TÉCNICO LISBOA

Advanced Quantum Field Theory
Chapter 3
Covariant Perturbation Theory

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Lecture 4

U matrix

- In this chapter we are going to develop a method to evaluate the Green functions of a given theory. As we have seen in the two previous chapters, we only know how to calculate for free fields, like the in an out fields.

- However, the Green functions we are interested in, are given in terms of the physical interacting fields, and we do not know how to operate with these. We are going to see how to express the physical fields as perturbative series in terms of free in fields.

- We start by defining the U matrix. We will be considering, for the moment, only scalar fields, later we will see the other cases. The physical interacting fields $\varphi(\vec{x},t)$ and their conjugate momenta $\pi(\vec{x},t)$, satisfy the same equal time commutation relations than the in fields, $\varphi_{in}(\vec{x},t)$ and $\pi_{in}(\vec{x},t)$.

- Also, both φ and φ_{in} form a complete set of operators, in the sense that any state, free or interacting, can be obtained by application of φ_{in} or φ in the vacuum. This implies that there should be an unitary transformation $U(t)$ that relates φ with φ_{in}, that is,

$$\varphi(\vec{x},t) = U^{-1}(t)\varphi_{in}(\vec{x},t)U(t)$$
$$\pi(\vec{x},t) = U^{-1}(t)\pi_{in}(\vec{x},t)U(t)$$

U matrix

- The dynamics of U can be obtained from the equations of motion for $\varphi(x)$ and $\varphi_{in}(x)$. These are,

$$\frac{\partial\varphi_{in}}{\partial t}(x) = i[H_{in}(\varphi_{in},\pi_{in}),\varphi_{in}]\,, \qquad \frac{\partial\pi_{in}}{\partial t}(x) = i[H_{in}(\varphi_{in},\pi_{in}),\pi_{in}]$$

and

$$\frac{\partial\varphi}{\partial t}(x) = i[H(\varphi,\pi),\varphi]\,, \qquad \frac{\partial\pi}{\partial t}(x) = i[H(\varphi,\pi),\pi]$$

- Then we get,

$$\begin{aligned}\dot\varphi_{in}(x) =& \frac{\partial}{\partial t}\left[U(t)\varphi(x)U^{-1}(t)\right]\\ =& \left[\dot U(t)U^{-1}(t),\varphi_{in}\right] + i\left[H(\varphi_{in},\pi_{in}),\varphi_{in}(x)\right]\\ =& \dot\varphi_{in}(x) + \left[\dot U U^{-1} + iH_I(\varphi_{in},\pi_{in}),\varphi_{in}\right]\end{aligned}$$

U matrix

- Where we defined

$$H_I(\varphi_{in}, \pi_{in}) = H(\varphi_{in}, \pi_{in}) - H_{in}(\varphi_{in}, \pi_{in}) \equiv H_I(t)$$

- In a similar way

$$\dot{\pi}_{in}(x) = \dot{\pi}_{in} + \left[\dot{U}U^{-1} + iH_I(\varphi_{in}, \pi_{in}), \pi_{in}\right]$$

- From these equations we obtain,

$$i\dot{U}U^{-1} = H_I(t) + E_0(t)$$

where $E_0(t)$ commutes with φ_{in} and π_{in} and is therefore a time dependent c-number, not an operator.

- Defining

$$H_I'(t) = H_I(t) + E_0(t)$$

we get a differential equation for $U(t)$, that reads,

$$i\frac{\partial U(t)}{\partial t} = H_I'(t)U(t)$$

U matrix

- The solution of this equation in terms of the in fields, is the basis of the covariant perturbation theory. To integrate the equation for U we need an initial condition. For that we introduce the operator

$$U(t,t') \equiv U(t)U^{-1}(t')$$

where $t \geq t'$, and that obviously satisfies

$$U(t,t) = 1$$

- It is easy to see that $U(t,t')$ also satisfies the differential equation, that is,

$$i\frac{\partial U(t,t')}{\partial t} = H'_I(t)U(t,t')$$

and has the correct initial condition.

- To proceed we start by transforming the differential equation in an equivalent integral equation, that is,

$$U(t,t') = 1 - i\int_{t'}^{t} dt_1 H'_I(t_1)U(t_1,t')$$

U matrix

- Notice that we have not solved the problem because $U(t,t')$ appears on both sides of the equation. However, we can iterate the equation to get the expansion,

$$\begin{aligned} U(t,t') =& 1 - i\int_{t'}^{t} dt_1 H_I'(t_1) + (-i)^2 \int_{t'}^{t} dt_1 H_I'(t_1) \int_{t'}^{t_1} dt_2 H_I'(t_2) \\ & + \cdots + (-i)^n \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \cdots \int_{t'}^{t_{n-1}} dt_n H_I'(t_1) \cdots H_I'(t_n) \\ & + \cdots \end{aligned}$$

- Of course this expansion can only be useful if H_I contains a small parameter and, because of that, we can truncate the expansion at certain order in that parameter.

- Coming back, as $t_1 \geq t_2 \geq, \cdots t_n$, the product is time-ordered and we can therefore write

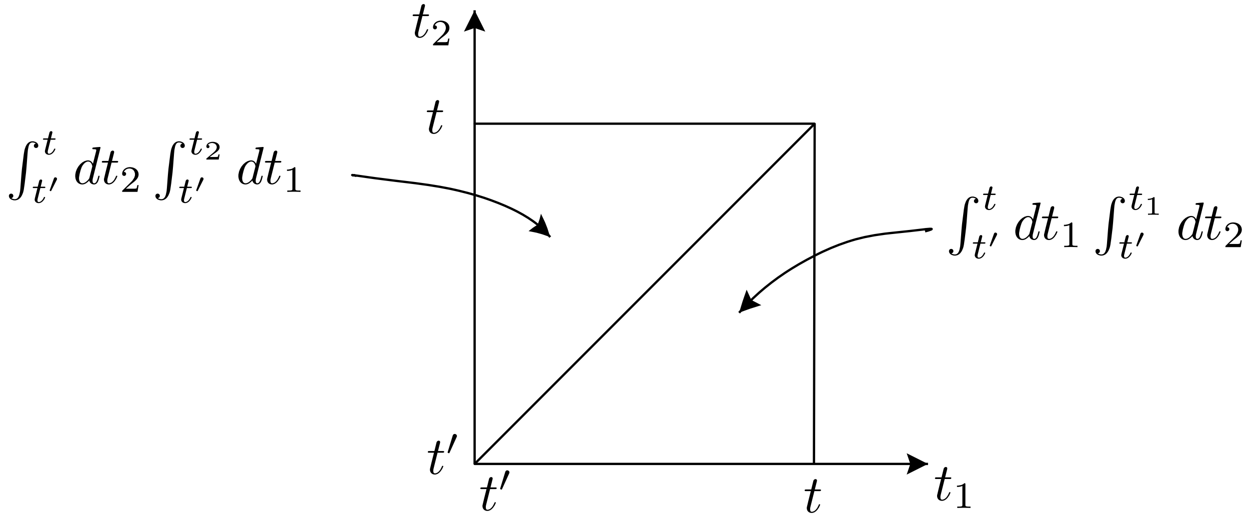
$$U(t,t') = 1 + \sum_{n=1}^{\infty} (-i)^n \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \cdots \int_{t'}^{t_{n-1}} dt_n T(H_I'(t_1) \cdots H_I'(t_n))$$

