I

1) Verdadeira. $C^\dagger = (AB)^\dagger + (BA)^\dagger = B^\dagger A^\dagger + A^\dagger B^\dagger =$

$= BA + AB = AB + BA = C$

2) Verdadeira. Os estados são alternadamente pares e ímpares.

3) Falsa $\lambda = \dfrac{2mV_0a^2}{\hbar^2} \rightarrow \lambda' = \dfrac{2mV_0/2\,(2a)^2}{\hbar^2} = 2\lambda$

4) Falsa $x = \alpha A + \beta A^\dagger$ e $\langle n|A|n+2\rangle = 0;\ \langle n|A^\dagger|n+2\rangle = 0$

II

1) $\psi(x,0) = \sum_n A_n u_n(x) \Rightarrow A_1 = -\frac{1}{\sqrt{2}}\ ;\ A_4 = B\ ;\ A_n = 0\quad n \geq 5, n=2,3$

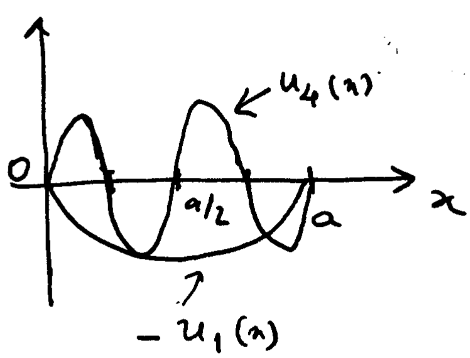
$\sum_n |A_n|^2 = 1 \Rightarrow |A_1|^2 + |A_4|^2 = 1 \Rightarrow \frac{1}{2} + |B|^2 = 1 \Rightarrow B = \frac{1}{\sqrt{2}}$

(real e positivo)

2) $\langle E\rangle = \sum_n |A_n|^2 E_n = \frac{1}{2}E_1 + \frac{1}{2}E_4 = \frac{1}{2}\frac{\pi^2\hbar^2}{2ma^2}(1+16)$

$= \frac{17}{4}\frac{\pi^2\hbar^2}{ma^2}$

3) Método Gráfico



Para $0<x<a/2$ há uma interferência positiva e para $a/2<x<a$ uma interferência negativa. Logo

$P(0<x<a/2) > \frac{1}{2}$

Método analítico

$$P(0<x<a/2) = \frac{1}{2}\int_0^{a/2} u_1^2(x)\,dx + \frac{1}{2}\int_0^{a/2} u_4^2(x)\,dx$$

$$- \int_0^{a/2} u_1(x)\,u_4(x)\,dx$$

$$\int_0^{a/2} u_1^2(x)\,dx = \frac{2}{a}\int_0^{a/2} dx\,\sin^2\left(\frac{\pi x}{a}\right) = \frac{2}{a}\frac{a}{\pi}\int_0^{\pi/2} dy\,\sin^2 y$$

$$= \frac{2}{\pi}\left[\frac{1}{2}y - \frac{1}{4}\sin 2y\right]_0^{\pi/2} = \frac{1}{2}$$

$$\int_0^{a/2} u_4^2(x)\,dx = \frac{2}{a}\int_0^{a/2} dx\,\sin^2\left(\frac{4\pi x}{a}\right) = \frac{2}{a}\frac{a}{4\pi}\int_0^{2\pi} dy\,\sin^2 y$$

$$= \frac{1}{2\pi}\left[\frac{1}{2}y - \frac{1}{4}\sin 2y\right]_0^{2\pi} = \frac{1}{2}$$

$$\int_0^{a/2} u_1 u_4\,dx = \frac{2}{a}\int_0^{a/2} dx\,\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{4\pi x}{a}\right)$$

$$= \frac{2}{a}\frac{a}{\pi}\int_0^{\pi/2} dy\,\sin y\,\sin 4y$$

$$= \frac{2}{\pi}\left[\frac{1}{6}\sin(3y) - \frac{1}{10}\sin(5y)\right]_0^{\pi/2}$$

$$= \frac{2}{\pi}\left(-\frac{1}{6} - \frac{1}{10}\right) = -\frac{8}{15\pi}$$

logo

$$P(0<x<a/2) = \frac{1}{2} + \frac{8}{15\pi} > \frac{1}{2}$$

4) $$\psi(x,t) = -\frac{1}{\sqrt{2}}\, u_1(x)\, e^{-\frac{i}{\hbar}E_1 t} + \frac{1}{\sqrt{2}}\, u_4(x)\, e^{-\frac{i}{\hbar}E_4 t}$$

Como $E_4 = 16 E_1$ se $\frac{E_1 T}{\hbar} = 2\pi \Rightarrow \frac{E_4 T}{\hbar} = 32\pi$

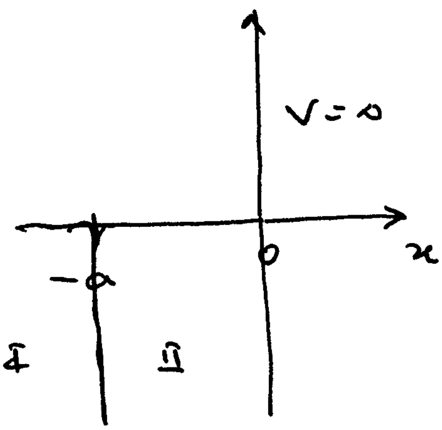
e portanto

$$\psi(x,T) = \psi(x,0)$$

Logo $T = \frac{2\pi\hbar}{E_1}$

III

1)



