TÉCNICO LISBOA

Advanced Quantum Field Theory
Chapter 4
Radiative Corrections

Jorge C. Romão
Instituto Superior Técnico, Departamento de Física & CFTP
A. Rovisco Pais 1, 1049-001 Lisboa, Portugal

Fall 2013

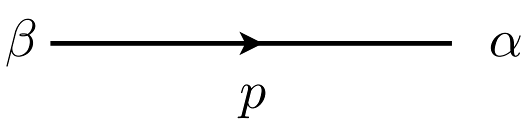
Lecture 6

QED Renormalization at one-loop

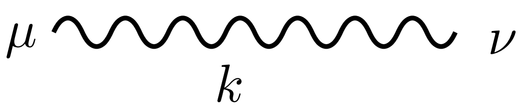
- We will consider the theory described by the Lagrangian

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial\cdot A)^2 + \overline{\psi}(i\partial\!\!\!/ + eA\!\!\!/ - m)\psi \ .$$

- The free propagators are



$$\left(\frac{i}{p\!\!\!/ - m + i\varepsilon}\right)_{\beta\alpha} \equiv S^0_{F\,\beta\alpha}(p)$$



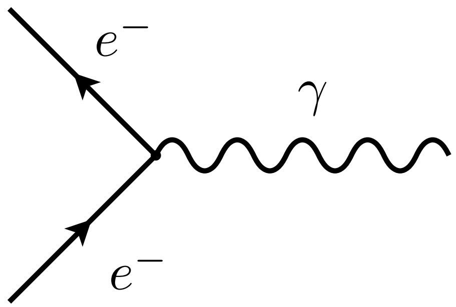
$$-i\left[\frac{g_{\mu\nu}}{k^2+i\varepsilon} + (1-\xi)\frac{k_\mu k_\nu}{(k^2+i\varepsilon)^2}\right]$$

$$= -i\left\{\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2+i\varepsilon} + \xi\frac{k_\mu k_\nu}{k^4}\right\}$$

$$\equiv G^0_{F\,\mu\nu}(k)$$

QED Renormalization at one-loop

- The only vertex is given by

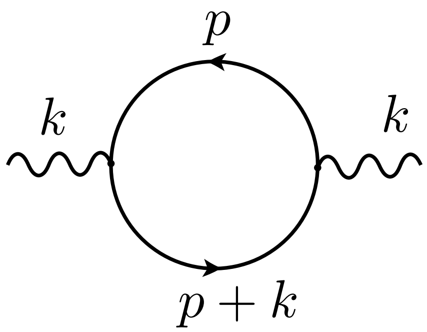


$$+ie(\gamma_\mu)_{\beta\alpha} \qquad e = |e| > 0$$

- We will now consider the one-loop corrections to the propagators and to the vertex. We will work in the Feynman gauge ($\xi = 1$).

Vacuum Polarization

- In first order the contribution to the photon propagator is given by the diagram



- We write it in the form

$$G^{(1)}_{\mu\nu}(k) \equiv G^0_{\mu\mu'}(k)\ i\,\Pi^{\mu'\nu'}(k) G^0_{\nu'\nu}(k)$$

where

$$i\,\Pi_{\mu\nu}(k) = -\,(+ie)^2 \int \frac{d^4p}{(2\pi)^4} \frac{\text{Tr}[\gamma_\mu(\not{p}+m)\gamma_\nu(\not{p}+\not{k}+m)]}{(p^2-m^2+i\varepsilon)((p+k)^2-m^2+i\varepsilon)}$$

$$= -\,4e^2 \int \frac{d^4p}{(2\pi)^4} \frac{[2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2+p\cdot k - m^2)}{(p^2-m^2+i\varepsilon)((p+k)^2-m^2+i\varepsilon)}$$

Vacuum Polarization

- Simple power counting indicates that this integral is quadratically divergent for large values of the internal loop momenta. In fact the divergence is milder, only logarithmic.

- The integral being divergent we have first to regularize it and then to define a renormalization procedure to cancel the infinities. For this purpose we will use the method of dimensional regularization.

- For a value of d small enough the integral converges. If we define $\epsilon = 4 - d$, in the end we will have a divergent result in the limit $\epsilon \to 0$.

- We note that μ is a parameter with dimensions of a mass that is introduced to ensure the correct dimensions of the coupling constant in dimension d, that is, $[e] = \frac{4-d}{2} = \frac{\epsilon}{2}$. We take then $e \to e\mu^{\frac{\epsilon}{2}}$

- We get therefore

$$
\begin{aligned}
i\,\Pi_{\mu\nu}(k,\epsilon) &= -\,4e^2\,\mu^\epsilon \int \frac{d^dp}{(2\pi)^d} \frac{[2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2 + p\cdot k - m^2)]}{(p^2 - m^2 + i\varepsilon)((p+k)^2 - m^2 + i\varepsilon)} \\
&= -\,4e^2\,\mu^\epsilon \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p,k)}{(p^2 - m^2 + i\varepsilon)((p+k)^2 - m^2 + i\varepsilon)}
\end{aligned}
$$

Vacuum Polarization

- We have defined a shorthand for the numerator

$$N_{\mu\nu}(p,k) = 2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2 + p\cdot k - m^2)$$

- To evaluate this integral we first use the Feynman parameterization to rewrite the denominator as a single term. For that we use (see Appendix)

$$\frac{1}{ab} = \int_0^1 \frac{dx}{\left[ax + b(1-x)\right]^2}$$

to get

$$\begin{aligned}
i\,\Pi_{\mu\nu}(k,\epsilon) =& -4e^2\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p,k)}{\left[x(p+k)^2 - xm^2 + (1-x)(p^2-m^2) + i\varepsilon\right]^2} \\
=& -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p,k)}{\left[p^2 + 2k\cdot px + xk^2 - m^2 + i\varepsilon\right]^2} \\
=& -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p,k)}{\left[(p+kx)^2 + k^2x(1-x) - m^2 + i\varepsilon\right]^2}
\end{aligned}$$

Vacuum Polarization

- For dimension d sufficiently small this integral converges and we can change variables

$$p \to p - kx$$

- We then get

$$i\,\Pi_{\mu\nu}(k,\epsilon) = -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d}\,\frac{N_{\mu\nu}(p-kx,k)}{\left[p^2 - C + i\epsilon\right]^2}$$

where

$$C = m^2 - k^2 x(1-x)$$

- $N_{\mu\nu}$ is a polynomial of second degree in the loop momenta. However as the denominator in only depends on p^2 is it easy to show that

$$\int \frac{d^dp}{(2\pi)^d}\,\frac{p^\mu}{\left[p^2 - C + i\epsilon\right]^2} = 0$$