U matrix

- Using the symmetry t_1, t_2 we can write,

$$\int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 T(H'_I(t_1)H'_I(t_2)) = \int_{t'}^{t} dt_2 \int_{t'}^{t_2} dt_1 T(H'_I(t_1)H'_I(t_2))$$
$$= \frac{1}{2}\int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 T(H'_I(t_1)H'_2(t_2))$$

- This follows from the following figure



U matrix

- In general, for n integrations, instead of $\frac{1}{2}$ we will have $\frac{1}{n!}$, and we get,

$$\begin{aligned} U(t,t') =& 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{t'}^{t} dt_1 \cdots \int_{t'}^{t} dt_n T(H'_I(t_1) \cdots H'_I(t_n)) \\ \equiv& T\left(\exp[-i\int_{t'}^{t} dt H'_I(t)]\right) \\ =& T\left(\exp[-i\int_{t'}^{t} d^4x \mathcal{H}_I(\varphi_{in})]\right) \end{aligned}$$

  where the time-ordered product is to be interpreted expanding the exponential.

- The operators U satisfy the following multiplication rule

$$U(t,t') = U(t,t'')U(t'',t')$$

  which can be seen using the definition, or from the explicit expression.

- From this property, we can obtain,

$$U(t,t') = U^{-1}(t',t)$$

Perturbative expansion of Green functions

- As we saw in the previous chapter, the LSZ technique reduces the evaluation of the elements of the S matrix to a basic ingredient, the so-called Green functions of the theory.

- These are expectation values of time-ordered products of the Heisenberg fields, $\varphi(x)$,

$$G(x_1, \cdots, x_n) \equiv \langle 0| T\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)|0\rangle$$

- The basic idea for the evaluation of the Green functions consists in expressing the fields $\varphi(x)$ in terms of the fields $\varphi_{in}(x)$, using the operator U. We get

$$\begin{aligned} G(x_1, \cdots, x_n) &= \langle 0| T(U^{-1}(t_1)\varphi_{in}(x_1)U(t_1,t_2)\varphi_{in}(x_2)U(t_2,t_3)\cdots \\ &\qquad \cdots U(t_{n-1},t_n)\varphi_{in}(x_n)U(t_n))|0\rangle \\ &= \langle 0| T(U^{-1}(t)U(t,t_1)\varphi_{in}(x_1)U(t_1,t_2)\cdots \\ &\qquad \cdots U(t_{n-1},t_n)\varphi_{in}(x_n)U(t_n,-t)U(-t))|0\rangle \end{aligned}$$

where t is a time that we will let go to ∞. When $t\to\infty$, t is later than all the t_i and $-t$ is earlier than all the times t_i.

Perturbative expansion of Green functions

- Therefore we can take $U^{-1}(t)$ e $U(-t)$ out of the time-ordered product. Using the multiplicative property of the operator U we can then write,

$$G(x_1, \cdots, x_n) = \langle 0| \, U^{-1}(t) T \left(\varphi_{in}(x_1) \cdots \varphi_{in}(x_n) \exp[-i \int_{-t}^{t} H_I'(t')dt'] \right) U(-t) \, |0\rangle$$

where the T is meant to be applied after expanding the exponential.

- If it were not for the presence of the operators $U^{-1}(t)$ and $U(-t)$, we would have expressed $G(x_1 \cdots x_n)$ completely in terms of the in fields.

- Now we are going to show that the vacuum is an eigenstate of the operator $U(t)$. We consider an arbitrary state $|\alpha p;\ in\rangle$ that contains one particle of momentum p, all the other quantum numbers being denoted by α. To simplify, we continue considering the case of the scalar field. We have then,

$$\langle \alpha p; in | U(-t) | 0 \rangle = \langle \alpha; in | a_{in}(p) U(-t) | 0 \rangle$$

$$= -\, i \int d^3x f_p^*(\vec{x}, -t') \left(\frac{\overrightarrow{\partial}}{\partial t'} - \frac{\overleftarrow{\partial}}{\partial t'} \right) \langle \alpha; in| \, \varphi_{in}(\vec{x}, -t') U(-t) \, |0\rangle$$

where $f_p(\vec{x}, t) = e^{-ip\cdot x}$.

Perturbative expansion of Green functions

- We use now express $\varphi_{in}(\vec{x},-t)$ in terms of $\varphi(\vec{x},-t)$. We get,

$$\langle \alpha p; in|U(-t)|0\rangle =$$

$$= -i \int d^3x f_p^*(\vec{x},-t') \overleftrightarrow{\partial}'_0 \langle \alpha; in|U(-t')\varphi(\vec{x},-t')U^{-1}(-t')U(-t)|0\rangle$$

$$= -i \int d^3x f_p^*(\vec{x},-t') \Big[-\overleftarrow{\partial}_{0'} \langle \alpha; in|\, U(-t')\varphi(\vec{x},-t')U^{-1}(-t')U(-t)\,|0\rangle$$

$$+ \langle \alpha; in|\, U(-t')\dot{\varphi}(\vec{x},-t')U^{-1}(-t')U(-t)\,|0\rangle \Big]$$

$$+ i \int' d^3x f_p^*(\vec{x},-t') \langle \alpha; in|\, \dot{U}(-t')\varphi(\vec{x},-t')U^{-1}(-t')U(-t)\,|0\rangle$$

$$+ i \int d^3x f_p^*(\vec{x},-t') \langle \alpha; in|\, U(-t')\varphi(\vec{x},-t')\dot{U}^{-1}(-t')U(-t)\,|0\rangle$$

- We take now the $t = t' \to \infty$ limit.

