Helicity and Chirality

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We explain the relation between helicity and chirality both for massless and massive fermions.

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I. HELICITY SPINORS

A good basis to do explicit calculations in scattering processes with fermions is the helicity spinor basis. This is a good basis, as the helicity defined by

$$h = \frac{\vec{S}\cdot\vec{p}}{|\vec{p}|} = \frac{1}{2}\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}. \tag{1}$$

commutes with the Dirac Hamiltonian, $\left[H_D, \vec{\Sigma}\cdot\vec{p}\right]=0$, where,

$$H_D = \vec{\alpha}\cdot\vec{p} + \beta m\ . \tag{2}$$

The solution of finding the helicity spinors for a particle moving in an arbitrary direction defined by the angles (θ,ϕ) can be found in standard texts[1, 2]. We just reproduce it here,

$$u_\uparrow = \sqrt{E+m}\begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right)e^{i\phi} \\ \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right)e^{i\phi} \end{bmatrix},\ u_\downarrow = \sqrt{E+m}\begin{bmatrix} -\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right)e^{i\phi} \\ \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right)e^{i\phi} \end{bmatrix}. \tag{3}$$

and

$$v_\uparrow = \sqrt{E+m}\begin{bmatrix} \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right)e^{i\phi} \\ -\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right)e^{i\phi} \end{bmatrix},\ v_\downarrow = \sqrt{E+m}\begin{bmatrix} \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right)e^{i\phi} \\ \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right)e^{i\phi} \end{bmatrix}. \tag{4}$$

II. HELICITY AND CHIRALITY PROJECTORS

It is useful to define an helicity projector. For this we use the spinor projector operator,

$$P(s) = \frac{1+\gamma_5 \not{s}}{2}\ , \tag{5}$$

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where s^μ is a four-vector representing the spin direction and satisfying $s^2 = -1, s \cdot p = 0$, as can be seen if we go to the rest frame of the particle. For a direction along the motion of the particle, s^μ takes the form,

$$s^\mu = (\gamma\beta, \gamma\hat{\beta}) \tag{6}$$

where β is the velocity of the particle, $\gamma = 1/\sqrt{1-\beta^2}$ and $\hat{\beta} = \vec{\beta}/\beta$. Then the projector for helicity $h = \pm$ is

$$P(h,s) = \frac{1 + h\,\gamma_5 \not{s}}{2} \tag{7}$$

We can check that this is indeed the case. Consider the example of $u_\uparrow$. Let use a simplified notation,

$$u_\uparrow = \sqrt{E+m} \begin{bmatrix} \chi_\uparrow(\theta,\phi) \\ \dfrac{\gamma\beta}{\gamma+1}\chi_\uparrow(\theta,\phi) \end{bmatrix} \tag{8}$$

where we have used

$$\frac{|\vec{p}|}{E+m} = \frac{\gamma\beta}{\gamma+1} \tag{9}$$

and $\chi_\uparrow(\theta,\phi)$ satisfies,

$$\vec{\sigma} \cdot \hat{\beta}\ \chi_\uparrow(\theta,\phi) = \chi_\uparrow(\theta,\phi)\ . \tag{10}$$

In the Dirac representation we obtain easily,

$$P(h,s) = \begin{bmatrix} \frac{1}{2}(1 + h\,\gamma\vec{\sigma}\cdot\hat{\beta}) & -\frac{h}{2}\gamma\beta \\ \frac{h}{2}\gamma\beta & \frac{1}{2}(1 - h\,\gamma\vec{\sigma}\cdot\hat{\beta}) \end{bmatrix}\ , \tag{11}$$

and therefore

$$\begin{aligned} P(h,s)u_\uparrow &= \begin{bmatrix} \frac{1}{2}(1 + h\,\gamma\vec{\sigma}\cdot\hat{\beta}) & -\frac{h}{2}\gamma\beta \\ \frac{h}{2}\gamma\beta & \frac{1}{2}(1 - h\,\gamma\vec{\sigma}\cdot\hat{\beta}) \end{bmatrix} \sqrt{E+m} \begin{bmatrix} \chi_\uparrow(\theta,\phi) \\ \dfrac{\gamma\beta}{\gamma+1}\chi_\uparrow(\theta,\phi) \end{bmatrix} \\ &= \sqrt{E+m} \begin{bmatrix} \frac{1}{2}(1 + h\,\gamma\,\vec{\sigma}\cdot\hat{\beta})\chi_\uparrow(\theta,\phi) - \frac{1}{2}h\,\dfrac{\gamma^2\beta^2}{\gamma+1}\chi_\uparrow(\theta,\phi) \\ \frac{1}{2}h\,\gamma\beta + \frac{1}{2}(1 - h\,\gamma\,\vec{\sigma}\cdot\hat{\beta})\dfrac{\gamma\beta}{\gamma+1}\chi_\uparrow(\theta,\phi) \end{bmatrix} \\ &= \sqrt{E+m} \begin{bmatrix} \frac{1}{2}\left(1 + h\,\gamma - h(\gamma-1)\right)\chi_\uparrow(\theta,\phi) \\ \dfrac{\gamma\beta}{\gamma+1}\dfrac{1}{2}\left(1 - h\,\gamma + h(\gamma+1)\right)\chi_\uparrow(\theta,\phi) \end{bmatrix} \\ &= \frac{1}{2}(1+h)u_\uparrow\ , \end{aligned} \tag{12}$$

and therefore

$$P(+,s)u_\uparrow = u_\uparrow, \quad P(-,s)u_\uparrow = 0\ , \tag{13}$$

as it should be for a projector. This can easily checked for the other cases.

Also useful projectors are the chirality projectors, P_L and P_R. In the Dirac representation that we are using, they are,

$$P_L = \frac{1}{2}(1-\gamma_5) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad P_R = \frac{1}{2}(1+\gamma_5) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\ . \tag{14}$$

It is clear from Eqs. (3) and (4) that the helicity spinors are not in general eigenstates of P_L and P_R.

III. HELICITY AND CHIRALITY FOR $m = 0$ AND $m \neq 0$

There is an important case when the helicity eigenstates are also chiral eigenstates. This is when the mass vanishes. One can easily check from Eqs. (3) and (4), that in the limit $m = 0$ we have,

$$P_R u_\uparrow = u_\uparrow \quad ; \quad P_L u_\uparrow = 0 \quad ; \quad P_R u_\downarrow = 0 \quad ; \quad P_L u_\downarrow = u_\downarrow \tag{15}$$

$$P_R v_\uparrow = 0 \quad ; \quad P_L v_\uparrow = v_\uparrow \quad ; \quad P_R v_\downarrow = v_\downarrow \quad ; \quad P_L v_\downarrow = 0 \tag{16}$$

which shows that chirality equals helicity in this limit, for particles. With our conventions chirality is minus the helicity for antiparticles.

