Representation-independent manipulations with Dirac matrices and spinors

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Abstract

Dirac matrices, also known as gamma matrices, are defined only up to a similarity transformation. Usually, some explicit representation of these matrices is assumed in order to deal with them. In this article, we show how it is possible to proceed without any explicit form of these matrices. Various important identities involving Dirac matrices and spinors have been derived without assuming any representation at any stage.

1 Introduction

In order to obtain a relativistically covariant equation for the quantum mechanical wave function, Dirac introduced a Hamiltonian that is linear in the momentum operator. In modern notation, it can be written as

$$H = \gamma^0 \Big(\boldsymbol{\gamma} \cdot \boldsymbol{p}_{\rm op} + m \Big) , \tag{1.1}$$

where m is the mass of the particle and $\boldsymbol{p}_{\rm op}$ the momentum operator. We will throughout use natural units with $c = \hbar = 1$ so that γ^0 and $\boldsymbol{\gamma}$ are dimensionless. Because of their anticommutation properties that we mention in § 2, they have to be matrices. The four matrices are written together as

$$\gamma^\mu \equiv \{ \gamma^0, \gamma^i \} , \tag{1.2}$$

where we have put a Lorentz index in the left hand side.* We will also define the corresponding matrices with lower indices in the usual way:

$$\gamma_\mu = g_{\mu\nu} \gamma^\nu , \tag{1.3}$$

*This does not mean that the matrices transform as vectors. They are, in fact, constant matrices which are frame-independent. The Lorentz index only implies that the four quantities obtained by sandwiching these matrices between spinors transform as components of a contravariant vector.

where $g_{\mu\nu}$ is the metric tensor. Our convention for it has been stated in Eq. (A.1).

Some properties of the Dirac matrices follow directly from their definition in Eq. (1.1), as shown in § 2. However, these properties do not specify the elements of the matrices uniquely. They only define the matrices up to a similarity transformation. Since spinors are plane-wave solutions of the equation

$$i\frac{\partial\psi}{\partial t} = H\psi\,, \tag{1.4}$$

and H contains the Dirac matrices which are not uniquely defined, the solutions also share this non-uniqueness.

In physics, whenever there is an arbitrariness in the definition of some quantity, it is considered best to deal with combinations of those quantities which do not suffer from the arbitrariness. For example, components of a vector depend on the choice of the axes of co-ordinates. Physically meaningful relations can either involve things like scalar products of vectors which do not depend on the choice of axes, or are in the form of equality of two quantities (say, two vectors) both of which transform the same way under a rotation of the axes, so that their equality is not affected. Needless to say, it is best if we can follow the same principles while dealing with Dirac matrices and spinors. However, in most texts dealing with them, this approach is not taken [1]. Most frequently, one chooses an explicit representation of the Dirac matrices and spinors, and works with it.

Apart from the fact that an explicit representation is aesthetically less satisfying, it must also be said that dealing with them can also lead to pitfalls. One might use some relation which holds in some specific representation but not in general, and obtain a wrong conclusion.

In this article, we show how, without using any explicit representation of the Dirac matrices or spinors, one can obtain useful relations involving them. The article is organized as follows. In § 2, we define the basic properties of Dirac matrices and spinors and mention the extent of arbitrariness in the definitions. In § 3, we recall some well-known associated matrices which are useful in dealing with Dirac matrices. In § 4, we derive some identities involving the Dirac matrices and associated matrices in a completely representation-independent way. In § 5, we show how spinors can be defined in a representation-independent fashion and identify their combinations on which normalization conditions can be imposed. We derive some important relations involving spinors in § 6, and involing spinor biliears in § 7. Concluding remarks appear in § 8.

2 Basic properties of Dirac matrices and spinors

Some properties of the Dirac matrices are immediately derived from Eq. (1.1). First, the relativistic Hamiltonian of a free particle is given by

$$H^2 = \boldsymbol{p}^2 + m^2\,, \tag{2.1}$$

and Eq. (1.1), when squared, must yield this relation. Assuming γ_0 and $\boldsymbol{\gamma}$ commute with the momentum operator, this gives a set of relations which can be summarized in the form

$$\Big[\gamma_\mu, \gamma_\nu\Big]_+ \quad = \quad 2g_{\mu\nu}\mathbb{1}\,, \tag{2.2}$$

where $g_{\mu\nu}$ is the metric defined in Eq. (A.1), and $\mathbb{1}$ is the unit matrix which will not be always explicitly written in the subsequent formulas. This relation requires that the Dirac matrices are at least 4×4 matrices, and we take them to be 4×4.

Hermiticity of the Hamiltonian of Eq. (1.1) gives some further conditions on the Dirac matrices, namely that γ_0 must be hermitian, and so should be the combinations $\gamma_0\gamma_i$. Both these relations can be summarized by writing

$$\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0\,. \tag{2.3}$$

Eqs. (2.2) and (2.3) are the basic properties which define the Dirac matrices. With these defining relations, the arbitrariness can be easily seen through the following theorems.

Theorem 1 *For any choice of the matrices γ_μ satisfying Eqs. (2.2) and (2.3), if we take another set defined by*

$$\tilde{\gamma}_\mu = U\gamma_\mu U^\dagger \tag{2.4}$$

for some unitary matrix U, then these new matrices satisfy the same anticommutation and hermiticity properties as the matrices γ_μ.

The proof of this theorem is straight forward and trivial. The converse is also true:

Theorem 2 *If two sets of matrices γ_μ and $\tilde{\gamma}_\mu$ both satisfy Eqs. (2.2) and (2.3), they are related through Eq. (2.4) for some unitary matrix U.*

The proof is non-trivial [2] and we will not give it here. The two theorems show that the Dirac matrices are defined only up to a similarity transformation with a unitary matrix.

To obtain the defining equation for the spinors, we multiply both sides of Eq. (1.4) by γ_0 and put $\boldsymbol{p}_{\rm op} = -i\boldsymbol{\nabla}$ into the Hamiltonian of Eq. (1.1). This gives the Dirac equation:

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0\,. \tag{2.5}$$

There are two types of plane-wave solutions:

$$\psi \sim \begin{cases} u(\!(\boldsymbol{p})\!)e^{-ip\cdot x}\,, \\ v(\!(\boldsymbol{p})\!)e^{+ip\cdot x}\,. \end{cases} \tag{2.6}$$