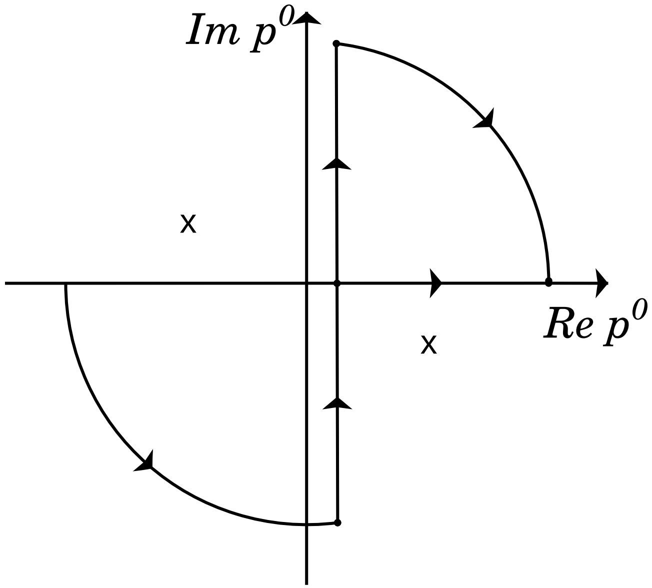
$$\int \frac{d^dp}{(2\pi)^d}\,\frac{p^\mu p^\nu}{\left[p^2 - C + i\epsilon\right]^2} = \frac{1}{d}\,g^{\mu\nu} \int \frac{d^dp}{(2\pi)^d}\,\frac{p^2}{\left[p^2 - C + i\epsilon\right]^2}$$

Vacuum Polarization

- This means that we only have to calculate integrals of the form

$$
\begin{aligned}
I_{r,m} &= \int \frac{d^d p}{(2\pi)^d} \frac{(p^2)^r}{[p^2 - C + i\epsilon]^m} \\
&= \int \frac{d^{d-1} p}{(2\pi)^d} \int dp^0 \frac{(p^2)^r}{[p^2 - C + i\epsilon]^m}
\end{aligned}
$$

- To make this integration we will use integration in the plane of the complex variable p^0 as described in the figure. The deformation of the contour corresponds to the so called Wick rotation,



Vacuum Polarization

- This means the replacement

$$p^0 \to i p_E^0 \qquad ; \qquad \int_{-\infty}^{+\infty} \to i \int_{-\infty}^{+\infty} dp_E^0$$

and $p^2 = (p^0)^2 - |\vec{p}|^2 = -(p_E^0)^2 - |\vec{p}|^2 \equiv -p_E^2$, where $p_E = (p_E^0, \vec{p})$ is an euclidean vector, that is

$$p_E^2 = (p_E^0)^2 + |\vec{p}|^2$$

- We can then write (see the Appendix for more details),

$$I_{r,m} = i(-1)^{r-m} \int \frac{d^d p_E}{(2\pi)^d} \frac{p_E^{2^r}}{[p_E^2 + C]^m}$$

where we do not need the $i\epsilon$ anymore because the denominator is positive definite $(C > 0)$

- The case when $C < 0$ is obtained by analytical continuation of the final result

Vacuum Polarization

- To proceed with the evaluation of $I_{r,m}$ we write,

$$\int d^d p_E = \int d\overline{p}\ \overline{p}^{\ d-1}\, d\Omega_{d-1}$$

where $\overline{p} = \sqrt{(p_E^0)^2 + |\vec{p}|^2}$ is the length of of vector p_E in the euclidean space with d dimensions and $d\Omega_{d-1}$ is the solid angle

- We can show (see Appendix) that

$$\int d\Omega_{d-1} = 2\,\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$$

- The $\overline{p}$ integral is done using the result,

$$\int_0^\infty dx\ \frac{x^p}{(x^n+a^n)^q} = \pi(-1)^{q-1}\,a^{p+1-nq}\ \frac{\Gamma(\frac{p+1}{n})}{n\sin(\pi\,\frac{p+1}{n})\,\Gamma(\frac{p+1}{2}-q+1)}$$

and we finally get

$$I_{r,m} = iC^{r-m+\frac{d}{2}}\ \frac{(-1)^{r-m}}{(4\pi)^{\frac{d}{2}}}\ \frac{\Gamma(r+\frac{d}{2})}{\Gamma(\frac{d}{2})}\ \frac{\Gamma(m-r-\frac{d}{2})}{\Gamma(m)}$$

Vacuum Polarization

- Note that the integral representation of $I_{r,m}$ is only valid for $d < 2(m-r)$ to ensure the convergence of the integral when $\overline{p} \to \infty$.
- However the final form of can be analytically continued for all the values of d except for those where the function $\Gamma(m-r-d/2)$ has poles, which are,

$$m - r - \frac{d}{2} \neq 0, -1, -2, \ldots$$

- For the application to dimensional regularization it is convenient to write $I_{r,m}$ after making the substitution $d = 4 - \epsilon$. We get

$$I_{r,m} = i\frac{(-1)^{r-m}}{(4\pi)^2} \left(\frac{4\pi}{C}\right)^{\frac{\epsilon}{2}} C^{2+r-m} \frac{\Gamma(2+r-\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})} \frac{\Gamma(m-r-2+\frac{\epsilon}{2})}{\Gamma(m)}$$

that has poles for $m - r - 2 \leq 0$ (see Appendix).

- We now go back to calculate $\Pi_{\mu\nu}$. First we notice that after the change of variables we get, neglecting linear terms that vanish,

$$N_{\mu\nu}(p-kx, k) = 2p_\mu p_\nu + 2x^2 k_\mu k_\nu - 2xk_\mu k_\nu - g_{\mu\nu}\left(p^2 + x^2k^2 - xk^2 - m^2\right)$$

Vacuum Polarization

- Therefore

$$\mathcal{N}_{\mu\nu} \equiv \mu^\epsilon \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p-kx,k)}{\left[p^2 - C + i\epsilon\right]^2}$$

$$= \left(\frac{2}{d} - 1\right) g_{\mu\nu}\mu^\epsilon I_{1,2} + \left[-2x(1-x)k_\mu k_\nu + x(1-x)k^2 g_{\mu\nu} + g_{\mu\nu}m^2 \right] \mu^\epsilon I_{0,2}$$

- Using now the results for I_{rm} we can write

$$\mu^\epsilon I_{0,2} = \frac{i}{16\pi^2} \left(\frac{4\pi\mu^2}{C}\right)^{\frac{\epsilon}{2}} \frac{\Gamma(\frac{\epsilon}{2})}{\Gamma(2)}$$

$$= \frac{i}{16\pi^2} \left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right) + \mathcal{O}(\epsilon)$$

- In doing this we have used the expansion of the Γ function

$$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)$$

γ being the Euler constant and we have defined

$$\Delta_\epsilon \equiv \frac{2}{\epsilon} - \gamma + \ln 4\pi$$

Vacuum Polarization

- In a similar way

$$\mu^\epsilon I_{1,2} = -\frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\frac{\epsilon}{2}} C\,\frac{\Gamma(3-\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})}\,\frac{\Gamma(-1+\frac{\epsilon}{2})}{\Gamma(2)}$$

$$=\frac{i}{16\pi^2}C\left(1+2\Delta_\epsilon-2\ln\frac{C}{\mu^2}\right)+\mathcal{O}(\epsilon)$$

- Due to the existence of a pole in $1/\epsilon$ in the previous equations we have to expand all quantities up to $\mathcal{O}(\epsilon)$. This means for instance, that

$$\frac{2}{d}-1=\frac{2}{4-\epsilon}-1=-\frac{1}{2}+\frac{1}{8}\epsilon+\mathcal{O}(\epsilon^2)$$