More interesting is to see what happens in the massive case. In this case the correct basis to use is the helicity basis (it commutes with the Hamiltonian). We can relate the helicity and chiral projectors in the following way. We start by using the identity,

$$\begin{aligned} P(h,s) =& \frac{1}{2}\left(1 + h\,\gamma_5\,\not{s}\right) = \frac{1}{2}\left(1 + h\,\gamma_5\,\not{s}\right)\left(P_L + P_R\right) \\ =& P_L\,\frac{1 - h\,\not{s}}{2} + P_R\,\frac{1 + h\,\not{s}}{2} \end{aligned} \tag{17}$$

Now in the Dirac representation we can write,

$$\frac{1 + q\,h\,\not{s}}{2} = \begin{bmatrix} \frac{1}{2}\left(1 + q\,h\,\gamma\beta\right) & -\frac{1}{2} q\,h\,\gamma\,\vec{\sigma}\cdot\hat{\beta} \\ -\frac{1}{2} q\,h\,\gamma\,\vec{\sigma}\cdot\hat{\beta} & \frac{1}{2}\left(1 - q\,h\,\gamma\beta\right) \end{bmatrix} \tag{18}$$

Now we define the two component spinors,

$$\chi_\uparrow = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) e^{i\phi} \end{bmatrix}, \quad \chi_\downarrow = \begin{bmatrix} -\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) e^{i\phi} \end{bmatrix}, \qquad \vec{\sigma}\cdot\hat{\beta}\ \chi_\uparrow = \chi_\uparrow, \quad \vec{\sigma}\cdot\hat{\beta}\ \chi_\downarrow = -\chi_\downarrow, \tag{19}$$

and use them to write the helicity spinors of Eqs. (3) and (4) in a simpler form as,

$$u_\uparrow = \sqrt{E+m} \begin{bmatrix} \chi_\uparrow \\ \frac{\gamma\beta}{\gamma+1}\chi_\uparrow \end{bmatrix}, \; u_\downarrow = \sqrt{E+m} \begin{bmatrix} \chi_\downarrow \\ -\frac{\gamma\beta}{\gamma+1}\chi_\downarrow \end{bmatrix}, \; v_\uparrow = \sqrt{E+m} \begin{bmatrix} -\frac{\gamma\beta}{\gamma+1}\chi_\downarrow \\ \chi_\downarrow \end{bmatrix}, \; v_\downarrow = \sqrt{E+m} \begin{bmatrix} \frac{\gamma\beta}{\gamma+1}\chi_\uparrow \\ \chi_\uparrow \end{bmatrix} \tag{20}$$

Now we expand to first order in m/E. We start with $u_\uparrow$. We get

$$P(+,s)u_\uparrow = P_L \frac{m}{2E}\sqrt{E} \begin{bmatrix} \chi_\uparrow \\ -\chi_\uparrow \end{bmatrix} + P_R \sqrt{E} \begin{bmatrix} \chi_\uparrow \\ \chi_\uparrow \end{bmatrix} \tag{21}$$

$$P(-,s)u_\uparrow = P_L \sqrt{E} \begin{bmatrix} \chi_\uparrow \\ \chi_\uparrow \end{bmatrix} + P_R \frac{m}{2E} \begin{bmatrix} \chi_\uparrow \\ -\chi_\uparrow \end{bmatrix} = 0\ . \tag{22}$$

Let us discuss the meaning of this result. From Eq. (21) we see that the projector of positive helicity is a linear combination of the two chiral projectors. Only in the massless limit it equals P_R. On the other hand Eq. (22) is to be expected, as the negative helicity projector applied to $u_\uparrow$ should always give zero for any value of the mass, see the second term in Eq. (13). Notice that we have used Eq. (14) in obtaining this result.

We now present the results for the other cases,

$$P(+,s)u_\downarrow = P_L \frac{m}{2E}\sqrt{E} \begin{bmatrix} \chi_\downarrow \\ \chi_\downarrow \end{bmatrix} + P_R \sqrt{E} \begin{bmatrix} \chi_\downarrow \\ -\chi_\downarrow \end{bmatrix} = 0, \qquad P(-,s)u_\downarrow = P_L \sqrt{E} \begin{bmatrix} \chi_\downarrow \\ -\chi_\downarrow \end{bmatrix} + P_R \frac{m}{2E}\sqrt{E} \begin{bmatrix} \chi_\downarrow \\ \chi_\downarrow \end{bmatrix}\ . \tag{23}$$

$$P(+,s)v_\uparrow = P_L \sqrt{E} \begin{bmatrix} -\chi_\downarrow \\ \chi_\downarrow \end{bmatrix} + P_R \frac{m}{2E}\sqrt{E} \begin{bmatrix} \chi_\downarrow \\ \chi_\downarrow \end{bmatrix}, \qquad P(-,s)v_\uparrow = P_L \frac{m}{2E}\sqrt{E} \begin{bmatrix} \chi_\downarrow \\ \chi_\downarrow \end{bmatrix} + P_R \sqrt{E} \begin{bmatrix} -\chi_\downarrow \\ \chi_\downarrow \end{bmatrix} = 0\ . \tag{24}$$

$$P(+,s)v_\downarrow = P_L\sqrt{E}\begin{bmatrix}\chi_\uparrow\\ \chi_\uparrow\end{bmatrix} + P_R\frac{m}{2E}\sqrt{E}\begin{bmatrix}-\chi_\uparrow\\ \chi_\uparrow\end{bmatrix} = 0, \qquad P(-,s)v_\downarrow = P_L\frac{m}{2E}\sqrt{E}\begin{bmatrix}-\chi_\uparrow\\ \chi_\uparrow\end{bmatrix} + P_R\sqrt{E}\begin{bmatrix}\chi_\uparrow\\ \chi_\uparrow\end{bmatrix} \ . \tag{25}$$

One can easily verify that in the massless limit chirality and helicity are identified and we obtain the same results as in Eqs. (15) and (16).

IV. THE PROCESS $e^- + e^+ \to \mu^- + \mu^+$ AS LABORATORY FOR HELICITY AND CHIRALITY

We are going to use the process $e^- + e^+ \to \mu^- + \mu^+$ in the CM frame to discuss various methods for the calculation the cross section. We use the kinematics in the CM frame,

$$\begin{aligned}
p_1 =& \frac{\sqrt{s}}{2}\ (1,0,0,\beta_1)\\
p_2 =& \frac{\sqrt{s}}{2}\ (1,0,0,-\beta_1)\\
p_3 =& \frac{\sqrt{s}}{2}\ (1,\beta_3\sin\theta,0,\beta_3\cos\theta)\\
p_4 =& \frac{\sqrt{s}}{2}\ (1,-\beta_3\sin\theta,0,-\beta_3\cos\theta)
\end{aligned} \tag{26}$$

as indicated in of Fig. 1, and where $\beta_1 = \sqrt{1-4m_e^2/s}$, $\beta_3 = \sqrt{1-4m_\mu^2/s}$.

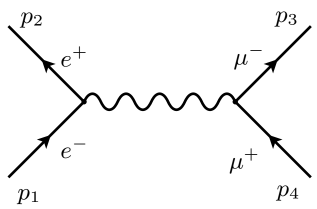

Figure 1: Scattering $e^-e^+ \to \mu^-\mu^+$ in QED.

This leads to the following amplitude

$$i\,\mathcal{M} = \overline{v}(p_2)(ie\gamma^\mu)u(p_1)\ \frac{-i\ g_{\mu\nu}}{(p_1+p_2)^2 + i\varepsilon}\ \overline{u}(p_3)(ie\gamma^\nu)v(p_4) \tag{27}$$

or

$$\mathcal{M} = \frac{e^2}{s}\overline{v}(p_2)\gamma^\mu u(p_1)\ \overline{u}(p_3)\gamma_\mu v(p_4)\ . \tag{28}$$

The differential cross section is then given by,

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\vec{p}_3|}{|\vec{p}_1|}\left\langle|\mathcal{M}|^2\right\rangle \tag{29}$$

Our goal is to evaluate $\left\langle|\mathcal{M}|^2\right\rangle$ using different techniques.

A. $e^- + e^+ \to \mu^- + \mu^+$ with Traces

We start by calculating the spin averaged squared amplitude using the well known trace technique. From Eq. (28) we obtain,

$$\left\langle|\mathcal{M}|^2\right\rangle = \frac{1}{4}\frac{e^4}{s^2}\mathrm{Tr}[(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu]\mathrm{Tr}[(\not{p}_3 + m_\mu)\gamma_\mu(\not{p}_4 - m_\mu)\gamma_\nu]$$

$$=(4\pi\alpha)^2\left(3-\beta_1^2-\beta_3^2+\beta_1^2\beta_3^2\cos^2\theta\right)$$
$$\to(4\pi\alpha)^2\left(1+\cos^2\theta\right) \tag{30}$$

where the last line corresponds to the well known result in the massless limit. For completeness we also write the expression for the differential cross section,

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2}{4s}\frac{\beta_3}{\beta_1}\left(3-\beta_1^2-\beta_3^2+\beta_1^2\beta_3^2\cos^2\theta\right)\to\frac{\alpha^2}{4s}\left(1+\cos^2\theta\right) \tag{31}$$

where the last expression is again in the massless limit.