Here and later, we indicate functional dependence in double parentheses so that it does not get confused with multiplicative factors in parentheses. The 4-vector p^μ is given by

$$p^\mu \equiv \{E_{\boldsymbol{p}}, \boldsymbol{p}\}\,, \tag{2.7}$$

where $E_{\boldsymbol{p}}$ is the positive energy eigenvalue:

$$E_{\boldsymbol{p}} = +\sqrt{\boldsymbol{p}^2 + m^2}\,. \tag{2.8}$$

Putting Eq. (2.6) into Eq. (2.5), we obtain the equations that define the u and v-spinors:

$$(\gamma_\mu p^\mu - m)u(\!(\boldsymbol{p})\!) \;=\; 0\,, \tag{2.9}$$

$$(\gamma_\mu p^\mu + m)v(\!(\boldsymbol{p})\!) \;=\; 0\,. \tag{2.10}$$

Obviously, if we change γ^μ to $\tilde{\gamma}_\mu$ through the prescription given in Eq. (2.4) and also change the spinors to

$$\tilde{u}(\!(\boldsymbol{p})\!) = Uu(\!(\boldsymbol{p})\!)\,, \qquad \tilde{v}(\!(\boldsymbol{p})\!) = Uv(\!(\boldsymbol{p})\!)\,, \tag{2.11}$$

Eqs. (2.9) and (2.10) are satisfied by the new matrices and the new spinors. Eq. (2.11) shows that the spinors themselves are representation-dependent.

3 Some associated matrices

In order to proceed, we recall the definitions of some matrices associated with the Dirac matrices. These definitions can be obtained in any textbook dealing with Dirac particles or fields, but are compiled here for the sake of completeness.

The sigma-matrices are defined as

$$\sigma_{\mu\nu} = \frac{i}{2}\Big[\gamma_\mu, \gamma_\nu\Big]\,. \tag{3.1}$$

The matrices $\frac{1}{2}\sigma_{\mu\nu}$ constitute a representation of the Lorentz group. The subgroup of rotation group has the generators $\frac{1}{2}\sigma_{ij}$, with both spatial indices. We define the spin matrices:

$$\Sigma^i = \frac{1}{2}\varepsilon^{ijk}\sigma_{jk}\,, \tag{3.2}$$

so that $\frac{1}{2}\Sigma^i$ represent the spin components.

The next important matrix is defined from the observation that the matrices $-\gamma_\mu^\top$ satisfy the same anticommutation and hermiticity properties as γ_μ. By Theorem 2, there must then exist a unitary matrix C such that

$$C^{-1}\gamma_\mu C = -\gamma_\mu^\top\,. \tag{3.3}$$

Note that the two definitions imply the relation

$$C^{-1}\sigma_{\mu\nu}C = -\sigma_{\mu\nu}^{\top}\,. \tag{3.4}$$

Another important matrix is γ_5, defined as

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3\,, \tag{3.5}$$

or equivalently as

$$\gamma_5 = \frac{i}{4!}\,\varepsilon_{\mu\nu\lambda\rho}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\,, \tag{3.6}$$

where ε stands for the completely antisymmetric rank-4 tensor.

From Eq. (2.2), it is easily seen that

$$\Big(\gamma_5\Big)^2 = \mathbb{1}\,. \tag{3.7}$$

It is also easy to see that γ_5 anticommutes with all γ_μ's and commutes with all $\sigma_{\mu\nu}$'s:

$$\Big[\gamma_\mu, \gamma_5\Big]_+ = 0\,, \tag{3.8}$$

$$\Big[\sigma_{\mu\nu}, \gamma_5\Big] = 0\,. \tag{3.9}$$

To illustrate why these matrices are useful, we prove one important result that can be found in all textbooks. Using Eq. (3.7), we can write

$$\mathrm{Tr}\,\Big(\gamma_\mu\Big) = \mathrm{Tr}\,\Big(\gamma_\mu\gamma_5\gamma_5\Big)\,. \tag{3.10}$$

Then, using the cyclic property of traces and Eq. (3.8), we obtain

$$\mathrm{Tr}\,\Big(\gamma_\mu\Big) = \mathrm{Tr}\,\Big(\gamma_5\gamma_\mu\gamma_5\Big) = -\,\mathrm{Tr}\,\Big(\gamma_\mu\gamma_5\gamma_5\Big)\,. \tag{3.11}$$

Comparing the two equations, we obtain

$$\mathrm{Tr}\,\Big(\gamma_\mu\Big) = 0\,, \tag{3.12}$$

a property that we will need very much in what follows. The usefulness of the matrices $\sigma_{\mu\nu}$ and C will be obvious as we proceed.

4 Identities involving Dirac matrices

First, there are the contraction formulas, e.g.,

$$\gamma^\mu\gamma_\mu = 4\,, \tag{4.1}$$

$$\gamma^\mu\gamma_\nu\gamma_\mu = -2\gamma_\nu\,, \tag{4.2}$$

$$\gamma^\mu\gamma_\nu\gamma_\lambda\gamma_\mu = 4g_{\nu\lambda}\,, \tag{4.3}$$

$$\gamma^\mu\gamma_\nu\gamma_\lambda\gamma_\rho\gamma_\mu = -2\gamma_\rho\gamma_\lambda\gamma_\nu\,, \tag{4.4}$$

and so on for longer strings of Dirac matrices, which can be proved easily by using the anticommutation relation of Eq. (2.2). There are also similar formulas involving contractions of the sigma matrices, like

$$\sigma^{\mu\nu}\sigma_{\mu\nu} = 12\,, \tag{4.5}$$

$$\sigma^{\mu\nu}\sigma^{\lambda\rho}\sigma_{\mu\nu} = -4\sigma^{\lambda\rho}\,, \tag{4.6}$$

and some other involving both gamma matrices and sigma matrices:

$$\sigma^{\mu\nu}\gamma^{\lambda}\sigma_{\mu\nu} = 0\,, \tag{4.7}$$

$$\gamma^{\lambda}\sigma^{\mu\nu}\gamma_{\lambda} = 0\,. \tag{4.8}$$

All of these can be easily proved by using the definition of the sigma matrices and the contraction formulas for the gamma matrices. There are also the trace formulas for strings of Dirac matrices. We do not give them here because they are usually proved in a representation-independent manner in textbooks.

