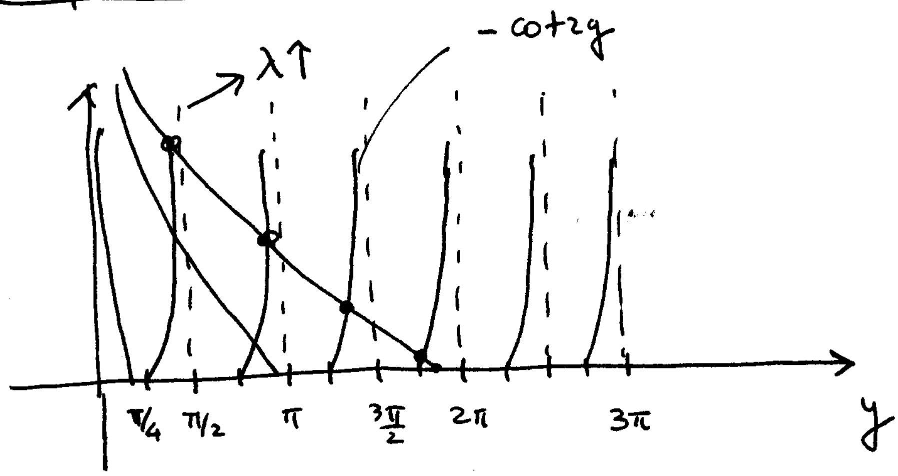
$$\boxed{-\cot 2y = \frac{\sqrt{\lambda - y^2}}{y}}$$

$y = qa$

$\alpha a = \sqrt{\lambda - y^2}$

$\lambda = \frac{2mV_0a^2}{\hbar^2}$

2) Graficamente:



Há portanto um valor mínimo de $\lambda > \left(\frac{\pi}{4}\right)^2 = \frac{\pi^2}{16}$. Logo

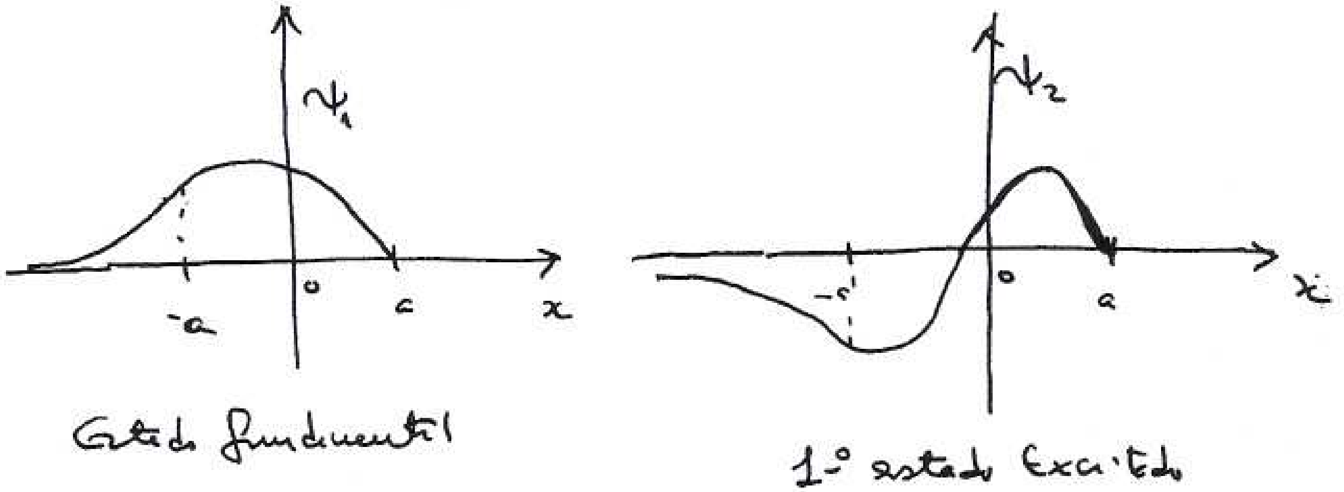
$$\frac{2mV_0a^2}{\hbar^2} > \frac{\pi^2}{16} \Rightarrow V_0 > \frac{\pi^2\hbar^2}{32ma^2}$$

3) $$\lambda = \frac{2mV_0a^2}{\hbar^2} = \frac{2ma^2}{\hbar^2}\frac{16\hbar^2}{ma^2} = 32$$

e $$\left(\frac{3\pi}{2}\right)^2 < \lambda < (2\pi)^2$$

pelo que há 4 estados ligados (ver figura anterior)

4)



5) $$\begin{cases} u_I = e^{ikx} + Re^{-ikx} \\ u_{II} = A\sin[q(x-a)] \end{cases} \quad \therefore \; q^2 = \frac{2m}{\hbar^2}(V_0+E)$$

$$\frac{ik(e^{-ika} - Re^{ika})}{e^{-ika} + Re^{ika}} = -q\cot(2qa)$$

$$Re^{ika}\left(q\cot(2qa) - ik\right) = e^{-ika}\left(-q\cot(2qa) - ik\right)$$

$$R = -e^{-2ika}\frac{q\cot(2qa) + ik}{q\cot(2qa) - ik} \Rightarrow |R|^2 = 1$$

o fluxo $j(x) = \frac{\hbar}{2im}\left[\psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx}\right]$

temos

$$j_I(x) = \frac{\hbar k}{m}\left(1 - |R|^2\right) = 0$$

$$j_{II}(x) = 0 \quad \Rightarrow \quad j_I(x) = j_{II}(x)$$

tudo o que incide é reflectido!