B. $e^-+e^+\to\mu^-+\mu^+$ with Helicity Amplitudes

With the trace technique we get an averaged result and the contribution from the various helicities is lost in the process. We can however explicitly calculate the sixteen helicity combinations that exist for the massive case. This simplifies a lot in the massless case as it is explained in Refs.[1, 2] for instance. What we have to do is to use the explicit form of Eqs. (3) and (4) for the kinematics of the process. We get for $p_1:\theta=0,\phi=0,\ p_2:\theta=\pi,\phi=\pi$,

$$u_\uparrow(p_1)=N\begin{bmatrix}1\\0\\\frac{\gamma_1\beta_1}{\gamma_1+1}\\0\end{bmatrix},\ u_\downarrow(p_1)=N\begin{bmatrix}0\\1\\0\\-\frac{\gamma_1\beta_1}{\gamma_1+1}\end{bmatrix},\ v_\uparrow(p_2)=N\begin{bmatrix}\frac{\gamma_1\beta_1}{\gamma_1+1}\\0\\-1\\0\end{bmatrix},\ v_\downarrow(p_2)=N\begin{bmatrix}0\\-\frac{\gamma_1\beta_1}{\gamma_1+1}\\0\\-1\end{bmatrix}. \tag{32}$$

where $N=\sqrt{E+m}$, and

$$u_\uparrow(p_3)=N\begin{bmatrix}c\\s\\\frac{\gamma_3\beta_3}{\gamma_3+1}c\\\frac{\gamma_3\beta_3}{\gamma_3+1}s\end{bmatrix},\ u_\downarrow(p_3)=N\begin{bmatrix}-s\\c\\\frac{\gamma_3\beta_3}{\gamma_3+1}s\\-\frac{\gamma_3\beta_3}{\gamma_3+1}c\end{bmatrix},\ v_\uparrow(p_4)=N\begin{bmatrix}\frac{\gamma_3\beta_3}{\gamma_3+1}c\\\frac{\gamma_3\beta_3}{\gamma_3+1}s\\-c\\-s\end{bmatrix},\ v_\downarrow(p_4)=N\begin{bmatrix}\frac{\gamma_3\beta_3}{\gamma_3+1}s\\-\frac{\gamma_1\beta_1}{\gamma_1+1}c\\s\\-c\end{bmatrix}. \tag{33}$$

where we have used the simplifying notation $c=\cos\theta/2, s=\sin\theta/2$, and the angles are such that $p_3:\theta,\phi=0$, $p_4:\theta=\pi-\theta,\phi=\pi$. Using these explicit spinors and the explicit form for the Dirac γ matrices, we can then obtain the helicity amplitudes that we write as

$$\mathcal{M}(h_1,h_2;h_3,h_4)=\frac{4\pi\alpha}{s}\ \overline{v}(p_2,h_2)\gamma^\mu u(p_1,h_1)\ \overline{u}(p_3,h_3)\gamma_\mu v(p_4,h_4)\ . \tag{34}$$

where $h_i=\uparrow,\downarrow$ for each particle. This is a straightforward but tedious calculation, that can be best done with a `mathematica` program[3]. The result is,

$$\mathcal{M}(\uparrow,\uparrow;\uparrow,\uparrow)=-(4\pi\alpha)\ \frac{4\,m_em_\mu}{s}\cos\theta \qquad \mathcal{M}(\downarrow,\uparrow;\uparrow,\uparrow)=-(4\pi\alpha)\ \frac{2\,m_\mu}{\sqrt{s}}\sin\theta \tag{35}$$

$$\mathcal{M}(\uparrow,\downarrow;\uparrow,\uparrow)=-(4\pi\alpha)\ \frac{2\,m_\mu}{\sqrt{s}}\sin\theta \qquad \mathcal{M}(\downarrow,\downarrow;\uparrow,\uparrow)=(4\pi\alpha)\ \frac{4\,m_em_\mu}{s}\cos\theta, \tag{36}$$

$$\mathcal{M}(\uparrow,\uparrow;\downarrow,\uparrow)=(4\pi\alpha)\ \frac{2\,m_e}{\sqrt{s}}\sin\theta \qquad \mathcal{M}(\downarrow,\uparrow;\downarrow,\uparrow)=-(4\pi\alpha)\,(1+\cos\theta) \tag{37}$$

$$\mathcal{M}(\uparrow,\downarrow;\downarrow,\uparrow)=(4\pi\alpha)\,(1-\cos\theta) \qquad \mathcal{M}(\downarrow,\downarrow;\downarrow,\uparrow)=-(4\pi\alpha)\ \frac{2\,m_e}{\sqrt{s}}\sin\theta, \tag{38}$$

$$\mathcal{M}(\uparrow,\uparrow;\uparrow,\downarrow)=(4\pi\alpha)\ \frac{2\,m_e}{\sqrt{s}}\sin\theta \qquad \mathcal{M}(\downarrow,\uparrow;\uparrow,\downarrow)=(4\pi\alpha)\,(1-\cos\theta) \tag{39}$$

$$\mathcal{M}(\uparrow,\downarrow;\uparrow,\downarrow) = -\,(4\pi\alpha)\,(1+\cos\theta) \qquad \mathcal{M}(\downarrow,\downarrow;\uparrow,\downarrow) = -(4\pi\alpha)\,\frac{2\,m_e}{\sqrt{s}}\sin\theta \tag{40}$$

$$\mathcal{M}(\uparrow,\uparrow;\downarrow,\downarrow) = (4\pi\alpha)\,\frac{4\,m_e m_\mu}{s}\cos\theta \qquad \mathcal{M}(\downarrow,\uparrow;\downarrow,\downarrow) = (4\pi\alpha)\,\frac{2\,m_\mu}{\sqrt{s}}\sin\theta \tag{41}$$

$$\mathcal{M}(\uparrow,\downarrow;\downarrow,\downarrow) = (4\pi\alpha)\,\frac{2\,m_\mu}{\sqrt{s}}\sin\theta \qquad \mathcal{M}(\downarrow,\downarrow;\downarrow,\downarrow) = -(4\pi\alpha)\,\frac{4\,m_e m_\mu}{s}\cos\theta \tag{42}$$

From these expressions one can calculate the spin averaged square of the amplitude defined by

$$\begin{aligned}\left\langle|\mathcal{M}|^2\right\rangle =& \frac{1}{4}\sum_{h_i}|\mathcal{M}(h_1,h_2;h_3,h_4)|^2 \\ =& (4\pi\alpha)^2\left(3-\beta_1^2-\beta_3^2+\beta_1^2\beta_3^2\cos^2\theta\right)\end{aligned} \tag{43}$$

exactly as we had obtained before in Eq. (30). In the massless limit we have only 4 non vanishing amplitudes,

$$\mathcal{M}(\uparrow,\downarrow;\uparrow,\downarrow) = \mathcal{M}(\downarrow,\uparrow;\downarrow,\uparrow) = -(4\pi\alpha)\,(1+\cos\theta) \tag{44}$$

$$\mathcal{M}(\uparrow,\downarrow;\downarrow,\uparrow) = \mathcal{M}(\downarrow,\uparrow;\uparrow,\downarrow) = (4\pi\alpha)\,(1-\cos\theta) \tag{45}$$

To understand this result, we notice that the only non-vanishing amplitudes in the massless limit are those where the spins add to ± 1, as shown in Fig. 2 and Fig. 3.