There are many other identities involving the Dirac matrices which are derived from the fact that the 16 matrices

$$\mathbb{1}, \gamma_\mu, \sigma_{\mu\nu}(\text{for } \mu < \nu), \gamma_\mu\gamma_5, \gamma_5 \tag{4.9}$$

constitute a complete set of 4×4 matrices, i.e., any 4×4 matrix can be expressed as a linear superposition of these 16 matrices. For example, we can write

$$\gamma_\mu\gamma_\nu = g_{\mu\nu}\mathbb{1} - i\sigma_{\mu\nu}\,, \tag{4.10}$$

which follows trivially from Eqs. (2.2) and (3.1). Eqs. (4.5) and (4.6) are also examples of this general theme. To see more examples of this kind, let us consider the combination $\varepsilon^{\mu\nu\lambda\rho}\sigma_{\mu\nu}\gamma_5$. We can use the definition of γ_5 from Eq. (3.6), and use the product of two Levi-Civita symbols given in Eq. (A.3). This gives

$$\varepsilon^{\mu\nu\lambda\rho}\sigma_{\mu\nu}\gamma_5 = -\frac{i}{4!}\,\sigma_{\mu\nu}\Big(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho + (-1)^P(\text{permutations})\Big)\,, \tag{4.11}$$

where the factor $(-1)^P$ is $+1$ if the permutation is even, and -1 if the permutation is odd. There are 24 possible permutations. Each of them can be simplified by using one or other of the contraction formulas given above, and the result is

$$\varepsilon^{\mu\nu\lambda\rho}\sigma_{\mu\nu}\gamma_5 = 2i\sigma^{\lambda\rho}\,, \tag{4.12}$$

or equivalently

$$\sigma^{\lambda\rho}\gamma_5 = -\frac{i}{2}\varepsilon^{\mu\nu\lambda\rho}\sigma_{\mu\nu}\,. \tag{4.13}$$

Another important identity can be derived by starting with the combination $\varepsilon_{\mu\nu\lambda\rho}\gamma^\rho\gamma_5$, and using Eqs. (3.6) and (A.3), as done for deducing Eq. (4.12). The final result can be expressed in the form

$$\gamma_\mu\gamma_\nu\gamma_\lambda = g_{\mu\nu}\gamma_\lambda + g_{\nu\lambda}\gamma_\mu - g_{\lambda\mu}\gamma_\nu - i\varepsilon_{\mu\nu\lambda\rho}\gamma^\rho\gamma_5\,. \tag{4.14}$$

With this very important identity, any string of three or more gamma matrices can be reduced to strings of smaller number of gamma matrices.

An important identity can be derived by multiplying Eq. (4.10) by γ_5, and using Eq. (4.13). This gives

$$\gamma_\mu \gamma_\nu \gamma_5 = g_{\mu\nu} \gamma_5 - \frac{1}{2} \varepsilon_{\mu\nu\lambda\rho} \sigma^{\lambda\rho} . \tag{4.15}$$

In particular, if the index μ is taken to be in the time direction and the index ν to be a spatial index, we obtain

$$\gamma_0 \gamma_i \gamma_5 = -\frac{1}{2} \varepsilon_{0ijk} \sigma^{jk} . \tag{4.16}$$

Taking the convention for the completely antisymmetric 3-dimensional tensor in such a way that $\varepsilon_{0ijk} = \varepsilon_{ijk}$, we can rewrite this equation by comparing the right hand side with the definition of the spin matrices in Eq. (3.2):

$$\Sigma_i = -\gamma_0 \gamma_i \gamma_5 . \tag{4.17}$$

With this form, it is easy to show that

$$\Big[\Sigma_i, \Sigma_j \Big]_+ = 2\delta_{ij} , \tag{4.18}$$

by using anticommutation properties of the gamma matrices.

5 Spinors

5.1 Eigenvectors of γ_0

Consider the matrix γ_0. It is a 4×4 matrix, so it has four eigenvalues and eigenvectors. It is hermitian, so the eigenvalues are real. In fact, from Eq. (2.2) we know that its square is the unit matrix, so that its eigenvalues can only be ± 1. Since γ_0 is traceless, as we have proved in § 3, there must be two eigenvectors with eigenvalue $+1$ and two with -1:

$$\gamma_0 \xi_s = \xi_s , \qquad \gamma_0 \chi_s = -\chi_s . \tag{5.1}$$

The subscripts on ξ and χ distinguishes two different eigenvectors of each kind. Of course this guarantees that

$$\xi_s^\dagger \chi_{s'} = 0 , \tag{5.2}$$

since they belong to different eigenvalues. But since the two ξ's are degenerate and so are the two χ's, there is some arbitrariness in defining them even for a given form of the matrix γ_0. In order to remove the arbitrariness, let us note that the matrices σ_{ij}, with both space indices, commute with γ_0. In particular, say,

$$\Big[\sigma_{12}, \gamma_0 \Big] = 0 . \tag{5.3}$$

Thus, we can choose the eigenstates of γ_0 such that they are simultaneously eigenstates of σ_{12}. From Eqs. (2.2) and (3.1), it is easy to see that

$$\Big(\sigma_{12}\Big)^2 = 1\,, \tag{5.4}$$

so that the eigenvalues of σ_{12} are ± 1 as well. Therefore, let us choose the eigenvectors of γ_0 such that

$$\sigma_{12}\xi_s = s\xi_s\,, \qquad \sigma_{12}\chi_s = s\chi_s\,, \tag{5.5}$$

with $s = \pm$. Once we fix the spinors in this manner, the four eigenvectors are mutually orthogonal, i.e., in addition to Eq. (5.2), the following relations also hold:

$$\xi_s^\dagger \xi_{s'} = \delta_{ss'}\,, \qquad \chi_s^\dagger \chi_{s'} = \delta_{ss'}\,. \tag{5.6}$$

One might wonder, why are we spending so much time in discussing the eigenvectors of γ_0? To see the reason, let us consider Eq. (2.9) for vanishing 3-momentum. In this case $E_{\boldsymbol{p}} = m$, so that Eq. (2.9) simply reduces to

$$(\gamma_0 - 1)u(\!(\mathbf{0})\!) = 0\,, \tag{5.7}$$

whereas Eq. (2.10) reduces to

$$(\gamma_0 + 1)v(\!(\mathbf{0})\!) = 0\,. \tag{5.8}$$

This shows that, at zero momentum, the u-spinors and the v-spinors are simply eigenstates of γ_0 with eigenvalues $+1$ and -1. Thus we can define the zero-momentum spinors as

$$u_s(\!(\mathbf{0})\!) \propto \xi_s\,, \qquad v_s(\!(\mathbf{0})\!) \propto \chi_{-s}\,, \tag{5.9}$$

apart from possible normalizing factors which will be specified later.

5.2 Spinors and their normalization

We now want to find the spinors for any value of $\boldsymbol{p}$. We know that these will have to satisfy Eqs. (2.9) and (2.10), and, for $\boldsymbol{p} \to 0$, should reduce to the zero-momentum solutions shown above. With these observations, we can try the following solutions:

$$\begin{aligned} u_s(\!(\boldsymbol{p})\!) &= N_{\boldsymbol{p}}(\gamma_\mu p^\mu + m)\xi_s\,, && (5.10)\\ v_s(\!(\boldsymbol{p})\!) &= N_{\boldsymbol{p}}(-\gamma_\mu p^\mu + m)\chi_{-s}\,, && (5.11) \end{aligned}$$

where $N_{\boldsymbol{p}}$ is a normalizing factor. One might wonder why we have put χ_{-s} and not χ_s in the definition of v_s. It is nothing more than a convention. It turns out that when we do quantum field theory, this convention leads to an easy interpretation of the subscript s. This issue will not be discussed here.