Estado ligado $E<0$. Então na região I e II temos

$$\frac{d^2u}{dx^2} - \alpha^2 u = 0 \quad \text{com } \alpha = \frac{2m|E|}{\hbar^2}$$

As soluções que se anulam para $x \to -\infty$ e para $x=0$ são

$$\begin{cases} u_I(x) = A e^{\alpha x} \\ u_{II}(x) = B \sinh \alpha x \end{cases}$$

Temos a continuidade em $x=-a$ da função e a descontinuidade da derivada

$$u'_{II}\Big|_{x=-a} - u'_{I}\Big|_{x=-a} = -\frac{\lambda}{a} u(-a)$$

Donde

$$\begin{cases} A\, e^{-\alpha a} = B \sinh(-\alpha a) = -B \sinh(\alpha a) \\ B\alpha \cosh(\alpha a) - \alpha A\, e^{-\alpha a} = -\dfrac{\lambda'}{a}\left(-B \sinh(\alpha a)\right) \end{cases}$$

ou seja

$$B\alpha \cosh(\alpha a) + \alpha B \sinh(\alpha a) = \frac{\lambda'}{a} B \sinh(\alpha a)$$

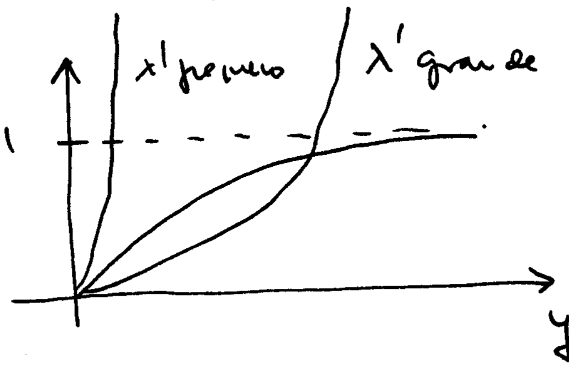
Donde

$$\tanh(\alpha a) = \frac{\alpha a}{\lambda' - \alpha a}$$

ou definindo $y = \alpha a$,

$$\boxed{\tanh y = \frac{y}{\lambda' - y}}$$

2)

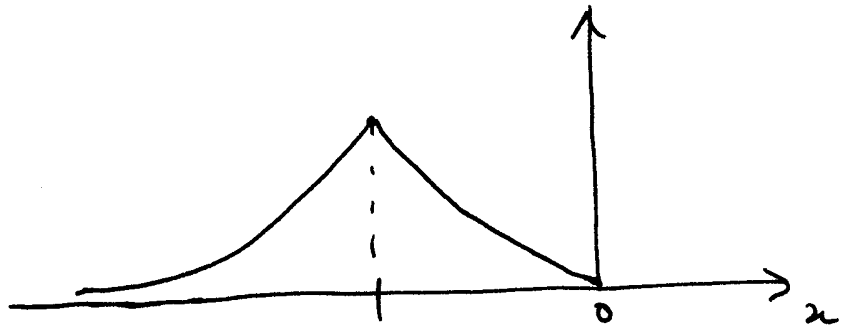


Só há soluções para λ' suficientemente grande. A condição é

$$\left(\frac{y}{\lambda' - y}\right)'\Bigg|_{y=0} < (\tanh y)'_{y=0} = 1$$

$$\frac{1}{\lambda'} < 1 \Rightarrow \boxed{\lambda' > 1}$$

3)



4) Agora as soluções da Equação

$$\frac{d^2u}{dx^2} + k^2 u = 0 \qquad k^2 = \frac{2mE}{\hbar^2}$$

serão

$$\begin{cases} u_I(x) = e^{ikx} + R e^{-ikx} \\ u_{II}(x) = A \sin kx \qquad (u(0)=0) \end{cases}$$

$$\begin{cases} e^{-ika} + R e^{ika} = -A \sin(ka) \\ Ak \cos ka - ik\left(e^{-ika} - Re^{ika}\right) = -\frac{\lambda'}{a}\left(-A \sin ka\right) \end{cases}$$

Donde

$$-k \cot(ka)\left(e^{-ika} + Re^{ika}\right) - ik\left(e^{-ika} - Re^{ika}\right) = -\frac{\lambda'}{a}\left(e^{-ika} + Re^{ika}\right)$$

ou

$$Re^{ika}\left(-ka\cot(ka) + \lambda' + ik\right) = e^{-ika}\left(ka\cot(ka) - \lambda' + ik\right)$$

$$R = -e^{-2ika}\,\frac{\lambda' - ka\cot(ka) - ik}{\lambda' - ka\cot(ka) + ik} \Rightarrow |R| = 1 \Rightarrow |R|^2 = 1$$

5) $$j(x) = \frac{\hbar}{2im}\left(\psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx}\right)$$

$$j_I(x) = \frac{\hbar}{2im}\Big[\left(e^{-ikx} + R^* e^{ikx}\right) ik \left(e^{+ikx} - Re^{-ikx}\right) - \text{c.c.}\Big]$$

$$= \frac{\hbar}{2im}\Big[ik\left(1 - Re^{-2ikx} + R^* e^{2ikx} - |R|^2\right) + ik\left(1 - R^* e^{2ikx} + R e^{-2ikx} - |R|^2\right)\Big]$$

$$= \frac{\hbar k}{m}\left(1 - |R|^2\right) = 0$$

$j_{II}(x) = 0$ pois a função é real.

Assim há conservação de fluxo

$$j_I(x) = j_{II}(x) = 0$$

Tudo o incidente é reflectido pois a onda não pode penetrar em $x > 0$.