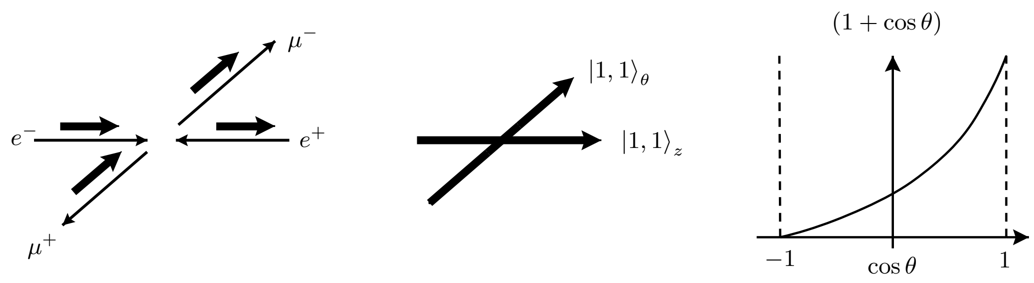

Figure 2: Amplitude $\mathcal{M}(\uparrow,\downarrow;\uparrow,\downarrow)$

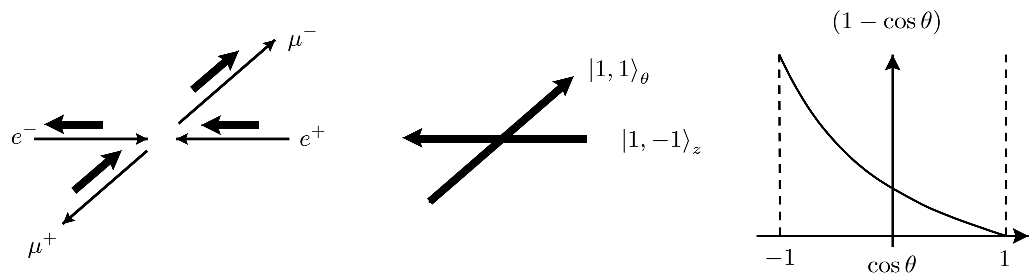

Figure 3: Amplitude $\mathcal{M}(\downarrow,\uparrow;\uparrow,\downarrow)$

For spin 1 in a direction θ we have

$$|1,+1\rangle_\theta = \frac{1}{2}(1-\cos\theta)\,|1,-1\rangle_z + \frac{1}{\sqrt{2}}\sin\theta)\,|1,0\rangle_z + \frac{1}{2}(1+\cos\theta)\,|1,1\rangle_z \tag{46}$$

and therefore

$$\mathcal{M}(\uparrow\downarrow;\uparrow\downarrow) \propto {}_\theta\langle 1,+1|1,+1\rangle_z = \frac{1}{2}(1+\cos\theta) \tag{47}$$

$$\mathcal{M}(\downarrow\uparrow;\uparrow\downarrow) \propto {}_{\theta}\langle 1,+1|1,-1\rangle_z = \frac{1}{2}(1-\cos\theta) \tag{48}$$

$$\tag{49}$$

We see that the non-vanishing amplitudes do not change the direction of the arrow of the spin. This is due to the

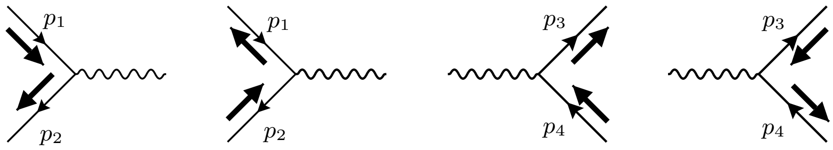

Figure 4: Non-vanishing amplitudes and the direction of the spin arrows.

fact that the QED interaction preserves chirality and in the limit $m \to 0$ chirality and helicity coincide.

C. $e^- + e^+ \to \mu^- + \mu^+$ with Traces and Helicity Projectors

The discussion of the previous section it is very nice because it clarifies much of the physics behind, but in practice the calculations have to be done in a particular basis, with a given representation for the Dirac matrices and are very time consuming. The question then arises if we can do the calculations using the traces and still preserve the helicities.

This can be done using the helicity projector operators, discussed before. Essentially one has to put an helicity projector before each u or v spinor. One can easily show the following substitution rules

$$u(p) \to P(h,s)u(p), \quad v(p) \to P(h,s)v(p), \quad \overline{u}(p) \to \overline{u}(p)P(h,s), \quad \overline{v}(p) \to \overline{v}(p)P(h,s) \tag{50}$$

where the spin direction of s is along the momentum direction, as in Eq. (6). Therefore the helicity amplitude reads now

$$\mathcal{M}(h_1,h_2;h_3,h_4) = \frac{4\pi\alpha}{s}\overline{v}(p_2)P(h_2,s_2)\gamma^\mu P(h_1,s_1)u(p_1)\overline{u}(p_3)P(h_3,s_3)\gamma_\mu P(h_4,s_4)v(p_4) \tag{51}$$

Now using the usual trace techniques one can show that

$$\begin{aligned}|\mathcal{M}(h_1,h_2;h_3,h_4)|^2 =& \frac{(4\pi\alpha)^2}{s^2}\mathrm{Tr}\left[\left(\not{p}_2 - m_e\right)P(h_2,s_2)\gamma^\mu P(h_1,s_1)\left(\not{p}_1 + m_e\right)P(h_1,s_1)\gamma^\nu P(h_2,s_2)\right] \\ & \mathrm{Tr}\left[\left(\not{p}_3 + m_\mu\right)P(h_3,s_3)\gamma_\mu P(h_4,s_4)\left(\not{p}_4 - m_\mu\right)P(h_4,s_4)\gamma_\nu P(h_3,s_3)\right]\end{aligned} \tag{52}$$

Now these traces involve a large number of Dirac matrices and are therefore not easy to evaluate by hand. But one can use well known software, like `FeynCalc` for `Mathematica` to handle this problem. A program can be found in[3]. We just give here the results so that we can compare with the helicity calculations.

$$|\mathcal{M}(\uparrow,\uparrow;\uparrow,\uparrow)|^2 = (4\pi\alpha)^2\ \frac{16\,m_e^2 m_\mu^2}{s^2}\cos^2\theta \qquad |\mathcal{M}(\downarrow,\uparrow;\uparrow,\uparrow)|^2 = (4\pi\alpha)^2\ \frac{4\,m_\mu^2}{s}\sin^2\theta \tag{53}$$

$$|\mathcal{M}(\uparrow,\downarrow;\uparrow,\uparrow)|^2 = (4\pi\alpha)^2\ \frac{4\,m_\mu^2}{s}\sin^2\theta \qquad |\mathcal{M}(\downarrow,\downarrow;\uparrow,\uparrow)|^2 = (4\pi\alpha)^2\ \frac{16\,m_e^2 m_\mu^2}{s}\cos^2\theta, \tag{54}$$

$$|\mathcal{M}(\uparrow,\uparrow;\downarrow,\uparrow)|^2 = (4\pi\alpha)^2\ \frac{4\,m_e^2}{s}\sin^2\theta \qquad |\mathcal{M}(\downarrow,\uparrow;\downarrow,\uparrow)|^2 = (4\pi\alpha)^2\,(1+\cos\theta)^2 \tag{55}$$

$$|\mathcal{M}(\uparrow,\downarrow;\downarrow,\uparrow)|^2 = (4\pi\alpha)^2\,(1-\cos\theta)^2 \qquad |\mathcal{M}(\downarrow,\downarrow;\downarrow,\uparrow)|^2 = (4\pi\alpha)^2\ \frac{4\,m_e^2}{s}\sin^2\theta, \tag{56}$$