It is easy to see that our choices for the spinors satisfy Eqs. (2.9) and (2.10) since

$$(\gamma_\mu p^\mu - m)(\gamma_\nu p^\nu + m) = p^2 - m^2 = 0\,. \tag{5.12}$$

It is also easy to see that in the zero-momentum limit, these solutions reduce to the eigenvalues of γ_0, apart from a normalizing factor. For example, putting $\boldsymbol{p} = 0$ and $E_{\boldsymbol{p}} = m$ into Eq. (5.10), we obtain

$$u_s(\!(\mathbf{0})\!) = N_{\mathbf{0}} m(\gamma_0 + 1)\xi_s = 2mN_{\mathbf{0}}\xi_s\,. \tag{5.13}$$

In order to determine a convenient normalization of the spinors, let us rewrite Eq. (5.10) more explicitly:

$$u_s(\!(\boldsymbol{p})\!) = N_{\boldsymbol{p}}(\gamma_0 E_{\boldsymbol{p}} - \gamma_i p_i + m)\xi_s = N_{\boldsymbol{p}}(E_{\boldsymbol{p}} + m - \gamma_i p_i)\xi_s\,, \tag{5.14}$$

using Eq. (5.1) in the last step. Similarly, we obtain

$$v_s(\!(\boldsymbol{p})\!) = N_{\boldsymbol{p}}(E_{\boldsymbol{p}} + m + \gamma_i p_i)\chi_{-s}\,. \tag{5.15}$$

Recalling that γ_i's are anti-hermitian matrices, we then obtain

$$u_s^\dagger(\!(\boldsymbol{p})\!) \;=\; N_{\boldsymbol{p}}^* \xi_s^\dagger (E_{\boldsymbol{p}} + m + \gamma_i p_i)\,, \tag{5.16}$$

$$v_s^\dagger(\!(\boldsymbol{p})\!) \;=\; N_{\boldsymbol{p}}^* \chi_{-s}^\dagger (E_{\boldsymbol{p}} + m - \gamma_i p_i)\,. \tag{5.17}$$

Thus,

$$u_s^\dagger(\!(\boldsymbol{p})\!) u_{s'}(\!(\boldsymbol{p})\!) = \Big| N_{\boldsymbol{p}} \Big|^2 \xi_s^\dagger \Big((E_{\boldsymbol{p}} + m)^2 - \gamma_i\gamma_j p_i p_j \Big) \xi_{s'}\,. \tag{5.18}$$

Since $p_i p_j = p_j p_i$, we can write

$$\gamma_i\gamma_j p_i p_j = \frac{1}{2}\Big[\gamma_i, \gamma_j\Big]_+ p_i p_j = -\delta_{ij} p_i p_j = -\boldsymbol{p}^2\,. \tag{5.19}$$

Using Eq. (2.8) then, we obtain

$$u_s^\dagger(\!(\boldsymbol{p})\!) u_{s'}(\!(\boldsymbol{p})\!) = 2E_{\boldsymbol{p}}(E_{\boldsymbol{p}} + m)\Big| N_{\boldsymbol{p}} \Big|^2 \xi_s^\dagger \xi_{s'}\,. \tag{5.20}$$

Choosing

$$N_{\boldsymbol{p}} = \frac{1}{\sqrt{E_{\boldsymbol{p}} + m}} \tag{5.21}$$

and using Eq. (5.6), we obtain the normalization conditions in the form

$$u_s^\dagger(\!(\boldsymbol{p})\!) u_{s'}(\!(\boldsymbol{p})\!) = 2E_{\boldsymbol{p}}\delta_{ss'}\,. \tag{5.22}$$

Through a similar procedure, one can obtain a similar condition on the v-spinors:

$$v_s^\dagger(\!(\boldsymbol{p})\!) v_{s'}(\!(\boldsymbol{p})\!) = 2E_{\boldsymbol{p}}\delta_{ss'}\,. \tag{5.23}$$

We now need a relation that expresses the orthogonality between a u-spinor and a v-spinor. In obtaining Eqs. (5.22) and (5.23), the linear terms in $\gamma_i p_i$, appearing in Eqs. (5.14) and (5.16) or in the similar set of equations involving the v-spinors, cancel. The same will not work in combinations of the form $u_s^\dagger(\!(\boldsymbol{p})\!) v_{s'}(\!(\boldsymbol{p})\!)$ because the $\gamma_i p_i$ terms have the same sign in both factors. However we notice that if we reverse the 3-momentum in one of the factors, these problematic terms cancel. We can then follow the same steps, more or less, and use Eq. (5.2) to obtain

$$u_s^\dagger(\!(-\boldsymbol{p})\!) v_{s'}(\!(\boldsymbol{p})\!) = v_s^\dagger(\!(-\boldsymbol{p})\!) u_{s'}(\!(\boldsymbol{p})\!) = 0\,. \tag{5.24}$$

Eq. (5.24) can be expressed in an alternative form by using bars rather than daggers, where $\bar{w} = w^\dagger \gamma_0$ for any spinor. Multiplying Eq. (2.9) from the left by $\bar{v}_{s'}(\!(\boldsymbol{p})\!)$ we obtain

$$\bar{v}_{s'}(\!(\boldsymbol{p})\!)(\gamma_\mu p^\mu - m) u_s(\!(\boldsymbol{p})\!) = 0\,. \tag{5.25}$$

Multiplying the hermitian conjugate of the equation for $v_{s'}(\!(\boldsymbol{p})\!)$ by $u_s(\!(\boldsymbol{p})\!)$ from the right, we get

$$\bar{v}_{s'}(\!(\boldsymbol{p})\!)(\gamma_\mu p^\mu + m) u_s(\!(\boldsymbol{p})\!) = 0\,. \tag{5.26}$$

Subtracting one of these equations from another, we find that

$$\bar{v}_{s'}(\!(\boldsymbol{p})\!) u_s(\!(\boldsymbol{p})\!) \quad = \quad 0 \tag{5.27}$$

provided $m \neq 0$. Similarly, one can also obtain the equation

$$\bar{u}_{s'}(\!(\boldsymbol{p})\!) v_s(\!(\boldsymbol{p})\!) \quad = \quad 0\,. \tag{5.28}$$

We will also show, in § 7.1, that Eqs. (5.22) and (5.23) are equivalent to the relations

$$\bar{u}_s(\!(\boldsymbol{p})\!) u_{s'}(\!(\boldsymbol{p})\!) \quad = \quad 2m\delta_{ss'}\,, \tag{5.29}$$

$$\bar{v}_s(\!(\boldsymbol{p})\!) v_{s'}(\!(\boldsymbol{p})\!) \quad = \quad -2m\delta_{ss'}\,. \tag{5.30}$$

Unless $m = 0$, these can be taken as the normalization conditions on the spinors.