Perturbative expansion of Green functions

- Then

$$\langle \alpha p; in|\, U(-t)\, |0\rangle = \sqrt{Z}\, \langle \alpha; in|\, U(-t) a_{in}(p)\, |0\rangle$$
$$+ \langle \alpha; in|\, \dot{U}(-t)\varphi(\vec{x},-t) + U(-t)\varphi(\vec{x},-t)\dot{U}^{-1}(-t)U(-t)\, |0\rangle$$

- Now the first term vanishes because $a_{in}(p)\,|0\rangle = 0$. The second term also vanishes because we have (we omit the arguments to simplify the notation),

$$\begin{aligned}\dot{U}\varphi + U\varphi\dot{U}^{-1}U =& \dot{U}U^{-1}\varphi_{in}U + \varphi_{in}U\dot{U}^{-1}U\\ =& \dot{U}U^{-1}\varphi_{in}U - \varphi_{in}\dot{U}U^{-1}U\\ =& [\dot{U}U^{-1}, \varphi_{in}]U = -i[H_I, \varphi_{in}]U = 0\end{aligned}$$

where we have assumed that the interactions have no derivative.

- We conclude then that,

$$\lim_{t\to\infty} \langle \alpha p; in|U(-t)|0\rangle = 0$$

for all states in that contain at least one particle.

Perturbative expansion of Green functions

- This means that,

$$\lim_{t\to\infty} U(-t)\left|0\right\rangle = \lambda_- \left|0\right\rangle$$

- In a similar way we could show that,

$$\lim_{t\to\infty} U(t)\left|0\right\rangle = \lambda_+ \left|0\right\rangle$$

- Returning now to the expression for the Green function, we can write,

$$G(x_1,\cdots x_n) = \lambda_-\lambda_+^* \left\langle 0\right| T\left(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n)\exp\left[-i\int_{-t}^{t} H_I'(t')dt'\right]\right)\left|0\right\rangle$$

  The dependence in the operator U disappeared from the expectation value.

- To proceed, let us evaluate the constants $\lambda_\pm$, or more to the point, the combination $\lambda_-\lambda_+^*$ that appears. We get (in the limit $\to\infty$),

$$\begin{aligned}\lambda_-\lambda_+^* &= \left\langle 0\right| U(-t)\left|0\right\rangle \left\langle 0\right| U^{-1}(t)\left|0\right\rangle = \left\langle 0\right| U(-t)U^{-1}(t)\left|0\right\rangle = \left\langle 0\right| U(-t,t)\left|0\right\rangle \\ &= \left\langle 0\right| T\left(\exp\left[+i\int_{-t}^{t} dt' H_I'(t')\right]\right)\left|0\right\rangle = \left\langle 0\right| T\left(\exp\left[-i\int_{-t}^{t} dt' H_I'(t')\right]\right)\left|0\right\rangle^{-1}\end{aligned}$$

Perturbative expansion of Green functions

- Using this result we can write the Green function in the form (when $t \to \infty$)

$$G(x_1, \cdots, x_n) = \frac{\langle 0| T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n)\exp[-i\int_{-t}^{t} dt' H'_I(t')]) |0\rangle}{\langle 0| T(\exp[-i\int_{-t}^{t} dt' H'_I(t')) |0\rangle}$$

- Before we write the final expression, we can now introduce the number $E_0(t)$. For that we recall that, $H'_I = H_I + E_0$ and noticing that E_0 is not an operator, we get a factor $\exp[-i\int_{-t}^{t} dt' E_0(t')]$ both in the numerator and denominator, canceling out in the final result.

- The final result can then be obtained substituting H'_I by H_I. We get,

$$G(x_1 \cdots, x_n) = \frac{\langle 0| T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n)\exp[-i\int_{-t}^{t} dt' H_I(t')]) |0\rangle}{\langle 0| T(\exp[-i\int_{-t}^{t} dt' H_I(t')) |0\rangle}$$

$$= \frac{\sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int_{-\infty}^{\infty} d^4y_1 \cdots d^4y_m \langle 0| T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n)\mathcal{H}_I(y_1)\cdots\mathcal{H}_I(y_m) |0\rangle}{\sum_{m=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} d^4y_1 \cdots d^4y_m \langle 0| T(\mathcal{H}_I(y_1)\cdots\mathcal{H}_I(y_m)) |0\rangle}$$

This equation is the fundamental result. The Green functions have been expressed in terms of the *in* fields whose algebra we know.

Wick's theorem

- To evaluate the amplitudes we have to move the annihilation operators to the right until they act on the vacuum.
- The final result from these manipulations can be stated in the form of a theorem, known as Wick's theorem, which reads,

$$T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n)) =$$

$$= :\varphi_{in}(x_1)\cdots\varphi_{in}(x_n): + [\langle 0|\, T(\varphi_{in}(x_1)\varphi_{in}(x_2))\,|0\rangle :\varphi_{in}(x_3)\cdots\varphi_{in}(x_n): + \text{perm.}$$

$$+ \langle 0|\, T(\varphi_{in}(x_1)\varphi_{in}(x_2))\,|0\rangle \langle 0|\, T(\varphi_{in}(x_3)\varphi_{in}(x_4))\,|0\rangle :\varphi_{in}(x_5)\cdots\varphi_{in}(x_n): + \text{perm.}$$

$$+ \cdots$$

$$+ \begin{cases} \langle 0|\, T(\varphi_{in}(x_1)\varphi_{in}(x_2))\,|0\rangle \cdots \langle 0|\, T(\varphi_{in}(x_{n-1})\varphi_{in}(x_n))\,|0\rangle + \text{perm.} & \text{n even} \\ \langle 0|\, T(\varphi_{in}(x_1)\varphi_{in}(x_2)\,|0\rangle \cdots \langle 0|\, T(\varphi_{in}(x_{n-2})\varphi_{in}(x_{n-1}))\,|0\rangle\, \varphi_{in}(x_n) + \text{perm.} & \text{n odd} \end{cases}$$

Wick's theorem: Proof

- The proof of the theorem is done by induction. For $n = 1$ it is certainly true (and trivial).
- Also for $n = 2$ we can shown that

$$T(\varphi_{in}(x_1)\varphi_{in}(x_2)) =: \varphi_{in}(x_1)\varphi_{in}(x_2) : +\text{c-number}$$

where the *c-number* comes from the commutations that are needed to move the annihilation operators to the right. To find this constant, we do not have to do any calculation, just to notice that

$$\langle 0| : \cdots : |0\rangle = 0$$

Then

$$T(\varphi_{in}(x_1)\varphi_{in}(x_2)) =: \varphi_{in}(x_1)\varphi_{in}(x_2) : + \langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_2)) |0\rangle$$

which is in agreement with the theorem.