- We now substitute these expressions to obtain $\mathcal{N}_{\mu\nu}$

Vacuum Polarization

- We obtain

$$
\begin{aligned}
\mathcal{N}_{\mu\nu} =& g_{\mu\nu}\left[-\frac{1}{2}+\frac{1}{8}\epsilon+\mathcal{O}(\epsilon^2)\right]\left[\frac{i}{16\pi^2}C\left(1+2\Delta_\epsilon-2\ln\frac{C}{\mu^2}\right)+\mathcal{O}(\epsilon)\right]\\
&+\left[-2x(1-x)k_\mu k_\nu+x(1-x)k^2 g_{\mu\nu}+g_{\mu\nu}m^2\right]\left[\frac{i}{16\pi^2}\left(\Delta_\epsilon-\ln\frac{C}{\mu^2}\right)+\mathcal{O}(\epsilon)\right]\\
=& -\frac{i}{16\pi^2}k_\mu k_\nu\left[\left(\Delta_\epsilon-\ln\frac{C}{\mu^2}\right)2x(1-x)\right]\\
&+\frac{i}{16\pi^2}g_{\mu\nu}k^2\left[\Delta_\epsilon\left(x(1-x)+x(1-x)\right)+\ln\frac{C}{\mu^2}\left(-x(1-x)-x(1-x)\right)\right.\\
&\left.\qquad+x(1-x)\left(\frac{1}{2}-\frac{1}{2}\right)\right]\\
&+\frac{i}{16\pi^2}g_{\mu\nu}m^2\left[\Delta_\epsilon(-1+1)+\ln\frac{C}{\mu^2}(1-1)+(-\frac{1}{2}+\frac{1}{2})\right]
\end{aligned}
$$

Vacuum Polarization

- Finally

$$\mathcal{N}_{\mu\nu} = \frac{i}{16\pi^2}\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right)\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right) 2x(1-x)$$

- Substituting in the expression for $\Pi_{\mu\nu}$ we get

$$\Pi_{\mu\nu}(k) = -\, 4e^2 \frac{1}{16\pi^2}\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\int_0^1 dx\ 2x(1-x)\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right)$$

$$= -\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\Pi(k^2,\epsilon)$$

where

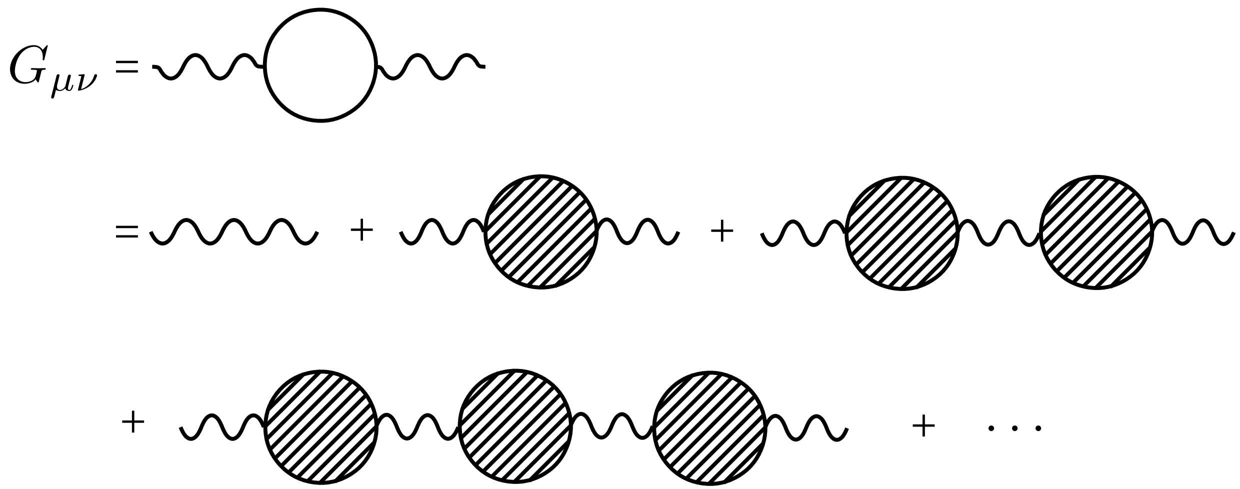
$$\Pi(k^2,\epsilon) \equiv \frac{2\alpha}{\pi}\int_0^1 dx\ x(1-x)\left[\Delta_\epsilon - \ln\frac{m^2 - x(1-x)k^2}{\mu^2}\right]$$

- This expression clearly diverges as $\epsilon \to 0$. Before we show how to renormalize it let us discuss the meaning of $\Pi_{\mu\nu}(k)$.

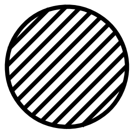
Vacuum Polarization

Lecture 6
QED Renormalization
Vacuum Polarization
Self-energy
The Vertex

- The full photon propagator is given by the series

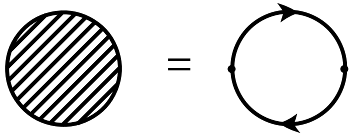


- Where we have defined



$\equiv i\,\Pi_{\mu\nu}(k) =$ sum of all one-particle irreducible (*proper*) diagrams to *all* orders

- In lowest order we have the contribution



which is what we have just calculated.

- To continue it is convenient to rewrite the free propagator of the photon (in an arbitrary gauge ξ) in the following form

$$iG^0_{\mu\nu} = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2} + \xi\frac{k_\mu k_\nu}{k^4} = P^T_{\mu\nu}\frac{1}{k^2} + \xi\frac{k_\mu k_\nu}{k^4}$$
$$\equiv iG^{0T}_{\mu\nu} + iG^{0L}_{\mu\nu}$$

where we have introduced the transversal projector $P^T_{\mu\nu}$ defined by

$$P^T_{\mu\nu} = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$

Vacuum Polarization

- This projector satisfies the relations,

$$\begin{cases} k^\mu P^T_{\mu\nu} = 0 \\ P^{T\,\nu}_{\mu} P^T_{\nu\rho} = P^T_{\mu\rho} \end{cases}$$

- The full photon propagator can also in general be written separating its transversal an longitudinal parts

$$G_{\mu\nu} = G^T_{\mu\nu} + G^L_{\mu\nu}$$

where $G^T_{\mu\nu}$ satisfies

$$G^T_{\mu\nu} = P^T_{\mu\nu} G_{\mu\nu}$$

- The result we got at one loop means that, to first order, the vacuum polarization tensor is transversal, that is

$$i\,\Pi_{\mu\nu}(k) = -ik^2 P^T_{\mu\nu}\,\Pi(k)$$

Vacuum Polarization

- This result is in fact valid to all orders of perturbation theory, a result that can be shown using the Ward-Takahashi identities. This means that the longitudinal part of the photon propagator is not renormalized,

$$G^L_{\mu\nu} = G^{0L}_{\mu\nu}$$

- For the transversal part we obtain from the series expansion

$$\begin{aligned} iG^T_{\mu\nu} =& P^T_{\mu\nu}\frac{1}{k^2} + P^T_{\mu\mu'}\frac{1}{k^2}(-i)k^2 P^{T\mu'\nu'}\Pi(k^2)(-i)P^T_{\nu'\nu}\frac{1}{k^2} \\ &+ P^T_{\mu\rho}\frac{1}{k^2}(-i)k^2 P^{T\rho\lambda}\,\Pi(k^2)(-i)P^T_{\lambda\tau}\frac{1}{k^2}(-i)k^2 P^{T\tau\sigma}\,\Pi(k^2)(-i)P^T_{\sigma\nu}\frac{1}{k^2} + \cdots \\ =& P^T_{\mu\nu}\frac{1}{k^2}\left[1-\Pi(k^2)+\Pi^2(k^2)+\cdots\right] \end{aligned}$$

- This gives, after summing the geometric series,

$$iG^T_{\mu\nu} = P^T_{\mu\nu}\frac{1}{k^2\left[1+\Pi(k^2)\right]}$$

Vacuum Polarization

- All that we have done up to this point is formal because the function $\Pi(k)$ diverges.