5.3 Spin sums

The spinors also satisfy some completeness relations, which can be proved without invoking their explicit forms [3]. Consider the sum

$$A_u(\!(\boldsymbol{p})\!) \equiv \sum_s u_s(\!(\boldsymbol{p})\!) \bar{u}_s(\!(\boldsymbol{p})\!)\,. \tag{5.31}$$

Note that, using Eq. (5.29), we get

$$\begin{aligned} A_u(\!(\boldsymbol{p})\!) u_{s'}(\!(\boldsymbol{p})\!) &= \sum_s u_s(\!(\boldsymbol{p})\!) \Big[\bar{u}_s(\!(\boldsymbol{p})\!) u_{s'}(\!(\boldsymbol{p})\!) \Big] \\ &= 2m u_{s'}(\!(\boldsymbol{p})\!)\,. \end{aligned} \tag{5.32}$$

And, using Eq. (5.28), we get

$$A_u(\!(\boldsymbol{p})\!) v_{s'}(\!(\boldsymbol{p})\!) = 0\,. \tag{5.33}$$

Recalling Eqs. (2.9) and (2.10), it is obvious that on the spinors $u_s(\!(\boldsymbol{p})\!)$ and $v_s(\!(\boldsymbol{p})\!)$, the operation of $A_u(\!(\boldsymbol{p})\!)$ produces the same result as the operation of $\gamma_\mu p^\mu + m$. Since any 4-component column vector can be written as a linear superposition of the basis spinors $u_s(\!(\boldsymbol{p})\!)$ and $v_s(\!(\boldsymbol{p})\!)$, it means that the action of $A_u(\!(\boldsymbol{p})\!)$ and of $\gamma_\mu p^\mu + m$ produces identical results on any 4-component column vector. The two matrices must therefore be the same:

$$\sum_s u_s(\!(\boldsymbol{p})\!) \bar{u}_s(\!(\boldsymbol{p})\!) = \gamma_\mu p^\mu + m\,. \tag{5.34}$$

Similar reasoning gives

$$\sum_s v_s(\!(\boldsymbol{p})\!) \bar{v}_s(\!(\boldsymbol{p})\!) = \gamma_\mu p^\mu - m\,. \tag{5.35}$$

6 Relations involving spinors

We now show some non-trivial properties of the spinors. In all textbooks, they are deduced in the Dirac-Pauli representation of the γ-matrices. Using Eq. (2.4), one can show that if they hold in one representation, they must hold in other representations as well. Here we derive them without using any representation at any stage of the proofs.

6.1 What γ_0 does on spinors

We first consider the effect of γ_0 acting on the spinors. From Eq. (5.14), we find

$$\begin{aligned} \gamma_0 u_s(\!(\boldsymbol{p})\!) &= N_{\boldsymbol{p}} \gamma_0 (E_{\boldsymbol{p}} + m - \gamma_i p_i) \xi_s \\ &= N_{\boldsymbol{p}} (E_{\boldsymbol{p}} + m + \gamma_i p_i) \gamma_0 \xi_s = N_{\boldsymbol{p}} (E_{\boldsymbol{p}} + m + \gamma_i p_i) \xi_s\,, \end{aligned} \tag{6.1}$$

using the anticommutation relations and Eq. (5.1). This shows that

$$\gamma_0 u_s(\!(\boldsymbol{p})\!) = u_s(\!(-\boldsymbol{p})\!)\,. \tag{6.2}$$

Following the same procedure, we can obtain the result

$$\gamma_0 v_s(\!(\boldsymbol{p})\!) = -v_s(\!(-\boldsymbol{p})\!)\,. \tag{6.3}$$

Eqs. (6.2) and (6.3) are very important relations for deducing behavior of fermions under the parity transformation. These relations can be used to deduce Eqs. (5.27) and (5.28) from Eq. (5.24), or vice versa.

6.2 What γ_5 does on spinors

Multiplying both sides of Eq. (2.9) by γ_5 from the left and using the anticommutation of γ_5 with all Dirac matrices, we obtain the equation

$$(\gamma_\mu p^\mu + m)\gamma_5 u(\!(\boldsymbol{p})\!) \quad = \quad 0 \,, \tag{6.4}$$

which clearly shows that $\gamma_5 u$ is a v-spinor. Similarly, $\gamma_5 v$ must be a u-spinor. However, this simple argument does not say whether $\gamma_5 u_+$ is v_+, or v_-, or a linear combination of the two.

To settle the issue, we note that

$$\gamma_0\gamma_5\xi_s = -\gamma_5\gamma_0\xi_s = -\gamma_5\xi_s \,, \tag{6.5}$$

since γ_5 anticommutes with γ_0. This equation shows that $\gamma_5\xi_s$ is an eigenvector of γ_0 with eigenvalue -1, i.e., it must be some combination of the χ-eigenvectors defined in Eq. (5.1). Moreover, since γ_5 commutes with σ_{12}, we observe that

$$\sigma_{12}\gamma_5\xi_s = \gamma_5\sigma_{12}\xi_s = s\gamma_5\xi_s \,. \tag{6.6}$$

This means that $\gamma_5\xi_s$ is an eigenstate of σ_{12} with eigenvalue s. Combining this information about the eigenvalues of γ_0 and σ_{12}, we conclude that $\gamma_5\xi_s$ must be equal to χ_s apart from a possible constant phase factor. However, for any choice of ξ_s, it is possible to choose the phase of the χ-eigenvectors such that the condition

$$\gamma_5\xi_s = \chi_s \tag{6.7}$$

is satisfied. Because of Eq. (3.7), this would also imply

$$\gamma_5\chi_s = \xi_s \,. \tag{6.8}$$

The action of γ_5 on the spinors can now be calculated easily. For example, it is easy to see that

$$\begin{aligned} \gamma_5 u_s(\!(\boldsymbol{p})\!) \quad &= \quad N_{\boldsymbol{p}}\gamma_5(\gamma_\mu p^\mu + m)\xi_s = N_{\boldsymbol{p}}(-\gamma_\mu p^\mu + m)\gamma_5\xi_s \\ &= \quad N_{\boldsymbol{p}}(-\gamma_\mu p^\mu + m)\chi_s = v_{-s}(\!(\boldsymbol{p})\!) \,. \end{aligned} \tag{6.9}$$

Through similar manipulations or through the use of Eq. (3.7), we can get

$$\gamma_5 v_s(\!(\boldsymbol{p})\!) = u_{-s}(\!(\boldsymbol{p})\!) \,. \tag{6.10}$$

