I

1) **falsa**: $[H, x] \propto p$ e $[p, p] = 0$

2) **Verdadeira** $\psi = A_1 u_1 + A_2 u_2 + A_3 u_3$ Com $A_1 = \frac{1}{2}$, $A_2 = \sqrt{\frac{3}{5}}$, $A_3 = \sqrt{\frac{3}{20}}$

$P(E=E_1) = 1/4$ $\langle E \rangle = 4 E_1$ (Corrected 18/12/2019 Thanks João P. Silva)

3) **Verdadeira**: $u'_{II}(0) - u'_{I}(0) > 0$

4) **Verdadeira**: $\langle \psi | H | \psi \rangle = \sum_n |A_n|^2 E_n = |A_0|^2 E_0 + \sum_{n\geq 1} |A_n|^2 E_n = E_0 + \sum_{n \geq 1} |A_n|^2 (E_n - E_0) \geq E_0$

5) **Verdadeira**: $\langle l'm' | L_z | lm \rangle = \hbar m \langle l'm' | lm \rangle = \hbar m \, \delta_{ll'} \delta_{mm'} = 0$

6) **Falsa** $|\psi\rangle = |\downarrow; x\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

7) **falsa** $P(L_z = -\hbar) = \frac{1}{2}$

8) **Verdadeira**. $\mu = \frac{1}{2} m_e \Rightarrow E_p = \frac{1}{2} E_H = -6.8\,eV$

II

1) $B = \sqrt{1 - A^2}$

2) $\langle H \rangle = E_0 (A^2 + 4B^2) = E_0 (4 - 3A^2)$; $E_0 = \frac{\pi^2 \hbar^2}{2ma^2}$

3) $$P(0<x<a/2) = \int_0^{a/2} dx \, (A^2 u_1^2 + B^2 u_2^2 + 2AB\, u_1 u_2)$$

$$= \frac{1}{2} + 2AB \underbrace{\int_0^{a/2} dx \, u_1 u_2}_{I}$$

$$I = \frac{2}{a} \int_0^{a/2} dx \, \sin\frac{\pi x}{a} \sin\frac{2\pi x}{a} = \frac{2}{\pi} \int_0^{\pi/2} dy \, \sin y \sin 2y$$

$$= \frac{4}{\pi} \int_0^{\pi/2} dy \, \sin^2 y \cos y = \frac{4}{3\pi} \left[\sin^3 y \right]_0^{\pi/2} = \frac{4}{3\pi}$$

$$P = \frac{1}{2} + \frac{8}{3\pi} A \sqrt{1 - A^2}$$

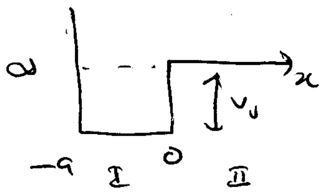
$$\frac{dP}{dA} = 0 = \frac{8}{3\pi} \left[\sqrt{1-A^2} - \frac{A^2}{\sqrt{1-A^2}} \right] \Rightarrow 1 - A^2 = A^2 \Rightarrow A = \pm \frac{1}{\sqrt{2}}$$

Mínimo $\Rightarrow A = -\frac{1}{\sqrt{2}}$

4) $\psi(x,t) = A\, u_1(x)\, e^{-i\frac{E_1}{\hbar}t} + B\, u_2(x)\, e^{-i\frac{E_2}{\hbar}t}$

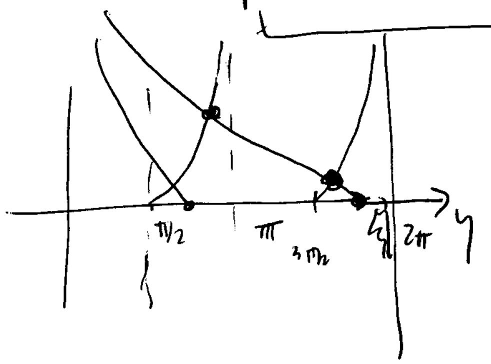
$$\langle H\rangle_t = \langle H\rangle_{t=0} = E_0\left(4 - 3A^2\right) = E_0\,\frac{5}{2} \text{ no mínimo}$$