- Continuing with the induction, let us assume that it is valid for a given n. We have to show that it remains valid for $n + 1$. Let us consider then $T(\varphi_{in}(x_1) \cdots \varphi_{in}(x_{n+1}))$ and let us assume that t_{n+1} is the earliest time.

Wick's theorem: Proof

- Then

$$
\begin{aligned}
&T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_{n+1})) = \\
&\qquad = T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n))\varphi_{in}(x_{n+1}) \\
&\qquad = :\varphi_{in}(x_1)\cdots\varphi_{in}(x_n):\varphi_{in}(x_{n+1}) \\
&\qquad\quad + \sum_{\text{perm}} \langle 0|\, T(\varphi_{in}(x_1)\varphi_{in}(x_2))\,|0\rangle :\varphi_{in}(x_3)\cdots\varphi_{in}(x_n):\varphi_{in}(x_{n+1}) \\
&\qquad\quad + \cdots
\end{aligned}
$$

- To write this equation in the form of the theorem it is necessary to find the rule showing how to introduce $\varphi_{in}(x_{n+1})$ inside the normal product. For that, we introduce the notation,

$$
\varphi_{in}(x) = \varphi_{in}^{(+)}(x) + \varphi_{in}^{(-)}(x)
$$

where $\varphi_{in}^{(+)}(x)$ contains the annihilation operator and $\varphi_{in}^{(-)}(x)$ the creation operator.

Wick's theorem: Proof

- Then we can write,

$$:\varphi_{in}(x_1)\cdots\varphi_{in}(x_n) := \sum_{A,B}\prod_{i\in A}\varphi_{in}^{(-)}(x_i)\prod_{j\in B}\varphi_{in}^{(+)}(x_j)$$

where the sum runs over all the sets A, B that constitute partitions of the n indices.

- Then

$$\begin{aligned}
&:\varphi_{in}(x_1)\cdots\varphi_{in}(x_n):\varphi_{in}(x_{n+1}) = \\
&\quad=\sum_{A,B}\prod_{i\in A}\varphi_{in}^{(-)}(x_i)\prod_{j\in B}\varphi_{in}^{(+)}(x_j)[\varphi_{in}^{(+)}(x_{n+1})+\varphi_{in}^{(-)}(x_{n+1})] \\
&\quad=\sum_{A,B}\prod_{i\epsilon A}\varphi_{in}^{(-)}(x_i)\prod_{j\in B}\varphi_{in}^{(+)}(x_j)\varphi_{in}^{(+)}(x_{n+1}) \\
&\qquad+\sum_{A,B}\prod_{i\in A}\varphi_{in}^{(-)}(x_i)\varphi_{in}^{(-)}(x_{n+1})\prod_{j\in B}\varphi_{in}^{(+)}(x_j) \\
&\qquad+\sum_{A,B}\prod_{i\in A}\varphi_{in}^{(-)}(x_i)\sum_{k\in B}\prod_{j\in B j\neq k}\varphi_{in}^{(+)}(x_j)\left\langle 0\right|\varphi_{in}^{(+)}(x_k)\varphi_{in}^{(-)}(x_{n+1})\left|0\right\rangle
\end{aligned}$$

Wick's theorem: Proof

- We can now write,

$$\langle 0|\, \varphi_{in}^{(+)}(x_k)\varphi_{in}^{(-)}(x_{n+1})\,|0\rangle = \langle 0|\, \varphi_{in}(x_k)\varphi_{in}(x_{n+1})\,|0\rangle$$

$$= \langle 0|\, T(\varphi_{in}(x_k)\varphi_{in}(x_{n+1}))\,|0\rangle$$

where we have used the fact that t_{n+1} is the earliest time.

- Therefore

$$: \varphi_{in}(x_1)\cdots\varphi_{in}(x_n) : \varphi_{in}(x_{n+1}) =: \varphi_{in}(x_1)\cdots\varphi_{in}(x_{n+1}) :$$

$$+ \sum_k : \varphi_{in}(x_1)\cdots\varphi_{in}(x_{k-1})\varphi_{in}(x_{k+1})\cdots\varphi_{in}(x_n) : \langle 0|\, T(\varphi_{in}(x_k)\varphi_{in}(x_{n+1}))\,|0\rangle$$

- With this result we obtain the general form of for the $n+1$ case, ending the proof of the theorem.

- To fully understand the theorem, it is important to do in detail the case $n=4$, to see how things work. The importance of the Wick's theorem for the applications comes from the following two corollaries.

Wick's theorem: Corollaries

- **Corollary 1** : If n is odd, then $\langle 0|\, T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n))\,|0\rangle = 0$, as results trivially from

$$\langle 0|\,\varphi_{in}(x)\,|0\rangle = 0$$

- **Corollary 2**: If n is even

$$\langle 0|\, T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n))\,|0\rangle =$$
$$= \sum_{\text{perm}} \delta_p \,\langle 0|\, T(\varphi_{in}(x_1)\varphi_{in}(x_2))\,|0\rangle \cdots \langle 0|\, T(\varphi_{in}(x_{n-1})\varphi_{in}(x_n))\,|0\rangle$$

where δ_p is the sign of the permutation that is necessary to introduce in case of fermion fields. This result, is in practice the most important one

- Therefore the vacuum expectation value of the time-ordered product of n operators that appear in the general formula, are obtained considering all the vacuum expectation values of the fields taken two by two (*contractions*) in all possible ways.