$$|\mathcal{M}(\uparrow,\uparrow;\uparrow,\downarrow)|^2 = (4\pi\alpha)^2\ \frac{4\,m_e^2}{s}\sin^2\theta \qquad |\mathcal{M}(\downarrow,\uparrow;\uparrow,\downarrow)|^2 = (4\pi\alpha)^2\,(1-\cos\theta)^2 \tag{57}$$

$$|\mathcal{M}(\uparrow,\downarrow;\uparrow,\downarrow)|^2 = (4\pi\alpha)^2\,(1+\cos\theta)^2 \qquad |\mathcal{M}(\downarrow,\downarrow;\uparrow,\downarrow)|^2 = (4\pi\alpha)^2\ \frac{4\,m_e^2}{s}\sin^2\theta \tag{58}$$

$$|\mathcal{M}(\uparrow,\uparrow;\downarrow,\downarrow)|^2 = (4\pi\alpha)^2\, \frac{16\, m_e^2 m_\mu^2}{s^2} \cos^2\theta \qquad |\mathcal{M}(\downarrow,\uparrow;\downarrow,\downarrow)|^2 = (4\pi\alpha)^2\, \frac{4\, m_\mu^2}{s} \sin^2\theta \tag{59}$$

$$|\mathcal{M}(\uparrow,\downarrow;\downarrow,\downarrow)|^2 = (4\pi\alpha)^2\, \frac{4\, m_\mu^2}{s} \sin^2\theta \qquad |\mathcal{M}(\downarrow,\downarrow;\downarrow,\downarrow)|^2 = (4\pi\alpha)^2\, \frac{16\, m_e^2 m_\mu^2}{s^2} \cos^2\theta \tag{60}$$

We can compare these results from those in Eqs. (35) to (42) and verify that they are in complete agreement.

V. ANOTHER VIEW AT HELICITY PROJECTORS

Although we have seen in the previous section that the projector (we are changing slightly the notation for the save of simplicity, see below)

$$P_s(h) = \frac{1 + h\,\gamma_5 \not{s}}{2} = \begin{bmatrix} \frac{1}{2}(1 + h\,\gamma\vec{\sigma}\cdot\hat{\beta}) & -\frac{h}{2}\gamma\beta \\ \frac{h}{2}\gamma\beta & \frac{1}{2}(1 - h\,\gamma\vec{\sigma}\cdot\hat{\beta}) \end{bmatrix} \tag{61}$$

correctly projects out the various helicity combinations, there is another way of defining the helicity projector, that we denote by,

$$P_h(h) = \frac{1 + h\,\vec{\Sigma}\cdot\hat{\beta}}{2} = \begin{bmatrix} \frac{1}{2}\left(1 + h\,\vec{\sigma}\cdot\hat{\beta}\right) & 0 \\ 0 & \frac{1}{2}\left(1 + h\,\vec{\sigma}\cdot\hat{\beta}\right) \end{bmatrix} \tag{62}$$

If we define the helicity spinor (for this section we just consider the positive energy spinors, but of course we could also discuss the other case)

$$u_h = \begin{bmatrix} \chi_h \\ h\dfrac{\gamma\beta}{\gamma+1}\chi_h \end{bmatrix} \tag{63}$$

where we use the notation $h = \uparrow, \downarrow$, and

$$\vec{\sigma}\cdot\hat{\beta}\chi_h = h\chi_h \tag{64}$$

as in Eq. (10). The form of u_h in Eq. (63) follows just from the Dirac equation $(\not{p} - m)u = 0$ [2]. Now we can easily show using the explicit expressions in Eqs. (61) and (62) that we have,

$$P_s(h')u_h = \frac{1}{2}(1 + hh')u_h, \quad P_h(h')u_h = \frac{1}{2}(1 + hh')u_h \tag{65}$$

that is, they both really project the helicity.

A. Relation between the projectors

To try to understand better the relation between both projectors we define the difference, by

$$P_s(h) = P_h(h) + \Delta(h) \tag{66}$$

where we have

$$\Delta(h) = \begin{bmatrix} \frac{1}{2}h(\gamma-1)\vec{\sigma}\cdot\hat{\beta} & -\frac{1}{2}h\,\gamma\beta \\ \frac{1}{2}h\,\gamma\beta & -\frac{1}{2}h(\gamma+1)\vec{\sigma}\cdot\hat{\beta} \end{bmatrix} \tag{67}$$

We can show the following results (not all of them independent):

1. The eigenvalues of Δ are zero, that is,

$$\Delta(h)u_h = 0 \tag{68}$$

2. The projectors $P_s(h), P_h(h)$ and the operator $\Delta(h)$ commute among themselves and therefore they have the same eigenvectors.
3. The operator $\Delta(h)$ is not a projector but it satisfies,

$$\Delta(h)+\Delta(-h)=0 \tag{69}$$

$$\Delta(h)^2+P_h(h)\Delta(h)+\Delta(h)P_h(h)=\Delta(h) \tag{70}$$

$$\Delta(h)^2-P_s(h)\Delta(h)-\Delta(h)P_s(h)=-\Delta(h) \tag{71}$$

$$\Delta(h)\Delta(-h)=-\Delta(h)P_h(-h)-P_h(h)\Delta(-h) \tag{72}$$

$$\Delta(h)\Delta(-h)=\Delta(h)P_s(-h)+P_s(h)\Delta(-h) \tag{73}$$

VI. THE PROCESS $e^- + e^+ \to \gamma + \gamma$ IN THE MASSLESS LIMIT

This process allow us to discuss a technique known as helicity amplitude calculations. We will discuss this in the limit of massless electrons and positrons for which it was designed [4–7].

A. $e^- + e^+ \to \gamma + \gamma$ with traces

We will consider as an example the process $e^-e^+ \to \gamma\gamma$ in QED, in the massless fermions limit. We have the two diagrams of Fig. 5. The amplitudes can easily be written as

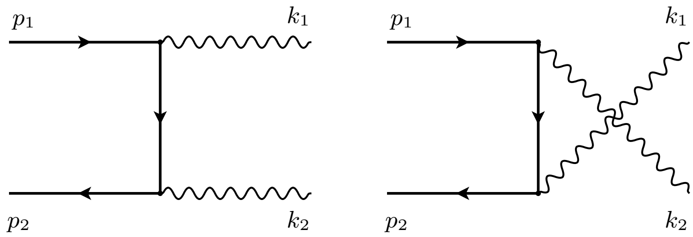

Figure 5: Diagrams for $e^-e^+ \to \gamma\gamma$

$$\begin{aligned}\mathcal{M}_1 &= -\,e^2\,\overline{v}(p_2)\not\epsilon^*(k_2)\,(\not p_1-\not k_1)\not\epsilon^*(k_1)u(p_1)\,\frac{1}{t}\\ \mathcal{M}_2 &= -\,e^2\,\overline{v}(p_2)\not\epsilon^*(k_1)\,(\not p_1-\not k_2)\not\epsilon^*(k_2)u(p_1)\,\frac{1}{u}\end{aligned} \tag{74}$$

where we used the Mandelstam variables $t=(p_1-k_1)^2$ and $u=(p_1-k_2)^2$. We first calculate using the well known trace technique. We get

$$\frac{1}{4}\sum_{\text{spins}}\sum_{\lambda_1,\lambda_2}|\mathcal{M}_1|^2=2e^4\,\frac{u}{t},\quad \frac{1}{4}\sum_{\text{spins}}\sum_{\lambda_1,\lambda_2}|\mathcal{M}_2|^2=2e^4\,\frac{t}{u} \tag{75}$$

$$\frac{1}{4}\sum_{\text{spins}}\sum_{\lambda_1,\lambda_2}\left[\mathcal{M}_1\mathcal{M}_2^\dagger+\mathcal{M}_1^\dagger\mathcal{M}_2\right]=-4e^4\frac{s(s+t+u)}{tu}=0 \tag{76}$$

where the last step follows because in the massless limit $s+t+u=0$.This means that there is no interference in the massless limit. We will get back to this point when using the helicity amplitudes. The final result is therefore,

$$\frac{1}{4}\sum_{\text{spins}}\sum_{\lambda_1,\lambda_2}|\mathcal{M}|^2=2e^4\left(\frac{u}{t}+\frac{t}{u}\right) \tag{77}$$