- The most satisfying way to solve this problem is the following. The initial lagrangian from which we started has been obtained from the classical theory and nothing tell us that it should be exactly the same in quantum theory. In fact, as we have just seen, the normalization of the wave functions is changed when we calculate *one-loop* corrections, and the same happens to the physical parameters of the theory, the charge and the mass.

- Therefore we can think that the correct lagrangian is obtained by adding corrections to the classical lagrangian, order by order in perturbation theory, so that we keep the definitions of charge and mass as well as the normalization of the wave functions. The terms that we add to the lagrangian are called *counterterms*

- This interpretation in terms of quantum corrections makes sense. In fact we can show that an expansion in powers of the coupling constant can be interpreted as an expansion in $\hbar^L$, where L is the number of the loops in the expansion term

Vacuum Polarization

- The total lagrangian is then,

$$\mathcal{L}_{\text{total}} = \mathcal{L}(e, m, ...) + \Delta\mathcal{L}$$

- Counterterms are defined from the normalization conditions that we impose on the fields and other parameters of the theory. In QED we have at our disposal the normalization of the electron and photon fields and of the two physical parameters, the electric charge and the electron mass.

- The normalization conditions are, to a large extent, arbitrary. It is however convenient to keep the expressions as close as possible to the free field case, that is, without radiative corrections.

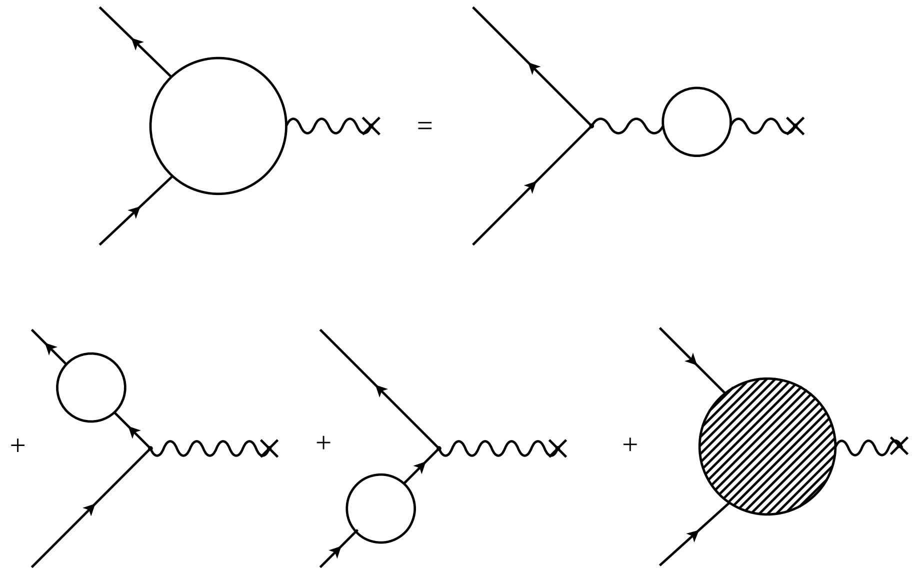
- We define therefore the normalization of the photon field as,

$$\lim_{k\to 0} k^2 i G^{RT}_{\mu\nu} = 1 \cdot P^T_{\mu\nu}$$

where $G^{RT}_{\mu\nu}$ is the renormalized propagator (the transversal part) obtained from the lagrangian $\mathcal{L}_{\text{total}}$.

Vacuum Polarization

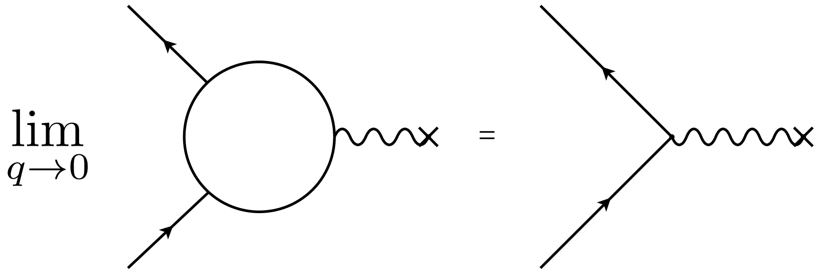
- The justification for this definition comes from the following argument. Consider the Coulomb scattering to all orders of perturbation theory. We have then the situation described in the figure



- Using the Ward-Takahashi identities one can show that the last three diagrams cancel in the limit $q = p' - p \to 0$. Then our normalization condition, means that the experimental value of the electric charge is determined in the limit $q \to 0$ of the Coulomb scattering.

Vacuum Polarization

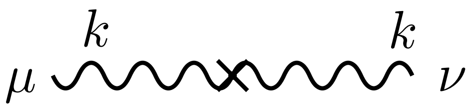
- We have then the situation described in



- The counterterm lagrangian has to have the same form as the classical lagrangian to respect the symmetries of the theory. For the photon field it is traditional to write

$$\Delta\mathcal{L} = -\frac{1}{4}(Z_3 - 1)F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\delta Z_3\, F_{\mu\nu}F^{\mu\nu}$$

corresponding to the following Feynman rule



$$-\,i\,\delta Z_3 k^2 \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right)$$

Vacuum Polarization

- We have then

$$\begin{aligned} i\Pi_{\mu\nu} =& i\Pi^{loop}_{\mu\nu} - i\,\delta Z_3 k^2 \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \\ =& -i\, \left(\Pi(k,\epsilon) + \delta Z_3 \right) k^2\, P^T_{\mu\nu} \end{aligned}$$

- Therefore we should make the substitution

$$\Pi(k,\epsilon) \to \Pi(k,\epsilon) + \delta Z_3$$

in the photon propagator. We obtain,

$$iG^T_{\mu\nu} = P^T_{\mu\nu} \frac{1}{k^2} \frac{1}{1 + \Pi(k,\epsilon) + \delta Z_3}$$

- The normalization condition implies

$$\Pi(0,\epsilon) + \delta Z_3 = 0$$

from which one determines the constant δZ_3.

Vacuum Polarization

- We get

$$\delta Z_3 = -\Pi(0,\epsilon) = -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\left[\Delta_\epsilon - \ln\frac{m^2}{\mu^2}\right]$$

$$= -\frac{\alpha}{3\pi}\left[\Delta_\epsilon - \ln\frac{m^2}{\mu^2}\right]$$

- The renormalized photon propagator can then be written as

$$iG_{\mu\nu}(k) = \frac{P^T_{\mu\nu}}{k^2[1+\Pi(k,\epsilon)-\Pi(0,\epsilon)]} + i\,G^L_{\mu\nu}$$

- Notice that the photon mass is not renormalized, that is the pole of the photon propagator remains at $k^2=0$.