- Now these *contractions* are nothing else than the Feynman propagators for free fields.

Wick's theorem: Examples

- For instance for four fields

$$
\begin{aligned}
&\langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_2)\varphi_{in}(x_3)\varphi_{in}(x_4)) |0\rangle \\
&\qquad = \langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_2)) |0\rangle \langle 0| T(\varphi_{in}(x_3)\varphi_{in}(x_4) |0\rangle \\
&\qquad\quad + \langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_3)) |0\rangle \langle 0| T(\varphi_{in}(x_2)\varphi_{in}(x_4)) |0\rangle \\
&\qquad\quad + \langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_4)) |0\rangle \langle 0| T(\varphi_{in}(x_2)\varphi_{in}(x_3)) |0\rangle \\
&\qquad =\Delta_F(x_1-x_2)\Delta_F(x_3-x_4)+\Delta_F(x_1-x_3)\Delta_F(x_2-x_4) \\
&\qquad\quad + \Delta_F(x_1-x_4)\Delta_F(x_2-x_3)
\end{aligned}
$$

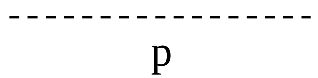
- In this expression

$$
\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\epsilon} e^{-ik(x-y)}
$$

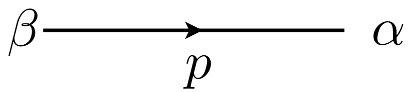
is the Feynman propagator for the free field theory in the case of scalar fields.

Wick's theorem: Examples

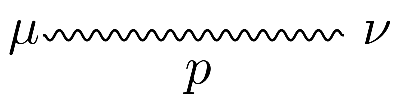
- It is convenient to use a graphical (diagrammatic) representation for these propagators. We have in configuration space,



$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x-y)}$$



$$S_F(x-y)_{\alpha\beta} = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p} + m)_{\alpha\beta}}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}$$



$$D_F^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{-ig^{\mu\nu}}{k^2 + i\epsilon} e^{-ik\cdot(x-y)}$$

respectively for scalar, spinor and photon (in the Feynman gauge) fields.

Wick's theorem: Examples

- As the interaction Hamiltonian is normal ordered, there will be no contractions between the fields that appear in $\mathcal{H}_I$. The fields in $\mathcal{H}_I$ can only contract with fields outside.

- In this way the contractions will connect the points corresponding to $\mathcal{H}_I$, the so-called vertices, to either external points or points in another $\mathcal{H}_I$, corresponding to another vertex.

- To illustrate this point let us consider the $\lambda\varphi^4$ theory where,

$$\mathcal{H}_I(x) = \frac{1}{4!}\lambda : \varphi^4_{in}(x) :$$

- Then a contribution of order λ^2 to $G(x_1, x_2, x_3, x_4)$ comes from the term,

$$\frac{\lambda^2}{(4!)^2} \langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_2)\varphi_{in}(x_3)\varphi_{in}(x_4) : \varphi^4_{in}(y_1) :: \varphi^4_{in}(y_2) : |0\rangle$$

and leads to the following diagrams

Wick's theorem: Examples

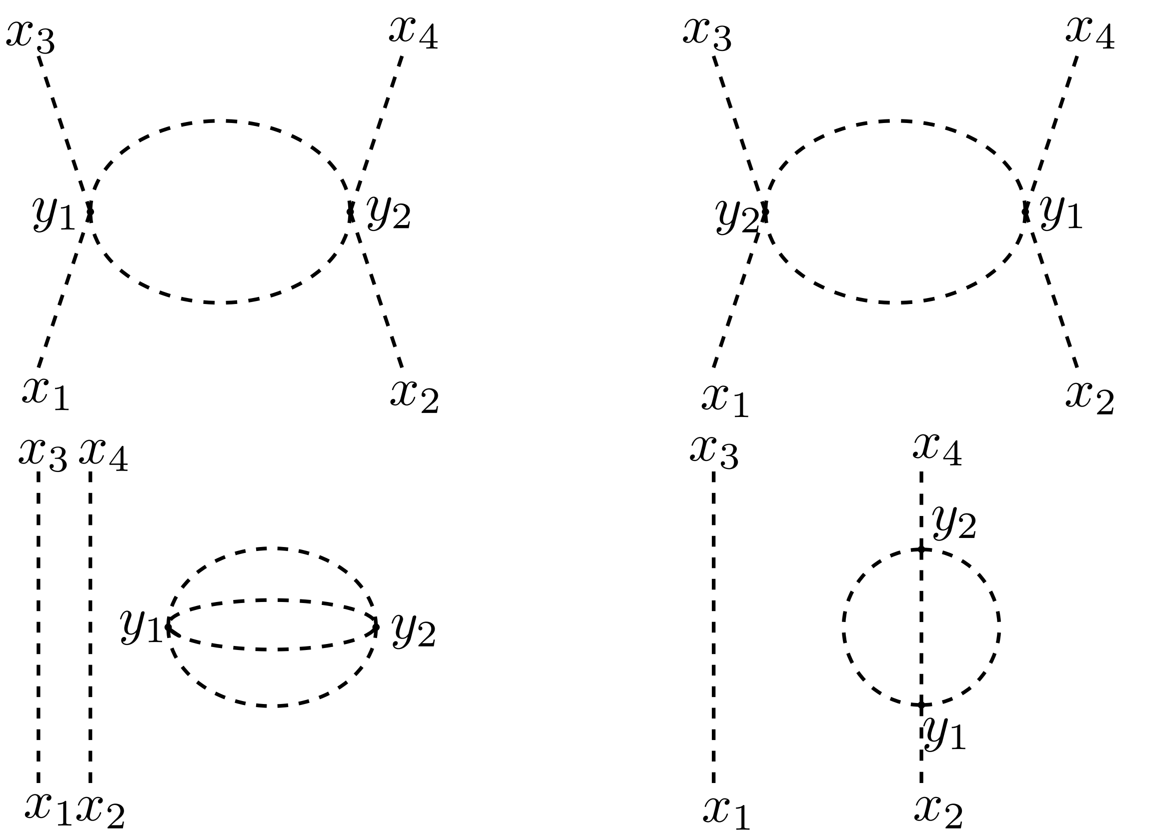


Figure 1: Some of the diagrams of order λ^2 to $G(x_1, x_2, x_3, x_4)$

- In these diagrams, the interaction is represented by four lines coming from one point, y_1 or y_2. These lines are contractions between one field from one $\mathcal{H}_I$ with other field that might belong either to another $\mathcal{H}_I$, or be one of the external fields in $G(x_1 \cdots x_4)$.