III

1) 

$$\begin{cases} u_I(x) = A \sin q(x+a) \\ u_{II}(x) = B\, e^{-\alpha x} \end{cases} \qquad \alpha^2 = \frac{2m|E|}{\hbar^2}$$

$$q \cot qa = -\alpha \quad \Rightarrow \quad \boxed{-\cot y = \frac{\sqrt{\lambda - y^2}}{y}}$$



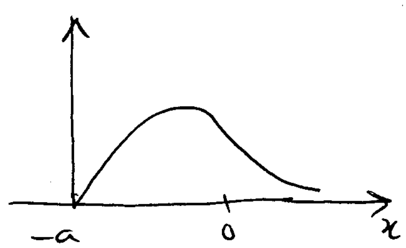
2) $\lambda \geqslant \frac{\pi^2}{4}$

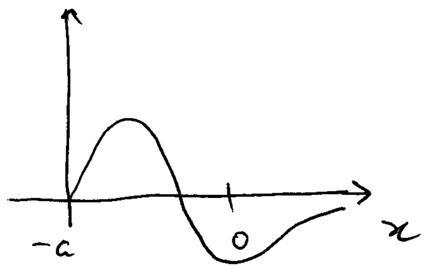
$$V_0 = \frac{\hbar^2}{2ma^2}\,\lambda$$

$$V_0 \geqslant \frac{\pi^2\hbar^2}{8ma^2}$$

3) $\lambda = \frac{2ma^2}{\hbar^2}\,\frac{49\pi^2\hbar^2}{32ma^2} = \frac{49\pi^2}{16} \Rightarrow \sqrt{\lambda} = \frac{7}{4}\pi \in \left[\frac{3\pi}{2}, 2\pi\right]$

$\Rightarrow$ 2 estados (figura)

4) 



5) $u_I = A \sin q(x+a)$; $q = \sqrt{\frac{2m(V_0+E)}{\hbar^2}}$

$$q \cot qa = \frac{-ik(1-R)}{1+R} \Rightarrow R = \frac{k - iq\cot qa}{k + iq\cot qa} \Rightarrow |R| = 1$$

$$V_0 \to 0 \Rightarrow q \to k \Rightarrow R \to \frac{1 - i\cot ka}{1 + i\cot ka} = -\frac{\cos ka + i \sin ka}{\cos ka - i\sin ka}$$

$$= -e^{+2ika}$$

Nesse limite não há região I e $u_{II}(-a) = 0 \Rightarrow R = -e^{2ika}$

(IV)

1) $\cos\theta = \sqrt{\frac{4\pi}{3}}\, Y_{10}$; $\sin\theta\cos\varphi = \frac{1}{2}\sqrt{\frac{8\pi}{3}}\,(Y_{1-1} - Y_{11})$

$$\psi = C\, r\, e^{-\frac{r}{2r_0}} \left(\alpha\, Y_{11} + \beta\, Y_{10} + \gamma\, Y_{1-1} \right)$$

$$\alpha = -\frac{1}{2}\sqrt{\frac{8\pi}{3}} \; ; \; \beta = \sqrt{\frac{4\pi}{3}} \; ; \; \gamma = \frac{1}{2}\sqrt{\frac{8\pi}{3}}$$

$$1 = \int d^3r\, |\psi|^2 = |C|^2 \int_0^\infty dr\, r^4 e^{-\frac{r}{r_0}} \left(|\alpha|^2 + |\beta|^2 + |\gamma|^2 \right)$$

$$= |C|^2\, 24\, r_0^5\, \frac{4\pi}{3} \left(1 + \frac{1}{2} + \frac{1}{2} \right)$$

$$= |C|^2\, 64\, r_0^5\, \pi \;\Rightarrow\; |C| = \left(\frac{1}{64\pi r_0^5} \right)^{1/2}$$

2) $P(L_z = 0) = \frac{|\beta|^2}{|\alpha|^2 + |\beta|^2 + |\gamma|^2} = \frac{1}{2}$; $P(L_z = \pm\hbar) = \frac{1}{4}$

(V)

1) $|\vec{p}|\,|\vec{\mathcal{E}}| \ll \frac{\hbar^2}{2I} \;\Rightarrow\; |\vec{\mathcal{E}}| \ll \frac{\hbar^2}{2I|\vec{p}|}$