Vacuum Polarization

- The *finite* radiative corrections are given through the function

$$\begin{aligned}\Pi^R(k^2) \equiv& \Pi(k^2,\epsilon) - \Pi(0,\epsilon)\\ =& -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\left[\frac{m^2 - x(1-x)k^2}{m^2}\right]\\ =& -\frac{\alpha}{3\pi}\left\{\frac{1}{3} + 2\left(1+\frac{2m^2}{k^2}\right)\left[\left(\frac{4m^2}{k^2}-1\right)^{1/2}\cot^{-1}\left(\frac{4m^2}{k^2}-1\right)^{1/2} - 1\right]\right\}\end{aligned}$$

  where the last equation is valid for $k^2 < 4m^2$.

- For values $k^2 > 4m^2$ the result for $\Pi^R(k^2)$ can be obtained by analytical continuation. Using $(k^2 > 4m^2)$

$$\cot^{-1} iz = i\left(-\tanh^{-1} z + \frac{i\pi}{2}\right)$$

  and

$$\left(\frac{4m^2}{k^2}-1\right)^{1/2} \to i\sqrt{1-\frac{4m^2}{k^2}}$$

Vacuum Polarization

- We get then

$$\Pi^R(k^2) = -\frac{\alpha}{3\pi}\left\{\frac{1}{3} + 2\left(1+\frac{2m^2}{k^2}\right)\left[-1+\sqrt{1-\frac{4m^2}{k^2}}\tanh^{-1}\left(1-\frac{4m^2}{k^2}\right)^{1/2} -i\frac{\pi}{2}\sqrt{1-\frac{4m^2}{k^2}}\right]\right\}$$

- The imaginary part of Π^R is given by

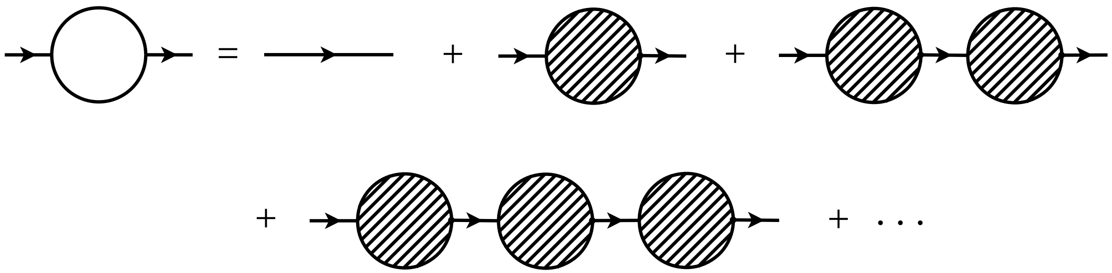
$$Im\ \Pi^R(k^2) = \frac{\alpha}{3}\left(1+\frac{2m^2}{k^2}\right)\sqrt{1-\frac{4m^2}{k^2}}\theta\left(1-\frac{4m^2}{k^2}\right)$$

and it is related to the pair production that can occur for $k^2 > 4m^2$.

- In fact, for $k^2 > 4m^2$ there is the possibility of producing one pair e^+e^-. Therefore on top of a virtual process (vacuum polarization) there is a real process (pair production).

Self-energy of the electron

- The electron full propagator is given by the diagrammatic series

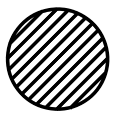


- This of can be written as,

$$S(p) = S^0(p) + S^0(p)\left(-i\,\Sigma(p)\right) S^0(p) + \cdots$$

$$= S^0(p)\left[1 - i\,\Sigma(p) S(p)\right]$$

where we have identified



$$\equiv -i\,\Sigma(p)$$

Self-energy of the electron

- Multiplying on the left with $S_0^{-1}(p)$ and on the right with $S^{-1}(p)$ we get

$$S_0^{-1}(p) = S^{-1}(p) - i\,\Sigma(p)$$

which we can rewrite as

$$S^{-1}(p) = S_0^{-1}(p) + i\,\Sigma(p)$$

- Using the expression for the free field propagator,

$$S_0(p) = \frac{i}{\not{p} - m} \Longrightarrow S_0^{-1}(p) = -i(\not{p} - m)$$

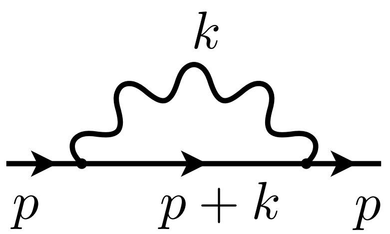
we can then write

$$S^{-1}(p) = S_0^{-1}(p) + i\Sigma(p) = -i\left[\not{p} - (m + \Sigma(p))\right]$$

- We conclude that it is enough to calculate $\Sigma(p)$ to all orders of perturbation theory to obtain the full electron propagator. The name *self-energy* given to $\Sigma(p)$ comes from the fact that it comes as an additional (momentum dependent) contribution to the mass.

Self-energy of the electron

- In lowest order there is only the diagram



- Therefore we get,

$$-i\Sigma(p) = (+ie)^2 \int \frac{d^4k}{(2\pi)^4} (-i) \frac{g_{\mu\nu}}{k^2 - \lambda^2 + i\varepsilon} \gamma^\mu \frac{i}{\not{p} + \not{k} - m + i\varepsilon} \gamma^\nu$$

where we have chosen the Feynman gauge ($\xi = 1$) for the photon propagator and we have introduced a small mass for the photon λ, in order to control the infrared divergences (IR) that will appear when $k^2 \to 0$ (see below).

- Using dimensional regularization and the results of the Dirac algebra in dimension d,

$$\gamma_\mu (\not{p} + \not{k}) \gamma^\mu = -(\not{p} + \not{k}) \gamma_\mu \gamma^\mu + 2(\not{p} + \not{k}) = -(d-2)(\not{p} + \not{k})$$

$$m \gamma_\mu \gamma^\mu = m\, d$$

Self-energy of the electron

- We get

$$
\begin{aligned}
-i\,\Sigma(p) =& - \mu^{\epsilon} e^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2-\lambda^2+i\varepsilon}\, \gamma_\mu\, \frac{\not{p}+\not{k}+m}{(p+k)^2-m^2+i\varepsilon}\, \gamma^\mu \\
=& - \mu^{\epsilon} e^2 \int \frac{d^d k}{(2\pi)^d} \frac{-(d-2)(\not{p}+\not{k})+m\,d}{[k^2-\lambda^2+i\varepsilon]\,[(p+k)^2-m^2+i\varepsilon]} \\
=& - \mu^{\epsilon} e^2 \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{-(d-2)(\not{p}+\not{k})+m\,d}{[(k^2-\lambda^2)\,(1-x)+x(p+k)^2-xm^2+i\varepsilon]^2} \\
=& - \mu^{\epsilon} e^2 \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{-(d-2)(\not{p}+\not{k})+m\,d}{[(k+px)^2+p^2x(1-x)-\lambda^2(1-x)-xm^2+i\varepsilon]^2} \\
=& - \mu^{\epsilon} e^2 \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{-(d-2)\,[\not{p}(1-x)+\not{k}]+m\,d}{[k^2+p^2x(1-x)-\lambda^2(1-x)-xm^2+i\varepsilon]^2} \\
=& - \mu^{\epsilon} e^2 \int_0^1 dx \left[-(d-2)\not{p}(1-x)+m\,d\right] I_{0,2}
\end{aligned}
$$