Wick's theorem: Examples

- To obtain the Feynman rules we are left with a combinatorial problem.
- We are not going to find them in configuration space, as they are much easier to express in momentum space, as we will see in the following.
- In the previous figure the diagrams a), b) and d) are called *connected* while the diagram c) is called *disconnected*. One diagram is *disconnected* when there is a part of the diagram that is not connected in any way to an external line. We will see in the following that these diagrams do not contribute to the Green functions.
- Diagram d) is *connected* but is also called *reducible* because it can be obtained by multiplication of simpler Green functions. As we will see only the *irreducible* diagrams are important.

Vacuum–Vacuum amplitudes

- We have seen in the previous section examples of the numerator of the fundamental equation for the Green functions.

- Let us now look at the denominator, the so-called vacuum-vacuum amplitudes. Continuing with the example of $\lambda\varphi^4$, some of the diagrams contributing for these amplitudes are shown below

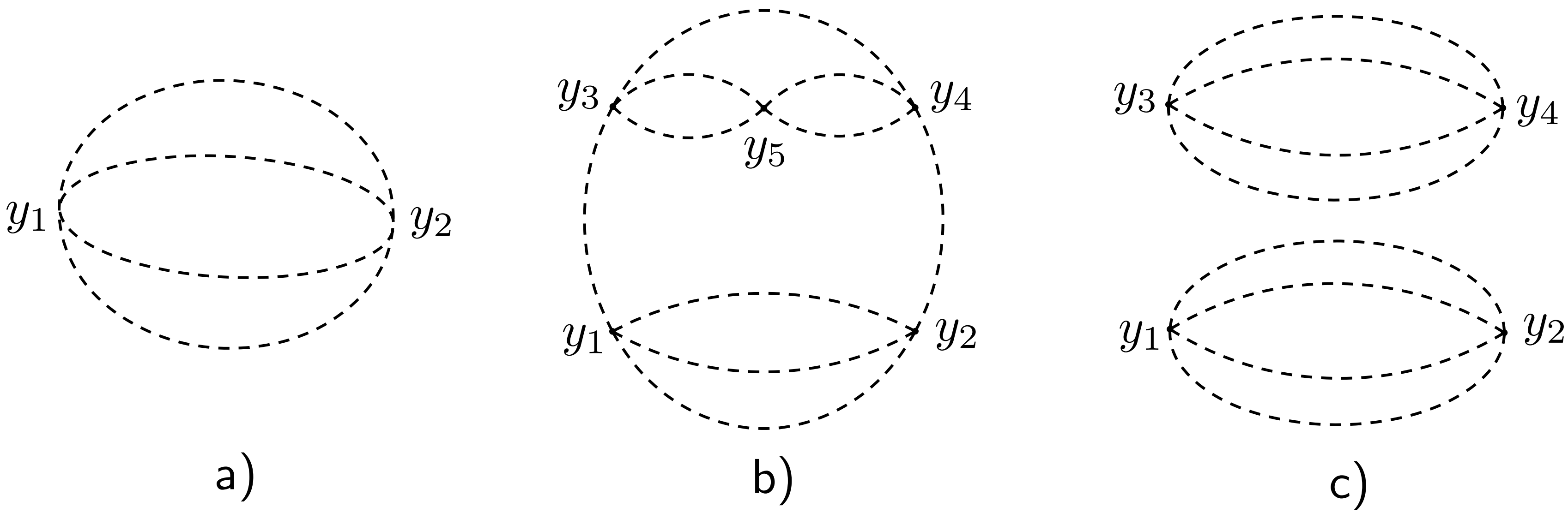


Figure 2: Some vacuum–vacuum amplitudes of order λ^2 in $\lambda\varphi^4$

Vacuum–Vacuum amplitudes

- The diagrams associated with the numerator can be separated into connected and disconnected parts.

- For all diagrams that have as connected part a contribution of order s in the interaction $\mathcal{H}_I$, the numerator of $G(x_1 \cdots x_n)$ takes the form,

$$\sum_{p=0}^{\infty} \frac{(-i)^p}{p!} \int d^4y_1 \cdots d^4y_p \, \langle 0| \, T(\varphi_{in}(x_1) \cdots \varphi_{in}(x_n) \mathcal{H}_I(y_1) \cdots \mathcal{H}_I(y_s)) \, |0\rangle_c$$

$$\times \frac{p!}{s!(p-s)!} \langle 0| \, T(\mathcal{H}_I(y_{s+1}) \cdots \mathcal{H}_I(y_p)) \, |0\rangle$$

where the subscript c indicates that only the connected parts are included.

- The combinatorial factor

$$\begin{pmatrix} p \\ s \end{pmatrix} = \frac{p!}{s!(p-s)!}$$

is the number of ways in which we can extract s terms $\mathcal{H}_I$ from a set of p terms.

Vacuum–Vacuum amplitudes

- We write then $(r = p - s)$

$$\frac{(-i)^s}{s!}\int d^4y_1\cdots d^4y_s \left\langle 0\right| T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n)\mathcal{H}_I(y_1)\cdots\mathcal{H}(y_s))\left|0\right\rangle_c$$
$$\times\sum_{r=0}^{\infty}\frac{(-i)^r}{r!}\int d^4z_1\cdots d^4z_r \left\langle 0\right| T(\mathcal{H}_I(z_1)\cdots\mathcal{H}_I(z_r))\left|0\right\rangle$$

- This equation has the form of a connected diagram of order s times an infinite series of vacuum-vacuum amplitudes, that cancels exactly against the denominator. This is true for all orders, and therefore we can write,

$$G(x_1,\cdots x_n) = \frac{\sum_i G_i(x_1\cdots x_n)}{\sum_k D_k} = \frac{(\sum_i G_i^c(x_1,\cdots x_n))(\sum_k D_k)}{\sum_k D_k}$$
$$= \sum_i G_i^c(x_1\cdots x_n)$$

where G_i^c are the connected diagrams and D_k the disconnected ones.

