PP Series 1 – Problem 1 (fisica Particula)

a) From the expansion Postulate we know that

$$|A|^2 + |B|^2 + |C^2| = 1 \qquad \left(\sum_n |A_n|^2 = 1 \right)$$

and as they are real (and positive)

$$\boxed{A^2 + B^2 + C^2 = 1}$$

Also $(L^2 = \hbar^2 \, l(l+1))$

$$\langle L^2 \rangle = A^2 \times 0 + B^2 \times 2\hbar^2 + C^2 \, 2\hbar^2$$

giving

$$\boxed{2B^2 + 2C^2 = 1}$$

Finally

$$\langle L_z \rangle = A^2 \times 0 + B^2 \hbar - C^2 \hbar$$

giving

$$\boxed{B^2 - C^2 = \frac{7}{18}}$$

Solution: $A = \frac{1}{\sqrt{2}}$; $B = \frac{2}{3}$; $C = \frac{1}{3\sqrt{2}}$

b) $\langle E \rangle = A^2 E_1 + B^2 E_2 + C^2 E_2$

$$= A^2 E_1 + (B^2 + C^2) E_2$$

$$A^2 = \frac{1}{2}; \quad B^2 + C^2 = \frac{1}{2}$$

$$E_1 = E_0 \qquad E_2 = E_0 \frac{1}{4} \qquad E_0 = -13.6\,\text{eV}$$

$$\langle E \rangle = E_0 \left(\frac{1}{2} + \frac{1}{2} \times \frac{1}{4} \right) = \frac{5}{8} E_0 = -8.5\,\text{eV}$$

c) $\langle r \rangle = \int d^3r \, \psi^* \, r \, \psi$

using

$$\int d\Omega \, Y^*_{lm} \, Y_{l'm'} = \delta_{ll'} \delta_{mm'}$$

we get

$$\langle r \rangle = \int_0^\infty dr \, r^2 \, r \left[A^2 R_{10}^2 + B^2 R_{21}^2 + C^2 R_{21}^2 \right]$$

$$= A^2 \int_0^\infty dr \, r^3 R_{10}^2 + (B^2 + C^2) \int_0^\infty dr \, r^3 R_{21}^2$$

$$\equiv \frac{1}{2} \langle r \rangle_{10} + \frac{1}{2} \langle r \rangle_{21}$$

③

$$\langle r_{10}\rangle = 4\left(\frac{1}{a_0}\right)^3 \int_0^\infty dr\, r^3\, e^{-\frac{2r}{a_0}}$$

$$= 4\left(\frac{1}{a_0}\right)^3 \int_0^\infty dy \left(\frac{a_0}{2}\right)^4 y^3 e^{-y}$$

$$= \frac{a_0}{4} \underbrace{\int_0^\infty dy\, y^3 e^{-y}}_{\Gamma(4) = 3! = 6}$$

so

$$\langle r_{10}\rangle = \frac{3}{2} a_0$$

$$\langle r_{21}\rangle = \frac{1}{3}\left(\frac{1}{2a_0}\right)^3 \int_0^\infty dr\, r^3 \left(\frac{r}{a_0}\right)^2 e^{-\frac{r}{a_0}}$$

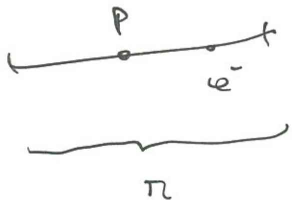
$$= \frac{1}{3}\left(\frac{1}{2a_0}\right)^3 (a_0)^4 \underbrace{\int_0^\infty dy\, y^5 e^{-y}}_{\Gamma(6) = 5! = 120}$$

$$= \frac{120}{3\times 8} a_0 = 5 a_0$$

$$\langle r\rangle = \frac{1}{2}\langle r\rangle_{10} + \frac{1}{2}\langle r\rangle_{21}$$

$$= \frac{1}{2}\,\frac{3}{2} a_0 + \frac{1}{2}\, 5a_0 = \frac{13}{4} a_0$$

2 – Considering that the electron is in an 1-D region of size r



The total energy of the electron is,

$$E = E_c + E_p = \frac{p^2}{2m} - \frac{1}{4\pi\varepsilon_0}\frac{q \cdot Q}{r} \quad (1)$$

↳ Coulomb potential

q is the electron charge and Q the proton charge

$q = Q = e$

From the Heisenberg uncertainty principle,

$$\Delta x \, \Delta p \geq \frac{\hbar}{2} \Leftrightarrow \Delta p \geq \frac{\hbar}{2\Delta x}$$