Self-energy of the electron

- Where

$$I_{0,2} = \frac{i}{16\pi^2}\left[\Delta_\epsilon - \ln\left[-p^2 x(1-x) + m^2 x + \lambda^2(1-x)\right]\right]$$

- The contribution from the *loop* to the electron self-energy $\Sigma(p)$ can then be written in the form,

$$\Sigma(p)^{loop} = A(p^2) + B(p^2)\,\not{p}$$

with

$$A = e^2\mu^\epsilon(4-\epsilon)m\frac{1}{16\pi^2}\int_0^1 dx\,\left[\Delta_\epsilon - \ln\left[-p^2x(1-x) + m^2x + \lambda^2(1-x)\right]\right]$$

$$B = -\,e^2\mu^\epsilon(2-\epsilon)\frac{1}{16\pi^2}\int_0^1 dx\,(1-x)\Bigg[\Delta_\epsilon$$
$$- \ln\left[-p^2x(1-x) + m^2x + \lambda^2(1-x)\right]\Bigg]$$

Self-energy of the electron

Lecture 6
QED Renormalization
Vacuum Polarization
Self-energy
The Vertex

- Using now the expansions

$$\mu^\epsilon(4-\epsilon) = 4\left[1+\epsilon\left(\ln\mu-\frac{1}{4}\right)+\mathcal{O}(\epsilon^2)\right]$$

$$\mu^\epsilon(4-\epsilon)\Delta_\epsilon = 4\left[\Delta_\epsilon+2\left(\ln\mu-\frac{1}{4}\right)+\mathcal{O}(\epsilon)\right]$$

$$\mu^\epsilon(2-\epsilon) = 2\left[1+\epsilon\left(\ln\mu-\frac{1}{2}\right)+\mathcal{O}(\epsilon^2)\right]$$

$$\mu^\epsilon(2-\epsilon)\Delta_\epsilon = 2\left[\Delta_\epsilon+2\left(\ln\mu-\frac{1}{2}\right)+\mathcal{O}(\epsilon)\right]$$

we can finally write,

$$A(p^2) = \frac{4\,e^2 m}{16\pi^2}\int_0^1 dx\left[\Delta_\epsilon-\frac{1}{2}-\ln\left[\frac{-p^2x(1-x)+m^2x+\lambda^2(1-x)}{\mu^2}\right]\right]$$

Self-energy of the electron

- And for B

$$B(p^2) = -\frac{2\,e^2}{16\pi^2}\int_0^1 dx\,(1-x)\left[\Delta_\epsilon - 1 - \ln\left[\frac{-p^2x(1-x)+m^2x+\lambda^2(1-x)}{\mu^2}\right]\right]$$

- To continue with the renormalization program we have to introduce the counterterm lagrangian and define the normalization conditions. We have

$$\Delta\mathcal{L} = i\,(Z_2-1)\,\overline{\psi}\gamma^\mu\partial_\mu\psi - (Z_2-1)\,m\,\overline{\psi}\psi + Z_2\delta m\,\overline{\psi}\psi + (Z_1-1)e\,\overline{\psi}\gamma^\mu\psi A_\mu$$

and therefore we get for the self-energy

$$-i\,\Sigma(p) = -i\,\Sigma^{loop}(p) + i\,(\not{p}-m)\,\delta Z_2 + i\,\delta m$$

- Contrary to the case of the photon we see that we have two constants to determine. In the *on-shell* renormalization scheme that is normally used in QED the two constants are obtained by requiring that the pole of the propagator corresponds to the physical mass (hence the name of *on-shell* renormalization), and that the residue of the pole of the renormalized electron propagator has the same value as the free field propagator.

Self-energy of the electron

- This implies,

$$\Sigma(\not p = m) = 0 \quad \Rightarrow \quad \delta m = \Sigma^{loop}(\not p = m)$$

$$\left.\frac{\partial \Sigma}{\partial \not p}\right|_{\not p = m} = 0 \quad \Rightarrow \quad \delta Z_2 = \left.\frac{\partial \Sigma^{loop}}{\partial \not p}\right|_{\not p = m}$$

- We then get for δm (δm is not IR divergent, so $\lambda \to 0$)

$$\begin{aligned}
\delta m =& A(m^2) + m\, B(m^2) \\
=& \frac{2\, m e^2}{16\pi^2} \int_0^1 dx \left\{ \left[2\Delta_\epsilon - 1 - 2\ln\left(\frac{m^2x^2 + \lambda^2(1-x)}{\mu^2} \right) \right] \right. \\
& \left. -(1-x) \left[\Delta_\epsilon - 1 - \ln\left(\frac{m^2x^2 + \lambda^2(1-x)}{\mu^2} \right) \right] \right\} \\
=& \frac{2\, m\, e^2}{16\pi^2} \left[\frac{3}{2}\Delta_\epsilon - \frac{1}{2} - \int_0^1 dx\, (1+x) \ln\left(\frac{m^2x^2 + \lambda^2(1-x)}{\mu^2} \right) \right] \\
=& \frac{3\alpha m}{4\pi} \left[\Delta_\epsilon - \frac{1}{3} - \frac{2}{3} \int_0^1 dx\, (1+x) \ln\left(\frac{m^2x^2}{\mu^2} \right) \right]
\end{aligned}$$

Self-energy of the electron

- In a similar way we get for δZ_2,

$$\delta Z_2 = \left.\frac{\partial \Sigma^{loop}}{\partial \not{p}}\right|_{\not{p}=m} = \left.\frac{\partial A}{\partial \not{p}}\right|_{\not{p}=m} + B + m \left.\frac{\partial B}{\partial \not{p}}\right|_{\not{p}=m}$$

where

$$\left.\frac{\partial A}{\partial \not{p}}\right|_{\not{p}=m} = \frac{4\, e^2\, m^2}{16\pi^2} \int_0^1 dx\, \frac{2(1-x)x}{-m^2 x(1-x) + m^2 x + \lambda^2 (1-x)}$$

$$= \frac{2\,\alpha\, m^2}{\pi} \int_0^1 dx\, \frac{(1-x)x}{m^2 x^2 + \lambda^2 (1-x)}$$

$$B = -\frac{\alpha}{2\pi} \int_0^1 dx\, (1-x) \left[\Delta_\epsilon - 1 - \ln\left(\frac{m^2 x^2 + \lambda^2 (1-x)}{\mu^2} \right) \right]$$

$$m \left.\frac{\partial B}{\partial \not{p}}\right|_{\not{p}=m} = -\frac{\alpha}{2\pi}\, m^2 \int_0^1 dx\, \frac{2x(1-x)^2}{m^2 x^2 + \lambda^2 (1-x)}$$

Self-energy of the electron

- Substituting we get,

$$\delta Z_2 = -\frac{\alpha}{2\pi}\left[\frac{1}{2}\Delta_\epsilon - \frac{1}{2} - \int_0^1 dx(1-x)\ln\left(\frac{m^2x^2}{\mu^2}\right) - 2\int_0^1 dx\, \frac{(1+x)(1-x)xm^2}{m^2x^2+\lambda^2(1-x)}\right]$$

$$= \frac{\alpha}{4\pi}\left[-\Delta_\epsilon - 4 + \ln\frac{m^2}{\mu^2} - 2\ln\frac{\lambda^2}{m^2}\right]$$

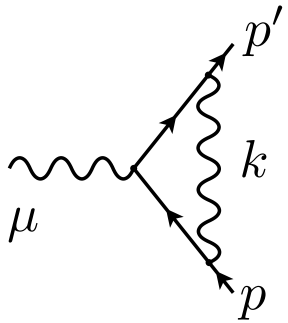
where we have taken the $\lambda \to 0$ limit in all cases that was possible.