- This result means that we can simply ignore completely the disconnected diagrams and consider only the connected ones when evaluating the Green functions. These are simply the sum of all connected diagrams, simplifying enormously the calculations.

Feynman rules for $\lambda\varphi^4$

- To understand how the Feynman rules appear, let us consider the case of a real scalar field with an interaction of the form,

$$\mathcal{H}_I = \frac{\lambda}{4!} : \varphi_{in}^4 := -\mathcal{L}_I$$

- To be more precise we consider two particles in the initial and final state. Then the S matrix element is,

$$\begin{aligned} S_{fi} =& \langle p_1' p_2'; out | p_1 p_2; in \rangle \\ =& (i)^4 \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{-ip_1\cdot x_1 - ip_2\cdot x_2 + ip_i' x_3 + ip_2'\cdot x_4} \\ & (\Box_{x_1} + m^2) \cdots (\Box_{x_4} + m^2) \langle 0 | T(\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)) | 0 \rangle \end{aligned}$$

- For the Green function we use the previous expressions and we obtain,

$$\begin{aligned} G(x_1, x_2, x_3, x_4) =& \sum_{p=0}^{\infty} \frac{(-i\lambda)^p}{p!} \int d^4z_1 \cdots d^4z_p \\ & \langle 0 | T(\varphi_{in}(x_1)\varphi_{in}(x_2)\varphi_{in}(x_3)\varphi_{in}(x_4) : \frac{\varphi_{in}^4(z_1)}{4!} : \cdots : \frac{\varphi_{in}^4(z_p)}{4!} :) | 0 \rangle_c \end{aligned}$$

Feynman rules for $\lambda\varphi^4$

- As the case $p=0$ is trivial (there is no interaction) we begin by the $p=1$ case. Then the Green function is,

$$G(x_1,x_2,x_3,x_4)=(-i\lambda)\int d^4z\,\langle 0|\,T\left(\varphi_{in}(x_1)\varphi_{in}(x_2)\varphi_{in}(x_3)\varphi_{in}(x_4) : \frac{\varphi^4_{in}(z)}{4!} :\right)|0\rangle$$
$$=(-i\lambda)\frac{4!}{4!}\int d^4z\Delta_F(x_1-z)\Delta_F(x_2-z)\Delta_F(x_3-z)\Delta_F(x_4-z)$$

- This corresponds, in the configuration space, the diagram

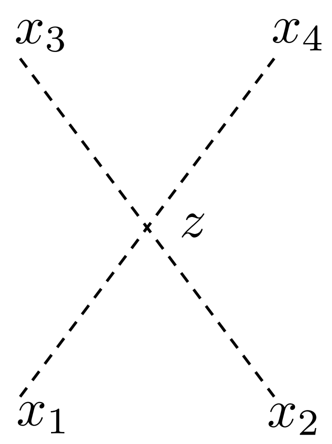


Figure 3: Vertex in the $\lambda\phi^4$ theory.

Feynman rules for $\lambda\varphi^4$

❒ To proceed, we introduce the Fourier transform of the propagators, that is,

$$\Delta_F(x_1 - z) = \int \frac{d^4 q_1}{(2\pi)^4} e^{-iq_1\cdot(x_1-z)} \Delta_F(q_1)$$

where

$$\Delta_F(q_1) = \frac{i}{q_1^2 - m^2}$$

❒ Then we get

$$\begin{aligned} G(x_1, \cdots x_4) =& (-i\lambda) \int d^4 z \frac{d^4 q_1}{(2\pi)^4} \cdots \frac{d^4 q_4}{(2\pi)^4} e^{-iq_1\cdot x_1 - iq_2 x_2 - iq_3 x_3 - iq_4 x_4 + i(q_1+q_2+q_3+q_4)\cdot z} \\ & \Delta_F(q_1)\Delta_F(q_2)\Delta_F(q_3)\Delta_F(q_4) \\ =& (-i\lambda) \int \frac{d^4 q_1}{(2\pi)^4} \cdots \frac{d^4 q_4}{(2\pi)^4} \; e^{-iq_1\cdot x_1 - iq_2\cdot x_2 - iq_3\cdot x_3 - iq_4\cdot x_4} \\ & (2\pi)^4 \delta^4(q_1+q_2+q_3+q_4) \Delta_F(q_1)\Delta_F(q_2)\Delta_F(q_3)\Delta_F(q_4) \end{aligned}$$

Feynman rules for $\lambda\varphi^4$

- If we now introduce the T matrix transition amplitude, defined by

$$S_{fi} = \delta_{fi} - i(2\pi)^4 \delta(P_f - P_i)\, T_{fi}$$

we obtain

$$-iT_{fi} = (-i\lambda)$$

- For this amplitude we draw the Feynman diagram

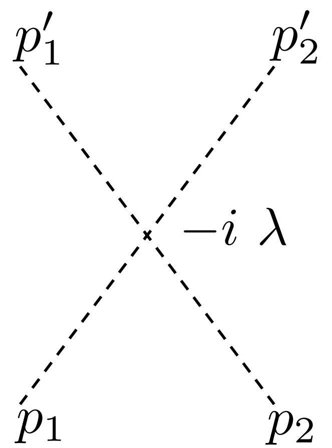


Figure 4: Vertex in momentum space.

and we associate to the vertex the number $(-i\lambda)$.

Feynman rules for $\lambda\varphi^4$

- Let us consider now a more complicated case, the evaluation of $G(x_1 \cdots x_4)$ in second order in the coupling λ. After this exercise we will be in position to be able to state the Feynman rules in momentum space with all generality.