B. $e^- + e^+ \to \gamma + \gamma$ with Gastmans's polarization vectors

We start by addressing the last point of the last section, that is why there is no interference between the two diagrams. The best way to see this is to use the helicity amplitude technique developed by Gastmans and collaborators [4–7]. The starting point for this technique is to realize that helicity commutes with the Hamiltonian and it is therefore a good basis to do the calculations. This means that, for our case, we have sixteen *independent* amplitudes. Then the final spin averaged amplitude squared is obtained by,

$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{1}{4} \sum_{h_i, \lambda_i} |\mathcal{M}(h_1, h_2, \lambda_1, \lambda_2)|^2 \tag{78}$$

where h_i, λ_i are the helicities of the four particles in the process. If it was only this there will be not much of a simplification. The big simplification comes from two observations:

1. In the massless limit helicity equals chirality, and because in QED the interaction preserves chirality, only a small number of amplitudes will be non-vanishing. This was the original idea.

2. If we can find a way of calculating the complex number that corresponds to each amplitude, we just have first to sum all the complex numbers coming from all the diagrams that contribute to a given helicity combination, and then get the absolute value to enter Eq. (78). This was done by Kleiss and collaborators [8, 9] and will be explained in the next section.

Here we just want to understand why there is no interference between the two diagrams of Fig. 5. From what we said above, that can only occur if the two diagrams contribute to different helicity combinations, that is, they contribute to different terms in the sum in Eq. (78). For this we use the original approach of Gastmans etal [4–7]. Their brilliant idea was to show that one could write the photon polarization vectors in terms of two null vectors $p^2 = 0, q^2 = 0$. As in QED the polarization vectors will always appear contracted with a Dirac gamma matrix, their main result is that, we can write,

$$\not{\epsilon}^+ = N \left[\not{p}\not{q}\not{k}\gamma_- - \not{k}\not{p}\not{q}\gamma_+ \right], \qquad \not{\epsilon}^- = N \left[\not{p}\not{q}\not{k}\gamma_+ - \not{k}\not{p}\not{q}\gamma_- \right] \tag{79}$$

where $\not{\epsilon}^\pm$ are the helicity $\pm$ polarization vectors for the photon (right and left circularly polarized photons), and we have defined

$$\gamma_+ = \frac{1+\gamma_5}{2}, \qquad \gamma_- = \frac{1-\gamma_5}{2}, \qquad N = \frac{1}{2} \Big[(p\cdot q)(p\cdot k)(q\cdot k) \Big]^{-1/2} \tag{80}$$

The four-vectors p, q are arbitrary, except that they are null and not Co-linear. The expressions in Eq. (79) are very useful because they express the helicities of the photon in terms of the chiralities of the fermions and in the massless limit these coincide with the helicities. In the following, although we will be using Gastmans's idea, we will depart from is notation in two respects:

1. In our Feynman rules the polarization vectors for the final state photons have a complex conjugation. This has no really effect because $\epsilon^{\pm *} = \epsilon^{\mp}$, but of course if we want to compare the final results there will be a difference between what we call positive or negative helicity and his definition.

2. In his original work Gastmans use really helicity eigenstates, that he denotes by $u_\pm(p), v_\pm(p)$, satisfying, in the massless limit that we consider here,

$$\gamma_\pm u_\pm(p) = u_\pm(p), \quad \gamma_\pm v_\mp(p) = v_\mp(p) \tag{81}$$

that is, in our notation, $u_+(p) = u_\uparrow(p), v_+(p) = v_\uparrow(p)$ and so on. That is why the method is called helicity amplitude technique. However, because we want to use in the future (see below) the method of Kleiss, we use a different convention for the $\pm$ signs. For us they are the eigenvalues of the chirality, that is,

$$\gamma_\pm u_\pm(p) = u_\pm(p), \quad \gamma_\pm v_\pm(p) = v_\pm(p) \tag{82}$$

which implies that for us, $v_+(p) = v_\downarrow(p)$, and so on. We will use always the conventions of Eq. (82), that is, our spinors are chirality eigenstates. In fact, following Kleiss, we will go one step further and will only use $u_\pm(p)$ as for chirality there is no distinction between u and v. Of course from the diagram we know which are the antiparticles and that for those helicity is the opposite of chirality.

We are now in position of applying Gastmans's method to our problem. In Eq. (74) we have an odd number of gamma matrices between the spinors, so both the electron and positron must have the same chiralities. Let us suppose that we start with the positive chirality, that is we write,

$$\begin{aligned}
\mathcal{M}_1(+,+;\lambda_1,\lambda_2) &= -\,e^2\,\overline{u}_+(p_2)\not\epsilon^*(k_2,\lambda_2)\,(\not p_1-\not k_1)\not\epsilon^*(k_1,\lambda_1)u_+(p_1)\,\frac{1}{t}\\
&= -\,e^2\,\overline{u}_+(p_2)\not\epsilon^{(-\lambda_2)}(k_2)\,(\not p_1-\not k_1)\not\epsilon^{(-\lambda_1)}(k_1)u_+(p_1)\,\frac{1}{t}\\
\mathcal{M}_2(+,+;\lambda_1,\lambda_2) &= -\,e^2\,\overline{u}_+(p_2)\not\epsilon^*(k_1,\lambda_1)\,(\not p_1-\not k_2)\not\epsilon^*(k_2,\lambda_2)u_+(p_1)\,\frac{1}{u}\\
&= -\,e^2\,\overline{u}_+(p_2)\not\epsilon^{(-\lambda_1)}(k_1)\,(\not p_1-\not k_2)\not\epsilon^{(-\lambda_2)}(k_2)u_+(p_1)\,\frac{1}{u}
\end{aligned}\tag{83}$$

The next step is to choose the vectors p, q in Eq. (79) to simplify the results after using the Dirac equation, $\not p_1 u_+(p_1) = 0, \overline{u}_+(p_2)\not p_2 = 0$. Therefore we choose $p = p_2, q = p_1$ to get,

$$\not\epsilon^{\pm}(k_1) = N_1\left[\not p_2\not p_1\not k_1\gamma_{\mp} - \not k_1\not p_2\not p_1\gamma_{\pm}\right], \quad \not\epsilon^{\pm}(k_2) = N_2\left[\not p_2\not p_1\not k_2\gamma_{\mp} - \not k_2\not p_2\not p_1\gamma_{\pm}\right]\tag{84}$$

By inspection we see that the photon helicities have to be opposite, and that the only non-vanishing amplitudes are for $\lambda_1 = +, \lambda_2 = -$ for $\mathcal{M}_1$ and $\lambda_1 = -, \lambda_2 = +$ for $\mathcal{M}_2$

$$\begin{aligned}
\mathcal{M}_1(+,+;+,-) &= -\,e^2 N_1N_2\;\overline{u}_+(p_2)\,(-1)\not k_2\not p_2\not p_1\gamma_+\,(\not p_1-\not k_1)\,\not p_2\not p_1\not k_1\gamma_+u_+(p_1)\,\frac{1}{t}\\
&= -\,e^2 N_1N_2\;\overline{u}_+(p_2)\,\not k_2\not p_2\not p_1\not k_1\not p_2\not p_1\not k_1\gamma_+u_+(p_1)\,\frac{1}{t}\\
\mathcal{M}_2(+,+;-,+) &= -\,e^2 N_1N_2\;\overline{u}_+(p_2)\,(-1)\not k_1\not p_2\not p_1\gamma_+\,(\not p_1-\not k_2)\,\not p_2\not p_1\not k_2\gamma_+u_+(p_1)\,\frac{1}{u}\\
&= -\,e^2 N_1N_2\;\overline{u}_+(p_2)\,\not k_1\not p_2\not p_1\not k_2\not p_2\not p_1\not k_2\gamma_+u_+(p_1)\,\frac{1}{u}
\end{aligned}\tag{85}$$

In a similar way we would find that only $\mathcal{M}_1(-,-;+,-)$ and $\mathcal{M}_2(-,-;+,-)$ would be non-vanishing. Therefore this shows that there is no interference between the two diagrams, as each non vanishing term in the sum in Eq. (78) will come either from one or the other diagram, but never form the two at the same time. We could now proceed to evaluate these amplitudes, transforming the squares in traces, but we will use other method below and leave this exercise here.