6.3 Conjugation relations

Let us now deduce another set of relations, which play an important role in deriving charge conjugation properties of fermions. To build up to these relations, let us first consider the object

$$\hat{\xi}_s = \gamma_0 C\xi_s^* \,, \tag{6.11}$$

where the matrix C was defined in Eq. (3.3). To find out about the nature of $\hat{\xi}_s$, we first consider the action of γ_0 on it:

$$\gamma_0 \hat{\xi}_s = \gamma_0 \gamma_0 C \xi_s^* = -\gamma_0 C \gamma_0^\top \xi_s^* \,, \tag{6.12}$$

using Eq. (3.3) again. However, the complex conjugate of Eq. (5.1) implies that

$$\gamma_0^\top \xi_s^* = \xi_s^* \,, \tag{6.13}$$

since

$$\gamma_0^* = \gamma_0^\top \tag{6.14}$$

because of the hermiticity of the matrix γ_0. Putting this in, we obtain

$$\gamma_0 \hat{\xi}_s = -\gamma_0 C \xi_s^* = -\hat{\xi}_s \,, \tag{6.15}$$

showing that $\hat{\xi}_s$ is an eigenvector of γ_0 with eigenvalue -1. Therefore, it must be a combination of the χ_s's.

To determine which combination of the χ_s's occur in $\hat{\xi}_s$, we use Eq. (3.4) and recall that σ_{12} commutes with γ_0 to obtain

$$\sigma_{12} \hat{\xi}_s = \gamma_0 \sigma_{12} C \xi_s^* = -\gamma_0 C \sigma_{12}^\top \xi_s^* \,. \tag{6.16}$$

It can be easily seen from Eqs. (2.3) and (3.1) that σ_{12} is hermitian. So, from Eq. (5.5), we obtain

$$\sigma_{12}^\top \xi_s^* = \Big(\sigma_{12} \xi_s \Big)^* = s \xi_s^* \,, \tag{6.17}$$

which gives

$$\sigma_{12} \hat{\xi}_s = -s \gamma_0 C \xi_s^* = -s \hat{\xi}_s \,. \tag{6.18}$$

This shows that $\hat{\xi}_s$ is also an eigenstate of σ_{12}, with eigenvalue $-s$. Recalling the result we found earlier about its eigenvalue of γ_0, we conclude that $\hat{\xi}_s$ must be proportional to χ_{-s}. Since both γ_0 and C are unitary matrices and ξ_s is normalized to have unit norm, the norm of $\hat{\xi}_s$ is also unity, so the proportionality constant can be a pure phase, of the form $e^{i\theta}$. But notice that the definition of the matrix C in Eq. (3.3) has a phase arbitariness as well. In other words, given a set of matrices γ_μ, the matrix C can be obtained only up to an overall phase from Eq. (3.3). We can utilize this arbitrariness by fixing $\hat{\xi}_s$ to be equal to χ_{-s}, i.e.,

$$\gamma_0 C \xi_s^* = \chi_{-s} \,. \tag{6.19}$$

Similarly one obtains

$$\gamma_0 C \chi_s^* = \xi_{-s} \,. \tag{6.20}$$

To see the implication of these relations between the eigenvectors of γ_0, we take the complex conjugate of Eq. (5.14). Remembering that the matrices γ_i are anti-hermititan so that $\gamma_i^* = -\gamma_i^\top$, we obtain

$$u_s^*(\!(\boldsymbol{p})\!) = N_{\boldsymbol{p}}(E_{\boldsymbol{p}} + m + \gamma_i^\top p_i)\xi_s^* = N_{\boldsymbol{p}}(E_{\boldsymbol{p}} + m - C^{-1}\gamma_i C p_i)\xi_s^* \,, \tag{6.21}$$

using the definition of the matrix C from Eq. (3.3). Multiplying from the left by $\gamma_0 C$, we obtain

$$\gamma_0 C u_s^*(\!(\boldsymbol{p})\!) \;=\; N_{\boldsymbol{p}}\Big[(E_{\boldsymbol{p}} + m)\gamma_0 C\xi_s^* - \gamma_0\gamma_i C p_i \xi_s^*\Big] \,. \tag{6.22}$$

Since γ_0 anticommutes with γ_i, this can be written as

$$\gamma_0 C u_s^*(\!(\boldsymbol{p})\!) \;=\; N_{\boldsymbol{p}}\Big[(E_{\boldsymbol{p}} + m) + \gamma_i p_i\Big]\gamma_0 C\xi_s^* = N_{\boldsymbol{p}}\Big[(E_{\boldsymbol{p}} + m) + \gamma_i p_i\Big]\chi_{-s} \,. \tag{6.23}$$

Using Eq. (5.15), we now obtain

$$\gamma_0 C u_s^*(\!(\boldsymbol{p})\!) = v_s(\!(\boldsymbol{p})\!) \,. \tag{6.24}$$

This is an important relation. Following similar steps, we can also prove the relation

$$\gamma_0 C v_s^*(\!(\boldsymbol{p})\!) = u_s(\!(\boldsymbol{p})\!) \,. \tag{6.25}$$

Because C appears in the conjugation properties of the spinors, we will sometimes refer to it as the conjugation matrix.

6.4 Alternative forms

The results obtained above can be combined to obtain some other relations. For example, multiply both sides of Eq. (6.24) from the left by C^{-1}. Using Eq. (3.3), the result can be written as

$$-\gamma_0^\top u_s^*(\!(\boldsymbol{p})\!) = C^{-1} v_s(\!(\boldsymbol{p})\!) \,. \tag{6.26}$$

But $\gamma_0^\top = \gamma_0^*$, so the left hand side is the complex conjugate of $\gamma_0 u_s(\!(\boldsymbol{p})\!)$. Using Eq. (6.2), we can then write

$$C^{-1} v_s(\!(\boldsymbol{p})\!) = -u_s^*(\!(-\boldsymbol{p})\!) \,. \tag{6.27}$$

Similar manipulations give the complimentary result,

$$C^{-1} u_s(\!(\boldsymbol{p})\!) = v_s^*(\!(-\boldsymbol{p})\!) \,. \tag{6.28}$$

We can also combine this result with the identities of Eqs. (6.9) and (6.10) to obtain

$$\begin{aligned} C^{-1}\gamma_5 v_s(\!(\boldsymbol{p})\!) &= v_{-s}^*(\!(-\boldsymbol{p})\!) \,, \\ C^{-1}\gamma_5 u_s(\!(\boldsymbol{p})\!) &= -u_{-s}^*(\!(-\boldsymbol{p})\!) \,. \end{aligned} \tag{6.29}$$

The matrix $C^{-1}\gamma_5$ plays a crucial role in the time-reversal properties of a fermion field [3].