2) $$\langle n | H_1 | m \rangle = -\frac{|\vec{p}|\,|\vec{\mathcal{E}}|}{4\pi} \int_0^{2\pi} d\varphi\, e^{i(m-n)\varphi} \left(e^{i\varphi} + e^{-i\varphi} \right)$$

$$= -\frac{|\vec{p}|\,|\vec{\mathcal{E}}|}{4\pi}\, 2\pi \left[\delta_{n,m+1} + \delta_{n,m-1} \right]$$

Estado não degenerado ($n=0$)

$$\langle 0 | H_1 | 0 \rangle = 0$$

Estados degenerados ($n \neq 0$)

$$\langle n|H_1|-n\rangle = -\frac{|\vec{p}||\vec{\mathcal{E}}|}{2}\left[\delta_{n,-n+1}+\delta_{n,-n-1}\right]$$

$\neq 0$ sse $n=\pm\frac{1}{2} \Rightarrow$ Impossível! Logo $\boxed{\text{Todas}}$ as correcções de 1ª ordem são nulas.

3) $$E_0^{(2)} = \sum_{k\neq 0}\frac{|\langle 0|H_1|k\rangle|^2}{0-E_k^{(0)}}$$

$$\langle 0|H_1|k\rangle = -\frac{|\vec{p}||\vec{\mathcal{E}}|}{2}\left[\delta_{0,k+1}+\delta_{0,k-1}\right]$$

portanto só contribuem os estados $k=\pm 1$.

$$E_0^{(2)} = -\frac{1}{E_{\pm 1}^{(0)}}\left(-\frac{|\vec{p}||\vec{\mathcal{E}}|}{2}\right)^2\times 2 = -\frac{I\,|\vec{p}|^2|\vec{\mathcal{E}}|^2}{\hbar^2}$$

VI

1) $$|\psi(0)\rangle = -i\frac{1}{\sqrt{2}}|\downarrow;z\rangle + \frac{1}{\sqrt{2}}|\uparrow;z\rangle \;;\; \begin{cases} H|\downarrow;z\rangle = -\hbar\omega|\downarrow;z\rangle \\ H|\uparrow;z\rangle = +\hbar\omega|\uparrow;z\rangle\end{cases}$$

$$|\psi(t)\rangle = -i\frac{1}{\sqrt{2}}|\downarrow;z\rangle e^{i\omega t} + \frac{1}{\sqrt{2}}|\uparrow;z\rangle e^{-i\omega t}$$

$$= \begin{pmatrix}\frac{1}{\sqrt{2}}e^{-i\omega t}\\ -\frac{i}{\sqrt{2}}e^{i\omega t}\end{pmatrix}$$

2) $$\langle S_x\rangle = \frac{\hbar}{2}\left(\frac{1}{\sqrt{2}}e^{i\omega t} + \frac{i}{\sqrt{2}}e^{-i\omega t}\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}e^{-i\omega t}\\ -\frac{i}{\sqrt{2}}e^{i\omega t}\end{pmatrix}$$

$$= \frac{\hbar}{2}\left(\frac{1}{\sqrt{2}}e^{i\omega t} \quad \frac{i}{\sqrt{2}}e^{-i\omega t}\right)\begin{pmatrix}-\frac{i}{\sqrt{2}}e^{i\omega t}\\ \frac{1}{\sqrt{2}}e^{-i\omega t}\end{pmatrix} = \frac{\hbar}{2}\,\frac{e^{i2\omega t}-e^{-i2\omega t}}{2i}$$

$$= \frac{\hbar}{2}\sin(2\omega t)$$

3) $|\psi(t)\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\omega t} \\ -\frac{i}{\sqrt{2}} e^{i\omega t} \end{pmatrix} = \alpha\,|\uparrow;x\rangle + \beta\,|\downarrow;x\rangle$

$$\alpha = \langle \uparrow; x | \psi(t)\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\omega t} \\ -\frac{i}{\sqrt{2}} e^{i\omega t} \end{pmatrix}$$

$$= \frac{1}{2}\left(e^{-i\omega t} - i e^{i\omega t}\right) = \frac{1}{2}\Big[(\cos\omega t + \sin\omega t) - i(\cos\omega t + \sin\omega t)\Big]$$

$$P\left(S_x = \tfrac{\hbar}{2}\right) = |\alpha|^2 = \frac{1}{4}\,(\cos\omega t + \sin\omega t|^2 \times 2 =$$

$$= \frac{1}{2}\left(1 + \sin 2\omega t\right)$$