C. The spinor products of Kleiss

We have seen that the helicity amplitudes method can reduce the number of calculations to be done. In the usual way this also means a smaller number of traces to be computed. But even the method of Gastmans, for more complex problems requires to calculate a large number of traces.

The complexity of the traces calculations grows with n^2 where n is the number of diagrams. The helicity approach reduces this but not completely. This where the method of Kleiss appears [9]. Suppose that we can assign a complex number to each diagram. Then for each independent helicity combination we just have to add the complex numbers that contribute to it and then add the absolute squares of those complex numbers. This can be turned in a very efficient numerical calculation. In this case the complexity just grows with n not with n^2. We now explain how we can assign a complex number to a given diagram.

For massless fermions we consider the eigenstates of chirality. As discussed above there is no distinction between u and v from the point of view of chirality. We denote these spinors by $u_\pm(p)$ with $p^2 = 0$ e $\not p u_\pm(p) = 0$. These spinors satisfy

$$\gamma_+\not p = u_+(p)\overline{u}_+(p), \quad \gamma_-\not p = u_-(p)\overline{u}_-(p), \quad \not p = u_+(p)\overline{u}_+(p) + u_-(p)\overline{u}_-(p)\tag{86}$$

With these spinors one can only form two non-vanishing independent *spinor products* [9],

$$s(p_1,p_2) \;=\; \overline{u}_+(p_1)u_-(p_2) = -s(p_2,p_1)$$

$$t(p_1,p_2) \;=\; \overline{u}_-(p_1)u_+(p_2) = s^*(p_2,p_1) \tag{87}$$

with the normalization

$$|s(p_1,p_2)|^2 = 2p_1\cdot p_2 \tag{88}$$

To make explicit calculations we need an explicit formula for them. This was given by [8],

$$s(p_1,p_2) = \left(p_1^2 + ip_1^3\right)\sqrt{\frac{p_2^0-p_2^1}{p_1^0-p_1^1}} - \left(p_2^2+ip_2^3\right)\sqrt{\frac{p_1^0-p_1^1}{p_2^0-p_2^1}} \tag{89}$$

Using this expression we can easily verify Eq. (88). Another important relation is ($\sigma = \pm$),

$$\overline{u}_\sigma(p_1)\gamma_\mu u_\sigma(p_2)\,\gamma^\mu = 2u_\sigma(p_2)\overline{u}_\sigma(p_1) + 2u_{-\sigma}(p_1)\overline{u}_{-\sigma}(p_2) \tag{90}$$

which shows that the spinors are normalized in such a way that

$$\overline{u}_\sigma(p)\gamma_\mu u_\sigma(p) = 2p_\mu\ . \tag{91}$$

Using Eq. (86) we can show the following relations

$$\begin{aligned}
\overline{u}_+(p_1)\not p_2\not p_3\cdots\not p_{2n-1}u_-(p_{2n}) &= s(p_1,p_2)s^*(p_3,p_2)s(p_3,p_4)\cdots s(p_{2n-1},p_{2n})\\
\overline{u}_-(p_1)\not p_2\not p_3\cdots\not p_{2n-1}u_+(p_{2n}) &= s^*(p_2,p_1)s(p_2,p_3)s^*(p_4,p_3)\cdots s^*(p_{2n},p_{2n-1})\\
\overline{u}_+(p_1)\not p_2\not p_3\cdots\not p_{2n}u_+(p_{2n+1}) &= s(p_1,p_2)s^*(p_3,p_2)s(p_3,p_4)\cdots s^*(p_{2n+1},p_{2n})\\
\overline{u}_-(p_1)\not p_2\not p_3\cdots\not p_{2n}u_-(p_{2n+1}) &= s^*(p_2,p_1)s(p_2,p_3)s^*(p_4,p_3)\cdots s(p_{2n},p_{2n+1})
\end{aligned} \tag{92}$$

where p_i are massless momenta, that is, $p_i^2 = 0$. When we have two fermion lines connected by a contraction of Dirac γ matrices we have to use Eq. (90). For instance

$$\overline{u}_+(p_1)\gamma_\mu u_+(p_2)\,\overline{u}_+(p_3)\gamma^\mu u_+(p_4) = 2s(p_3,p_1)s^*(p_4,p_2)\ . \tag{93}$$

With these expressions we can transform all the helicity (chirality) amplitudes for massless fermions into a complex number written in terms of spinor products.

Up to now we have not considered photons in the external lines. Kleiss showed that this can also be implemented in the spinor product formalism [9, 10] with the definition

$$\epsilon^\mu(k,\lambda) = \frac{1}{(4k\cdot p)^{1/2}}\,\overline{u}_\lambda(k)\gamma^\mu u_\lambda(p) \tag{94}$$

where p is any null four-vector ($p^2=0$) that is not proportional to k. For this definition to be consistent we should have,

$$\sum_\lambda \epsilon^\mu(k,\lambda)\epsilon^{*\mu}(k,\lambda) = -g^{\mu\nu} + \text{terms proportional to } k. \tag{95}$$

The terms proportional to k, can then be neglected due to gauge invariance. In fact we have ($N=\sqrt{4k\cdot p}$),

$$\begin{aligned}
\sum_\lambda \epsilon^\mu(k,\lambda)\epsilon^{*\nu}(k,\lambda) =& \frac{1}{N^2}\left[\overline{u}_+(k)\gamma^\mu u_+(p)\overline{u}_+(p)\gamma^\nu u_+(k) + \overline{u}_-(k)\gamma^\mu u_-(p)\overline{u}_-(p)\gamma^\nu u_-(k)\right]\\
=&\frac{1}{N^2}\left(\mathrm{Tr}\left[\gamma^\mu\not p\gamma^\nu\gamma_+\not k\right] + \mathrm{Tr}\left[\gamma^\mu\not p\gamma^\nu\gamma_-\not k\right]\right)\\
=&-g^{\mu\nu} + \frac{p^\mu k^\nu + p^\nu k^\mu}{k\cdot p}
\end{aligned} \tag{96}$$

as we wanted to shown. As this polarization vector will always appear contracted with a Dirac γ_μ we can use Eq. (90) to write

$$\not\epsilon(k,\lambda) = \frac{1}{N}\left[2u_\lambda(p)\overline{u}_\lambda(k) + 2u_{-\lambda}(k)\overline{u}_{-\lambda}(p)\right],\quad \not\epsilon^*(k,\lambda) = \frac{1}{N}\left[2u_\lambda(k)\overline{u}_\lambda(p) + 2u_{-\lambda}(p)\overline{u}_{-\lambda}(k)\right] \tag{97}$$