6.5 Antisymmetry of C

As a bi-product of the discussion about the spinors, we show here an interesting property of the conjugation matrix C. Taking the expression for $v_s(\!(\boldsymbol{p})\!)$ from Eq. (6.24) and putting it into Eq. (6.25), we obtain

$$\gamma_0 C \gamma_0^* C^* u_s(\!(\boldsymbol{p})\!) = u_s(\!(\boldsymbol{p})\!) \,. \tag{6.30}$$

Using Eqs. (6.14) and (3.3), this can be written as

$$-\, CC^* u_s(\!(\boldsymbol{p})\!) = u_s(\!(\boldsymbol{p})\!) \,. \tag{6.31}$$

Thus, both u-spinors are eigenvectors of the matrix $-CC^*$, with eigenvalue $+1$. Similarly, substituting the expression for the u-spinor from Eq. (6.25) into Eq. (6.24), we obtain that both v-spinors are also eigenvectors of the matrix $-CC^*$, with eigenvalue $+1$. Any column vector can be expressed as a linear superposition of the u and v spinors, so any column vector is an eigenvector of the matrix $-CC^*$, with eigenvalue $+1$. This can happen only if $-CC^*$ is the unit matrix, i.e., if

$$C^* = -C^{-1} \,. \tag{6.32}$$

Using the unitarity of the matrix C, this relation can also be written as

$$C^\top = -C \,, \tag{6.33}$$

i.e., C must be an antisymmetric matrix in any representation of the Dirac matrices. It should be noted that this conclusion is obtained irrespective of the choice of the overall phase of C that was utilized to arrive at Eq. (6.19), or equivalently to Eqs. (6.24) and (6.25).

7 Fermion field bilinears

Whenever fermion fields have to be used in Lorentz invariant combinations, we must encounter pairs of them in order that the overall combination conserves angular momentum. For this reason, fermion field bilinears deserve some attention.

7.1 Identities involving bilinears

A vector p_λ can be rewritten as

$$p_\lambda = g_{\lambda\rho} p^\rho = \Big(\gamma_\lambda \gamma_\rho + i\sigma_{\lambda\rho}\Big) p^\rho = \gamma_\lambda \not{p} + i\sigma_{\lambda\rho} p^\rho \,. \tag{7.1}$$

Alternatively, we can write

$$p'_\lambda = g_{\lambda\rho} p'^\rho = \Big(\gamma_\rho \gamma_\lambda + i\sigma_{\rho\lambda}\Big) p^\rho = \not{p}' \gamma_\lambda - i\sigma_{\lambda\rho} p'^\rho \,. \tag{7.2}$$

Adding these two equation, sandwiching the result between two spinors, and using Eq. (2.9) and its hermitian conjugate, we obtain the relation

$$\bar{u}(\!(\boldsymbol{p}')\!)\gamma_\lambda u(\!(\boldsymbol{p})\!) = \frac{1}{2m}\bar{u}(\!(\boldsymbol{p}')\!)\Big[(p+p')_\lambda - i\sigma_{\lambda\rho}q^\rho\Big]u(\!(\boldsymbol{p})\!)\,, \tag{7.3}$$

where

$$q = p - p'\,. \tag{7.4}$$

This result is called the Gordon identity.

Variants of this identity can be easily derived following the same general technique. For example, suppose the two spinors on the two sides belong to different particles, with masses m and m'. In this case, it is easy to see that

$$\bar{u}(\!(\boldsymbol{p}')\!)\Big[(p+p')_\lambda - i\sigma_{\lambda\rho}q^\rho\Big]u(\!(\boldsymbol{p})\!) = (m'+m)\bar{u}(\!(\boldsymbol{p}')\!)\gamma_\lambda u(\!(\boldsymbol{p})\!)\,. \tag{7.5}$$

Similarly, one can obtain the identity

$$\bar{u}(\!(\boldsymbol{p}')\!)\Big[(p+p')_\lambda - i\sigma_{\lambda\rho}q^\rho\Big]\gamma_5 u(\!(\boldsymbol{p})\!) = (m'-m)\bar{u}(\!(\boldsymbol{p}')\!)\gamma_\lambda\gamma_5 u(\!(\boldsymbol{p})\!)\,. \tag{7.6}$$

It should be noted that the normalization relations of Eqs. (5.22) and (5.23) can be written in an alternative form by using the Gordon identity. For this, we put $\boldsymbol{p} = \boldsymbol{p}'$ in Eq. (7.3) and take only the time component of the equation. This gives

$$2m\, u^\dagger_{s'}(\!(\boldsymbol{p})\!)u_s(\!(\boldsymbol{p})\!) = 2E_{\boldsymbol{p}}\bar{u}_{s'}(\!(\boldsymbol{p})\!)u_s(\!(\boldsymbol{p})\!)\,, \tag{7.7}$$

where we have put the indices s, s' on the spinors in order to distinguish the different solutions. This shows that Eqs. (5.22) and (5.29) are equivalent. The proof of the equivalence of Eqs. (5.23) and (5.30) is similar.

7.2 Non-relativistic reduction

In field-theoretical manipulations, sometimes we encounter expressions which can be interpreted easily by making a non-relativistic reduction. For example, in Quantum Electrodynamics (QED), the matrix element of the electromagnetic current operator turns out to be superposition of two bilinears of the form $\bar{u}(\!(\boldsymbol{p}')\!)\gamma_\lambda u(\!(\boldsymbol{p})\!)$ and $\bar{u}(\!(\boldsymbol{p}')\!)\sigma_{\lambda\rho}q^\rho u(\!(\boldsymbol{p})\!)$, and an intuitive feeling for these bilinears can be obtained by going to the non-relativistic limit. With this in mind, here we give the non-relativistic reduction of all possible fermion bilinears.

A general bilinear is of the form

$$\bar{u}(\!(\boldsymbol{p}')\!)Fu(\!(\boldsymbol{p})\!) \tag{7.8}$$

for some matrix F. Any such matrix can be written as a superposition of the following 16 basis matrices:

$$\mathbb{1}, \gamma_\lambda, \sigma_{\lambda\rho}\ (\text{for } \mu<\nu),\ \gamma_\lambda\gamma_5, \gamma_5\,. \tag{7.9}$$

So it is enough to obtain non-relativistic reduction with the bilinears involving these basis matrices only.

The leading term in the non-relativistic approximation can be obtained by using the zero-momentum solutions for the spinors. With the normalization defined in Eq. (5.22), we obtain

$$\bar{u}_{s'}(\!(\boldsymbol{p}')\!) F u_s(\!(\boldsymbol{p})\!) \xrightarrow{\rm NR} 2m\, \xi^\dagger_{s'} F \xi_s \,. \tag{7.10}$$

Therefore, it is enough to obtain expressions for $\xi^\dagger_{s'} F \xi_s$ for the basis matrices, which we now set out to do. We divide the discussion into five parts, depending on the five types of basis matrices as shown in Eq. (7.9).