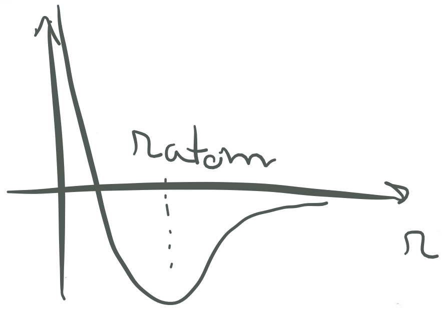
As the electron can be anywhere within r then its momentum is

$$\Delta p \geq \frac{\hbar}{2r}$$

substituting this in (1)

$$E = \frac{\hbar^2}{4mr^2} - \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r} \quad (2)$$

Plotting (2),



The r (distance) with minimum energy can be obtained through

$$\frac{\partial E}{\partial r} = 0 \Rightarrow r = \frac{2\hbar\pi\varepsilon_0}{mq} \quad (3)$$

Putting values at (3) one gets that the electron should within

$$r = 2{,}6 \times 10^{-11} = 26 \text{ pm}$$

Experimentally $r_{atom} = 31 \pm 5$ pm

3 - $E_\gamma = 0{,}37 \text{ meV}$ $\qquad \rho \sim 400 \text{ cm}^{-3}$

a) Considering the reaction $p\gamma \to \Delta$

One has in the LAB: $P_\mu = (E_p + E_\gamma ; \vec{P}_p - \vec{P}_\gamma)$

The minimal energy for the process to occur happens when the proton and the photon have opposite momentum directions.

After the interaction only 1 particle is created. Taking it at rest, $P_\mu = (m_\Delta, \vec{0})$

Computing $P_\mu P^\mu = E^2 - \vec{P}.\vec{P}$ one has

$$(E_p + E_\gamma)^2 - (P_p - P_\gamma)^2 = m_\Delta^2 \Leftrightarrow$$

$$E_p^2 + \cancel{E_\gamma^2} + 2E_p E_\gamma - P_p^2 - \cancel{E_\gamma^2} + 2 P_p E_\gamma = m_\Delta^2 \Leftrightarrow$$

$$m_p^2 + 2 E_p E_\gamma + 2\underbrace{\sqrt{E_p^2 - m_p^2}}_{\approx E_p} E_\gamma = m_\Delta^2 \Leftrightarrow \qquad \Big| \; E_\gamma = P_\gamma$$

$$E_p = \frac{m_\Delta^2 - m_p^2}{4 E_\gamma} \simeq 4{,}31 \times 10^{14} \text{ MeV} = 4{,}31 \times 10^{20} \text{ eV}$$

b) The mean free path is given by

$$L = \frac{1}{\sigma \rho} = \frac{1}{0{,}6 \times 10^{-27} \text{ cm} \times 400 \text{ cm}^{-3}} = 4{,}17 \times 10^{24} \text{ cm}$$

$$= 4{,}17 \times 10^{19} \text{ km}$$

$$= 1{,}35 \text{ Mpc}$$

Where it was used

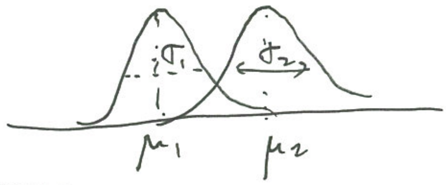
$1 \text{ mb} = 10^{-27} \text{ cm}^2$

$1 \text{ km} = 3{,}24 \times 10^{-14} \text{ pc}$

4- The number of electrons produced by $N\gamma$ photons is $N^{pe} = G \cdot N^{\gamma}$ (1), where G is the PMT gain

The condition to have resolved peaks is

$$\mu_1 - \mu_2 \geq \sigma_1 + \sigma_2 \quad (2)$$



where μ is the distribution mean and σ its standard deviation.