D. $e^- + e^+ \to \gamma + \gamma$ with helicity amplitudes

We are now going to use this spinor product technique to redo the calculations for the amplitudes for $e^- + e^+ \to \gamma + \gamma$. As before the electron and positron have the same chiralities. So we start, as before from Eq. (83), but now we are going to use Kleiss representation of the photon polarization vector, given in Eqs. (94) and (97). Here the main point is how to choose the four-vector p. We have total freedom, except that p and k cannot be co-linear. From the experience with Gastmans polarization vector we see that we get the simplest expressions if we use in $\mathcal{M}_1$ $p = p_2$ for ϵ_2 and $p = p_1$ for ϵ_1. This is because we can use the Dirac equation to simplify some terms. Also if we look at $\mathcal{M}_2$ in Eq. (83) we see that we should make the opposite choice there, that is $p = p_1$ for ϵ_2 and $p = p_2$ for ϵ_1. With this choice we get easily that the non-vanishing amplitudes are,

$$
\begin{aligned}
\mathcal{M}(++;+-) =& -4e^2 \frac{s(p_1,k_1)s(p_2,k_2)s^*(k_1,p_2)s^*(p_1,k_1)}{t} \frac{1}{N(p_1,k_1)^2} \\
=& -2e^2 \frac{s(p_2,k_2)s^*(k_1,p_2)}{t} \\
\mathcal{M}(++;-+) =& 4e^2 \frac{s(p_1,k_2)s(p_2,k_1)s^*(k_2,p_2)s^*(p_1,k_2)}{u} \frac{1}{N(p_1,k_2)^2} \\
=& 2e^2 \frac{s(p_2,k_1)s^*(k_2,p_2)}{u} \\
\mathcal{M}(--;-+) =& -4e^2 \frac{s(p_1,k_1)s(p_2,k_1)s^*(k_2,p_2)s^*(p_1,k_1)}{t} \frac{1}{N(p_1,k_1)^2} \\
=& -2e^2 \frac{s(p_2,k_1)s^*(k_2,p_2)}{t} \\
\mathcal{M}(--;+-) =& 4e^2 \frac{s(p_1,k_2)s(p_2,k_2)s^*(k_1,p_2)s^*(p_1,k_2)}{u} \frac{1}{N(p_1,k_2)^2} \\
=& 2e^2 \frac{s(p_2,k_2)s^*(k_1,p_2)}{u}
\end{aligned}
\tag{98}
$$

From these expressions we get

$$
\frac{1}{4} \sum_{\sigma_1,\sigma_2,\lambda_1,\lambda_2} |\mathcal{M}(\sigma_1,\sigma_2;\lambda_1,\lambda_2)|^2 = 2e^4 \frac{u}{t} + 8e^4 \frac{u}{t} = 2e^4 \frac{u^2+t^2}{ut}
\tag{99}
$$

in agreement with Eq. (77). There is a subtle point here. What would have happened if we had made another choice for p? To illustrate this we take the same convention for both $\mathcal{M}_2$ and $\mathcal{M}_1$, that is, $p = p_2$ for ϵ_2 and $p = p_1$ for ϵ_1. Then we can show that we obtain,

$$
\begin{aligned}
\mathcal{M}(++;--) =& 0 \\
\mathcal{M}(++;-+) =& -4e^2 \ \frac{s(p_2,k_2)s(p_2,k_1)s^*(k_2,p_1)s^*(p_1,k_2)}{u} \ \frac{1}{N(p_1,k_1)^2} \\
\mathcal{M}(++;+-) =& 4e^2 \ \frac{s(p_1,k_2)s(p_2,p_1)s^*(p_1,k_1)s^*(p_1,p_2)}{u} \ \frac{1}{N(p_1,k_1)^2} \\
& -4e^2 \ \frac{s(p_1,k_1)s(p_2,k_2)s^*(k_1,p_2)s^*(p_1,k_1)}{t} \ \frac{1}{N(p_1,k_1)^2} \\
\mathcal{M}(++;++) =& 4e^2 \ \frac{s(p_1,p_2)s(p_2,p_1)s^*(p_1,k_1)s^*(p_1,k_2)}{u} \ \frac{1}{N(p_1,k_1)^2} \\
& +4e^2 \ \frac{s(p_2,k_2)s(p_2,p_1)s^*(k_2,k_1)s^*(p_1,k_2)}{u} \ \frac{1}{N(p_1,k_1)^2} \\
\mathcal{M}(--;--) =& \mathcal{M}(++;++)^* \\
\mathcal{M}(--;-+) =& \mathcal{M}(++;+-)^* \\
\mathcal{M}(--;+-) =& \mathcal{M}(++;-+)^*
\end{aligned}
\tag{100}
$$

$$\mathcal{M}(--;++) = 0$$

These expressions seem different form those in Eq. (98). However choosing the CM kinematics and using Eq. (33a) we can show that they give the same result. For instance, in Eq. (98) the amplitude $\mathcal{M}(++;++)$ vanishes, while in Eq. (100) it seems that i does not. Choosing the kinematics,

$$\begin{aligned} p_1 &= (\sqrt{s}/2, 0, 0, \sqrt{s}/2) \\ p_2 &= (\sqrt{s}/2, 0, 0, -\sqrt{s}/2) \\ k_1 &= (\sqrt{s}/2, 0, \sqrt{s}/2 \sin\theta, \sqrt{s}/2 \cos\theta) \\ k_2 &= (\sqrt{s}/2, 0, -\sqrt{s}/2 \sin\theta, -\sqrt{s}/2 \cos\theta) \end{aligned} \tag{101}$$

we get from Eq. (89)

$$s(p_1,p_2) = i\,2\frac{\sqrt{s}}{2} \qquad s(p_1,k_1) = i\,\frac{\sqrt{s}}{2} - i\,\frac{\sqrt{s}}{2}e^{-i\theta}$$

$$s(p_1,k_2) = i\,\frac{\sqrt{s}}{2} + i\,\frac{\sqrt{s}}{2}e^{-i\theta} \qquad s(p_2,k_1) = -i\,\frac{\sqrt{s}}{2} - i\,\frac{\sqrt{s}}{2}e^{-i\theta}$$

$$s(p_2,k_2) = -i\,\frac{\sqrt{s}}{2} + i\,\frac{\sqrt{s}}{2}e^{-i\theta} \qquad s(k_2,k_1) = -i\,2\,\frac{\sqrt{s}}{2}e^{-i\theta}$$

Using these explicit expressions we can show that

$$\begin{aligned} s(p_1,p_2)s^*(p_1,k_1) \ &+\ s(p_2,k_2)s^*(k_2,k_1) = \\ &= \left(\frac{\sqrt{s}}{2}\right)^2 \left[2i\left(-i + ie^{i\theta}\right) + \left(-i + ie^{-i\theta}\right)\left(2\,ie^{i\theta}\right)\right] \\ &= \left(\frac{\sqrt{s}}{2}\right)^2 \left(2 - 2e^{i\theta} + 2e^{i\theta} - 2\right) = 0 \end{aligned} \tag{102}$$

which is a necessary condition to verify that $\mathcal{M}(++;++)$ in Eq. (100) vanishes. Ina similar way we could verify the other cases. In summary the choice of the reference four-vector does not affect the final result.

E. Conclusion

Let us summarize the results that we just obtained.

1. In a given process, the number of traces to be calculated grows like n^2, where n is the number of diagrams. This makes it difficult to use for large number of diagrams.
2. The helicity amplitudes allow us to separate the sum of polarizations in independent combinations.
3. In the massless limit many of the helicity combinations vanish simplifying the calculations. Also in this limit helicity equals chirality for fermions and we can exploit the properties of the chirality projectors.
4. The spinor product technique allows us to calculate a complex number for each Feynman diagram.
5. Therefore the complexity of the problem only grows linearly with the number of diagrams n. Also it is very convenient to implement in numerical evaluations, we just have to add complex numbers.

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