7.2.1 Scalar bilinear

This corresponds to the case $F = \mathbb{1}$. The non-relativistic reduction is trivial: the relevant formula is already given in Eq. (5.6).

7.2.2 Vector bilinears

These corresponds to $F = \gamma_\lambda$ for some index λ. Consider first the case when λ is a spatial index. Note that, using the definition of ξ_s from Eq. (5.1) and the fact that γ_0 anticommutes with all γ_i, we can write

$$\xi^\dagger_{s'} \gamma_i \xi_s = \xi^\dagger_{s'} \gamma_0 \gamma_i \xi_s = -\xi^\dagger_{s'} \gamma_i \gamma_0 \xi_s = -\xi^\dagger_{s'} \gamma_i \xi_s \,, \tag{7.11}$$

so that

$$\xi^\dagger_{s'} \gamma_i \xi_s = 0 \,. \tag{7.12}$$

As for the temporal part of the matrix element, we simply obtain

$$\xi^\dagger_{s'} \gamma_0 \xi_s = \xi^\dagger_{s'} \xi_s = \delta_{ss'} \,, \tag{7.13}$$

using the fact that ξ_s is an eigenvector of γ_0 with eigenvalue $+1$, as written down in Eq. (5.1).

7.2.3 Tensor bilinears

When F is of the form $\sigma_{\lambda\rho}$, there can be two kinds of non-relativistic limit. If one of the indices is the temporal index, we can use Eqs. (5.1) and (7.12) to show that

$$\xi^\dagger_{s'} \sigma_{0i} \xi_s = \xi^\dagger_{s'} \sigma_{i0} \xi_s = 0 \,. \tag{7.14}$$

If both indices are space indices, the matrix is essentially a spin matrix, as mentioned in Eq. (3.2). The combination is then the matrix element of the spin operator. In particular, if the spinors on the two sides have $s = s'$, then the bilinear is the expectation value of spin in that state.

7.2.4 Pseudoscalar bilinear

This corresponds to the case $F = \gamma_5$. Since γ_5 anticommutes with γ_0, we can use the steps shown in Eq. (7.11), with γ_i replaced by γ_5, to show that

$$\xi^\dagger_{s'} \gamma_5 \xi_s = 0\,. \tag{7.15}$$

7.2.5 Axial vector bilinears

Finally, we discuss the cases when F is of the form $\gamma_\lambda\gamma_5$. Two different cases arise, as in the case with vector or tensor bilinears. For $F = \gamma_0\gamma_5$, we can use the hermitian conjugate of Eq. (5.1) to write $\xi^\dagger_{s'}\gamma_0 = \xi^\dagger_{s'}$, so that

$$\xi^\dagger_{s'} \gamma_0\gamma_5 \xi_s = \xi^\dagger_{s'} \gamma_5 \xi_s = 0\,, \tag{7.16}$$

borrowing the result of Eq. (7.15). On the other hand, for $F = \gamma_i\gamma_5$, we find

$$\xi^\dagger_{s'} \gamma_i\gamma_5 \xi_s = \xi^\dagger_{s'} \gamma_0\gamma_i\gamma_5 \xi_s = -\xi^\dagger_{s'} \Sigma_i \xi_s\,, \tag{7.17}$$

using Eq. (4.17). Once again, these are matrix elements of the spin operator, which reduce to the expectation value of spin if the two spinors on both sides are the same.

8 Concluding remarks

The aim of the article was to show that some important identities involving Dirac spinors can be proved without invoking any specific form for the spinors. As we mentioned earlier, the specific forms depend on the representation of the Dirac matrices. For the sake of elegance and safety, it is better to deal with the spinors in a representation-independent manner.

The analysis can be extended to quantum field theory involving Dirac fields. Properties of Dirac field under parity, charge conjugation and time reversal can be derived in completely representation-independent manner. This has been done at least in one textbook of quantum field theory [3], to which we refer the reader for details.

Appendix

A The metric tensor and the Levi-Civita symbol

Our convention for the metric tensor is:

$$g_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)\,. \tag{A.1}$$

The Levi-Civita symbol is the completely antisymmetric rank-4 tensor, with

$$\varepsilon_{0123} = 1\,. \tag{A.2}$$

Product of two Levi-Civita symbols can be expressed in terms of the metric tensor:

$$\varepsilon^{\mu\nu\lambda\rho}\,\varepsilon_{\mu'\nu'\lambda'\rho'} = - \left\| \begin{matrix} \delta^\mu_{\mu'} & \delta^\mu_{\nu'} & \delta^\mu_{\lambda'} & \delta^\mu_{\rho'} \\ \delta^\nu_{\mu'} & \delta^\nu_{\nu'} & \delta^\nu_{\lambda'} & \delta^\nu_{\rho'} \\ \delta^\lambda_{\mu'} & \delta^\lambda_{\nu'} & \delta^\lambda_{\lambda'} & \delta^\lambda_{\rho'} \\ \delta^\rho_{\mu'} & \delta^\rho_{\nu'} & \delta^\rho_{\lambda'} & \delta^\rho_{\rho'} \end{matrix} \right\| \equiv -\delta^\mu_{[\mu'}\delta^\nu_{\nu'}\delta^\lambda_{\lambda'}\delta^\rho_{\rho']}\,, \tag{A.3}$$

where the pair of two vertical lines on two sides of the matrix indicates the determinant of the matrix, and the square brackets appearing among the indices imply an antisymmetrization with respect to the enclosed indices. By taking successive contractions of this relation, we can obtain the following relations:

$$\varepsilon^{\mu\nu\lambda\rho}\,\varepsilon_{\mu\nu'\lambda'\rho'} = -\delta^\nu_{[\nu'}\delta^\lambda_{\lambda'}\delta^\rho_{\rho']}\,, \tag{A.4}$$

$$\varepsilon^{\mu\nu\lambda\rho}\,\varepsilon_{\mu\nu\lambda'\rho'} = -2\delta^\lambda_{[\lambda'}\delta^\rho_{\rho']}\,, \tag{A.5}$$

$$\varepsilon^{\mu\nu\lambda\rho}\,\varepsilon_{\mu\nu\lambda\rho'} = -6\delta^\rho_{\rho'}\,, \tag{A.6}$$

$$\varepsilon^{\mu\nu\lambda\rho}\,\varepsilon_{\mu\nu\lambda\rho} = -24\,. \tag{A.7}$$

References

[1] See, for example,

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[2] See, e.g., S. S. Schweber, *Introduction to relativistic quantum field theory*, (Harper & Row, 1962).

[3] A. Lahiri and P. B. Pal, *A first book of Quantum Field Theory*, (Narosa Publishing House, New Delhi, 2nd edition 2004).