If the distributions follow Poisson statistics then

$\mu_i = N_i$ and $\sigma_i = \sqrt{N_i}$

Therefore,

$$N_{i+1}^{pe} - N_i^{pe} \geq \sqrt{N_{i+1}^{pe}} + \sqrt{N_i^{pe}} \quad (3)$$

using (1)

$$G\left(N_{i+1}^{\gamma} - N_i^{\gamma}\right) \geq \sqrt{G}\left(\sqrt{N_{i+1}^{\gamma}} + \sqrt{N_i^{\gamma}}\right)^{(2)}$$

and so,

$$G \geq \left(\frac{\sqrt{N_{i+1}^{\gamma}} + \sqrt{N_i^{\gamma}}}{N_{i+1}^{\gamma} - N_i^{\gamma}}\right)^2$$

Hence, for $N_i^{\gamma} = 2$ and $N_{i+1}^{\gamma} = 3$

$$G \geq 9{,}9 \simeq 10$$

and for $N_i^{\gamma} = 3$ and $N_{i+1}^{\gamma} = 4$

$$G \geq 13{,}93 \cong 14$$

Therefore the gain should be greater that 14.

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a) Using the information in the table for the "incident photon" one has,

$$X_{max} = X_0 \, t_{max} = X_0 \left(\log\left(\frac{E}{E_c}\right) - 0{,}5 \right) \simeq 325 \text{ g cm}^{-2}$$

The altitude can be obtained using the provided atmosphere model,

$$x \simeq X e^{-h/7} \Leftrightarrow h_M = -7 \log\left(\frac{X_{max}}{X}\right) \simeq 8{,}1 \text{ km}$$

b) The total number of produced photons is

$$N_\gamma^{total} = \underbrace{\left(\frac{E}{E_c}\right)}_{\text{number of radiation lengths}} \underbrace{N_\gamma^{ch}}_{\text{2000 ph/rad. length}} \simeq 2{,}27 \times 10^7 \text{ photons}$$

c) Assuming that all photons are coming from the center of gravity of the shower, t_{med}, one has that

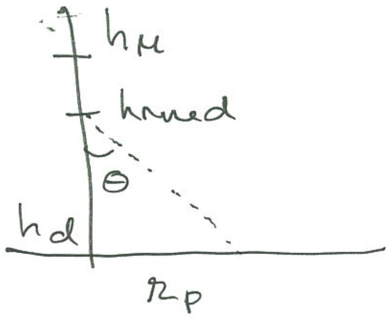
$$t_{med} = t_{max} + 1{,}7 = \log\left(\frac{E}{E_c}\right) - 0{,}5 + 1{,}7 \simeq 10{,}5$$

or in traversed matter units,

$$X_{med} = t_{med} X_0 = 385{,}7 \text{ g cm}^{-2}$$

Using the atmosphere model,

$$h_{med} = -7 \log\left(\frac{X_{med}}{X}\right) \simeq 6{,}88 \text{ km}$$



Assuming that the light pool can be derived from an opening angle of $\sim 1{,}3°$, one gets r_p as,

$$r_p = tg(\theta)(h_{med} - h_d) \simeq 110{,}6\ m$$

Therefore the number of photons at ground per area'

$$n_\gamma = \frac{N_\gamma|_{ground}}{A} = \frac{N_\gamma\, e^{-(h_{med}-h_d)/3}}{\pi r_p^2} \simeq 116\ ph/m^2$$

Where N_γ is the number of produced photons calculated previously.

The number of detected photons is then,

$$N_\gamma^{det} = A_{det}\, n_\gamma\, \varepsilon_{ref}\, \varepsilon_{acq} =$$

$$= A_{det}(116 \times 0{,}7 \times 0{,}2) = A_{det} \cdot 16{,}24$$

Thus, to detect at least one photon the detector should have at least,

$$A_{det} = \frac{1}{16{,}24}\ m^2 \simeq 0{,}06\ m^2 = 600\ cm^2$$

d) All the quantities, calculated previously, have to be re-evaluated as the shower, having less energy, has a smaller Xmax (higher in the atmosphere) and produces less light. Hence,

$n_\gamma = 1\ \text{photon}/m^2$ and

$$N_\gamma^{det} = A_{det}\, n_\gamma\, \varepsilon_{ref}\, \varepsilon_{acq} = A_{det} \cdot 0{,}14$$

Thus, to detect at least one photon $A_{det} = 7{,}14\,m^2$

Comparing with the previous result for 1 TeV, this means that for 50 GeV the detector should be 119 times bigger.

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Note: Usually 1 photon is not enough to detect a shower as there are uncertainties that affect the measurement such as: electronics noise, photon sky background...

Experiments like MAGIC or VERITAS have detection areas of around 100 m². For these,

50 GeV shower ⇒ 17 photons

1 TeV shower ⇒ 1629 photons

Note that although for 50 GeV gamma ~~show~~ induced showers the experiment "sees" 17 photons these showers are in the threshold of detectability due to the reasons mentioned above.