- It is clear that the final result diverges in that limit, therefore implying that Z_2 is IR divergent. This is not a problem for the theory because δZ_2 is not a physical parameter. We will see that the IR diverges cancel for real processes.

- If we had taken a general gauge $(\xi \neq 1)$ we would find out that δm would not be changed but that Z_2 would show a gauge dependence. Again, in physical processes this should cancel in the end.

The Vertex

- The diagram contributing to the QED vertex at one-loop is



- In the Feynman gauge ($\xi = 1$) this gives a contribution,

$$
\begin{aligned}
ie\,\mu^{\epsilon/2}\Lambda_\mu^{loop}(p',p) =&(ie\,\mu^{\epsilon/2})^3 \int \frac{d^dk}{(2\pi)^d}(-i)\frac{g_{\rho\sigma}}{k^2-\lambda^2+i\varepsilon} \\
&\gamma^\sigma \frac{i[(\not{p}'+\not{k})+m]}{(p'+k)^2-m^2+i\varepsilon}\gamma_\mu \frac{i[(\not{p}+\not{k})+m]}{(p+k)^2-m^2+i\varepsilon}\gamma^\rho
\end{aligned}
$$

where Λ_μ is related to the full vertex Γ_μ through the relation

$$
\begin{aligned}
i\Gamma_\mu =&ie\,(\gamma_\mu + \Lambda_\mu^{loop} + \gamma_\mu\delta Z_1) \\
=&ie\,\left(\gamma_\mu + \Lambda_\mu^R\right)
\end{aligned}
$$

The Vertex

- The integral that defines $\Lambda_\mu^{loop}(p',p)$ is divergent. As before we expect to solve this problem by regularizing the integral, introducing counterterms and normalization conditions. The counterterm has the same form as the vertex
- The normalization constant is determined by requiring that in the limit $q = p' - p \to 0$ the vertex reproduces the tree level vertex because this is what is consistent with the definition of the electric charge in the $q \to 0$ limit of the Coulomb scattering.
- Also this should be defined for on-shell electrons. We have therefore that the normalization condition gives,

$$\overline{u}(p) \left(\Lambda_\mu^{loop} + \gamma_\mu \delta Z_1 \right) u(p) \Big|_{\not{p}=m} = 0$$

- If we are interested only in calculating δZ_1 and in showing that the divergences can be removed with the normalization condition then the problem is simpler. It can be done in two ways.

The Vertex: Calculating δZ_1: 1st method

- We use the fact that δZ_1 is to be calculated on-shell and for $p = p'$. Then

$$i\Lambda_\mu^{loop}(p,p) = e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2-\lambda^2+i\varepsilon}\gamma_\rho \frac{1}{\not p+\not k-m+i\varepsilon}\gamma_\mu \frac{1}{\not p+\not k-m+i\varepsilon}\gamma^\rho$$

- However we have

$$\frac{1}{\not p+\not k-m+i\varepsilon}\gamma_\mu \frac{1}{\not p+\not k-m+i\varepsilon} = -\frac{\partial}{\partial p^\mu}\frac{1}{\not p+\not k-m+i\varepsilon}$$

and therefore

$$\begin{aligned} i\Lambda_\mu^{loop}(p,p) =& -e^2\mu^\epsilon \frac{\partial}{\partial p^\mu}\int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2-\lambda^2+i\varepsilon}\gamma_\rho \frac{\not p+\not k+m}{(p+k)^2-m^2+i\varepsilon}\gamma^\rho \\ =& -i\frac{\partial}{\partial p^\mu}\Sigma^{loop}(p) \end{aligned}$$

The Vertex: Calculating δZ_1: 1st method

- We conclude then, that $\Lambda_\mu^{loop}(p,p)$ is related to the self-energy of the electron

$$\Lambda_\mu^{loop}(p,p) = -\frac{\partial}{\partial p^\mu}\Sigma^{loop}$$

- This result is one of the forms of the Ward-Takahashi identity
- On-shell we have

$$\Lambda_\mu^{loop}(p,p)\bigg|_{\not p=m} = -\frac{\partial \Sigma^{loop}}{\partial p^\mu}\bigg|_{\not p=m} = -\delta Z_2 \gamma_\mu$$

and the normalization condition gives

$$\delta Z_1 = \delta Z_2$$

- As we have already calculated δZ_2 then δZ_1 is determined.

The Vertex: Calculating δZ_1: $2^{\rm nd}$ method

- In this second method we do not rely in the Ward identity but just calculate the integrals for the vertex. For the moment we do not put $p'=p$ but we will assume that the vertex form factors are to be evaluated for on-shell spinors.

- Then we have

$$
\begin{aligned}
i\,\overline{u}(p')\Lambda_\mu^{loop}u(p) =& e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\,\frac{\overline{u}(p)\gamma_\rho\left[\not p' + \not k + m)\right]\gamma_\mu\left[\not p + \not k + m)\right]\gamma^\rho u(p)}{D_0 D_1 D_2}\\
=& e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\,\frac{\mathcal{N}_\mu}{D_0 D_1 D_2}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{N}_\mu =& \overline{u}(p)\Big[(-2+d)k^2\gamma_\mu + 4p\cdot p'\gamma_\mu + 4(p+p')\cdot k\,\gamma_\mu + 4m\,k_\mu\\
&- 4\not k\,(p+p')_\mu + 2(2-d)\not k k_\mu\Big]u(p)\\
D_0 =& k^2 - \lambda^2 + i\epsilon\\
D_1 =& (k+p')^2 - m^2 + i\epsilon\\
D_2 =& (k+p)^2 - m^2 + i\epsilon
\end{aligned}
$$

The Vertex: Calculating δZ_1: 2$^{\rm nd}$ method

- Now we use the results of the Appendix to express the ingrals with three denominators

$$
\begin{aligned}
r_1^\mu =& p'^\mu \quad ; \quad r_2^\mu = p^\mu \,, \ P^\mu = x_1 p'^\mu + x_2 p^\mu \\
C =& (x_1+x_2)^2 m^2 - x_1 x_2\, q^2 + (1-x_1-x_2)\lambda^2 \,, \quad q = p' - p
\end{aligned}
$$

- We get,

$$
\begin{aligned}
i\,\overline{u}(p')\Lambda_\mu^{loop} u(p) =& i\,\frac{\alpha}{4\pi}\Gamma(3)\int_0^1 dx_1 \int_0^{1-x_1} dx_2\ \frac{1}{2C} \\
&\Big\{\overline{u}(p')\gamma_\mu u(p)\Big[-(-2+d)(x_1^2 m^2 + x_2^2 m^2 + 2x_1x_2 p'\cdot p) - 4p'\cdot p \\
&+4(p+p')\cdot(x_1 p' + x_2 p) + \frac{(2-d)^2}{2} C\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right)\Big] \\
&+\overline{u}(p)u(p)m\Big[4(x_1p'+x_2p)_\mu - 4(p'+p)_\mu(x_1+x_2) \\
&-2(2-d)(x_1+x_2)(x_1p'+x_2p)_\mu\Big]\Big\} \\
=& \, i\,\overline{u}(p)\left[G(q^2)\,\gamma_\mu + H(q^2)\,(p+p')\right]u(p)
\end{aligned}
$$

The Vertex: Calculating δZ_1: 2$^{\rm nd}$ method

- In the last equation we have defined

$$G(q^2) \equiv \frac{\alpha}{4\pi}\left[\Delta_\epsilon - 2 - 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \ \ln \frac{(x_1+x_2)^2 m^2 - x_1 x_2 q^2 + (1-x_1-x_2)\lambda^2}{\mu^2}\right.$$

$$+\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \left(\frac{-2(x_1+x_2)^2 m^2 - x_1 x_2 q^2 - 4m^2 + 2q^2}{(x_1+x_2)^2 m^2 - x_1 x_2 q^2 + (1-x_1-x_2)\lambda^2}\right.$$

$$\left.\left. + \frac{2(x_1+x_2)(4m^2-q^2)}{(x_1+x_2)^2 m^2 - x_1 x_2 q^2 + (1-x_1-x_2)\lambda^2}\right)\right]$$

$$H(q^2) \equiv \frac{\alpha}{4\pi}\left[\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \ \frac{-2m\,(x_1+x_2) + 2m\,(x_1+x_2)^2}{(x_1+x_2)^2 m^2 - x_1 x_2 q^2 + (1-x_1-x_2)\lambda^2}\right]$$

- Finally we get for the renormalized vertex,

$$\overline{u}(p')\Lambda^R_\mu(p',p)u(p) = \overline{u}(p')\left[\left(G(q^2)+\delta Z_1\right)\gamma_\mu + H(q^2)\,(p+p')_\mu\right]u(p)$$

The Vertex: Calculating δZ_1: 2nd method

- As δZ_1 is calculated in the limit of $q = p' - p \to 0$ it is convenient to use the Gordon identity to get rid of the $(p'+p)^\mu$ term. We have

$$\overline{u}(p')\,(p'+p)_\mu\, u(p) = \overline{u}(p')\left[2m\gamma_\mu - i\sigma_{\mu\nu}\, q^\nu\right] u(p)$$

and therefore,

$$\begin{aligned}\overline{u}(p')\Lambda^R_\mu(p',p)u(p) =&\overline{u}(p')\left[\left(G(q^2)+2mH(q^2)+\delta Z_1\right)\gamma_\mu - i\,H(q^2)\,\sigma_{\mu\nu}\,q^\nu\right]u(p)\\ =&\overline{u}(p')\left[\gamma_\mu F_1(q^2)+\frac{i}{2m}\sigma_{\mu\nu}q^\nu F_2(q^2)\right]u(p)\end{aligned}$$

- In the last expression we have introduced the usual notation for the vertex form factors,

$$\begin{aligned}F_1(q^2) \equiv& G(q^2)+2mH(q^2)+\delta Z_1\\ F_2(q^2) \equiv& -2mH(q^2)\end{aligned}$$

The Vertex: Calculating δZ_1: 2$^{\text{nd}}$ method

- The normalization condition implies $F_1(0)=0$, that is,

$$\delta Z_1 = -G(0) - 2m\, H(0)$$

- We have therefore to calculate $G(0)$ and $H(0)$. In this limit the integrals are much simpler. We get (we change variables $x_1+x_2 \to y$),

$$G(0) = \frac{\alpha}{4\pi}\left[\Delta_\epsilon - 2 - 2\int_0^1 dx_1 \int_{x_1}^1 dy\, \ln\frac{y^2m^2+(1-y)\lambda^2}{\mu^2} \right.$$
$$\left. + \int_0^1 dx_1 \int_{x_1}^1 dy\, \frac{-2y^2m^2-4m^2+8ym^2}{y^2m^2+(1-y)\lambda^2}\right]$$

$$H(0) = \frac{\alpha}{4\pi}\int_0^1 dx_1 \int_{x_1}^1 dy\, \frac{-2m\,y+2m\,y^2}{y^2m^2+(1-y)\lambda^2}$$

- Now using

$$\int_0^1 dx_1 \int_{x_1}^1 dy\, \ln\frac{y^2m^2+(1-y)\lambda^2}{\mu^2} = \frac{1}{2}\left(\ln\frac{m^2}{\mu^2} - 1\right)$$

The Vertex: Calculating δZ_1: 2nd method

- And

$$\int_0^1 dx_1 \int_{x_1}^1 dy \, \frac{-2y^2m^2 - 4m^2 + 8ym^2}{y^2m^2 + (1-y)\lambda^2} = 7 + 2\ln\frac{\lambda^2}{m^2}$$

$$\int_0^1 dx_1 \int_{x_1}^1 dy \, \frac{-2m\,y + 2m\,y^2}{y^2m^2 + (1-y)\lambda^2} = -\frac{1}{m}$$

(where we took the limit $\lambda \to 0$ if possible) we get,

$$G(0) = \frac{\alpha}{4\pi}\left[\Delta_\epsilon + 6 - \ln\frac{m^2}{\mu^2} + 2\ln\frac{\lambda^2}{m^2}\right]$$

$$H(0) = -\frac{\alpha}{4\pi}\frac{1}{m}$$

- Substituting the previous expressions we get finally,

$$\delta Z_1 = \frac{\alpha}{4\pi}\left[-\Delta_\epsilon - 4 + \ln\frac{m^2}{\mu^2} - 2\ln\frac{\lambda^2}{m^2}\right]$$

in agreement with the previous result

The Vertex: Form factors

- The general form of the form factors $F_i(q^2)$, for $q^2 \neq 0$, is quite complicated. We give here only the result for $q^2 < 0$

$$F_1(q^2) = \frac{\alpha}{4\pi}\left\{\left(2\ln\frac{\lambda^2}{m^2}+4\right)(\theta\coth\theta-1)-\theta\tanh\frac{\theta}{2}-8\coth\theta\int_0^{\theta/2}\beta\tanh\beta d\beta\right\}$$

$$F_2(q^2) = \frac{\alpha}{2\pi}\frac{\theta}{\sinh\theta}$$

where

$$\sinh^2\frac{\theta}{2} = -\frac{q^2}{4m^2}.$$

- In the limit of zero transferred momenta $(q = p' - p = 0)$ we get

$$\begin{cases} F_1(0) = 0 \\ F_2(0) = \dfrac{\alpha}{2\pi} \end{cases}$$

a result that we will use while discussing the anomalous magnetic moment of the electron.