- We get in second order in λ,

$$G(x_1, \cdots x_4) =$$

$$= \frac{(-i\lambda)^2}{2!} \int d^4z_1 d^4z_2 \, \langle 0| \, T\left(\varphi_{in}(x_1)\varphi_{in}(x_2)\varphi_{in}(x_3)\varphi_{in}(x_4) :\frac{\varphi^4_{in}(z_1)}{4!} :: \frac{\varphi^4_{in}(z_2)}{4!} : \right) |0\rangle,$$

$$= \frac{(-i\lambda)^2}{2!} \int d^4z_1 d^4z_2 \left(\frac{4\times 3}{4!} \right) \times \left(\frac{4\times 3}{4!} \right) \times 2$$

$$\Big\{ \Delta_F(x_1-z_1)\Delta_F(x_2-z_1)\Delta_F(z_1-z_2)\Delta_F(z_1-z_2)\Delta_F(z_2-x_3)\Delta_F(z_2-x_4)$$

$$+ \Delta_F(x_1-z_1)\Delta_F(x_2-z_2)\Delta_F(z_1-z_2)\Delta_F(z_1-z_2)\Delta_F(z_1-x_3)\Delta_F(z_2-x_4)$$

$$+ \Delta_F(x_1-z_1)\Delta_F(x_2-z_2)\Delta_F(z_1-z_2)\Delta_F(z_1-z_2)\Delta_F(z_1-x_4)\Delta_F(z_2-x_3)$$

$$+ \Delta_F(x_1-z_2)\Delta_F(x_2-z_2)\Delta_F(z_2-z_1)\Delta_F(z_2-z_1)\Delta_F(z_1-x_3)\Delta_F(z_1-x_4)$$

$$+ \Delta_F(x_1-z_2)\Delta_F(x_2-z_1)\Delta_F(z_1-z_2)\Delta_F(z_2-z_1)\Delta_F(z_1-x_3)\Delta_F(z_2-x_4)$$

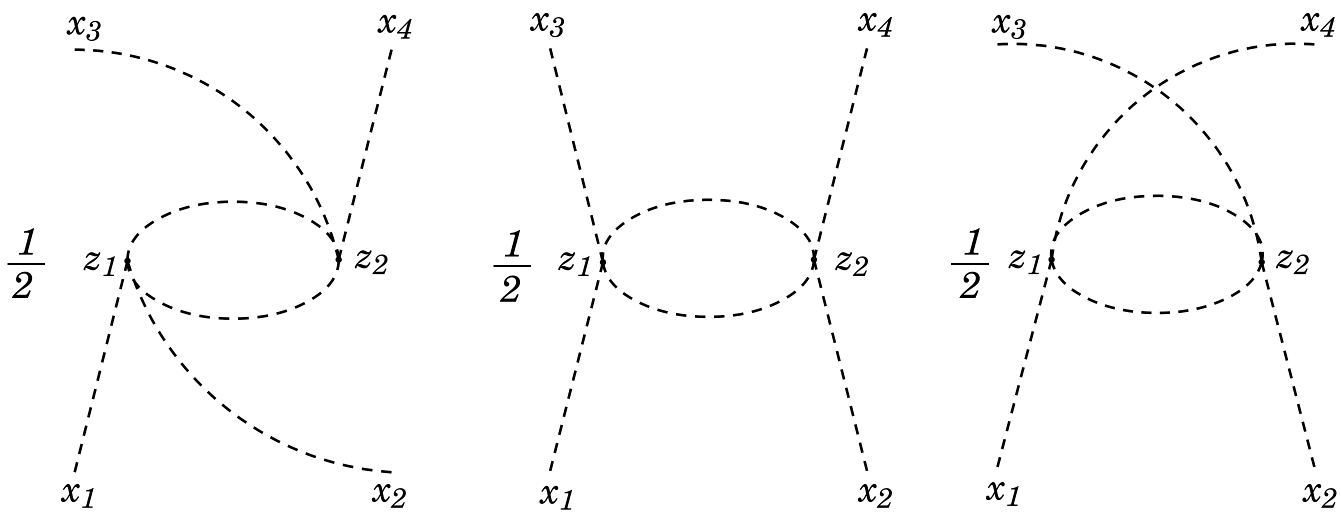
$$+ \Delta_F(x_1-z_2)\Delta_F(x_2-z_1)\Delta_F(z_1-z_2)\Delta_F(z_1-z_2)\Delta_F(z_1-x_4)\Delta_F(z_1-x_3) \Big\}$$

Feynman rules for $\lambda\varphi^4$

- In terms of diagrams we have

$$G(x_1,\cdots x_4) = \frac{(-i\lambda)^2}{2!}\int d^4z_1 d^4z_2$$

$$\Bigg\{$$



$$+\left(z_1 \leftrightarrow z_2\right)\Bigg\}$$

Feynman rules for $\lambda\varphi^4$

- Let us now go into momentum space, by introducing the Fourier transform of the propagators. We start by diagram a),

$$G^{(a)}(x_1,x_2,x_3,x_4) =$$

$$= \frac{(-i\lambda)^2}{2!}\frac{1}{2}\int d^4z_1 d^4z_2 \Delta_F(x_1-z_1)\Delta_F(x_2-z_1)\Delta_F(z_1-z_2)\Delta_F(z_1-z_2)$$
$$\Delta_F(z_2-x_3)\Delta_F(z_2-x_4)$$

$$= \frac{(-i\lambda)^2}{2!}\frac{1}{2}\int d^4z_1 d^4z_2 \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{d^4q_3}{(2\pi)^4}\frac{d^4q_4}{(2\pi)^4}\frac{d^4q_5}{(2\pi)^4}\frac{d^4q_6}{(2\pi)^4}$$
$$e^{i[(q_1\cdot x_1+q_2\cdot x_2-q_3\cdot x_3-q_4\cdot x_4)+z_1\cdot(q_5-q_1-q_2+q_6)+z_2\cdot(q_3+q_4-q_5-q_6)]}$$
$$\Delta_F(q_1)\Delta_F(q_2)\Delta_F(q_3)\Delta_F(q_4)\Delta_F(q_5)\Delta_F(q_6)$$

$$= \frac{(-i\lambda)^2}{2!}\frac{1}{2}(2\pi)^4\int \frac{d^4q_1}{(2\pi)^4}\cdots\frac{d^4q_5}{(2\pi)^4}\delta^4(q_1+q_2-q_3-q_4)\ e^{i[q_1\cdot x_1+q_2\cdot x_2-q_2\cdot x_3-q_4\cdot x_4]}$$
$$\Delta_F(q_1)\Delta_F(q_2)\Delta_F(q_3)\Delta_F(q_4)\Delta_F(q_5)\Delta_F(q_1+q_2-q_5)$$

Feynman rules for $\lambda\varphi^4$

- Now we insert the last equation into the reduction formula. We get

$$S_{fi}^{(a)} = (i)^4 \int d^4x_1 \cdots d^4x_4 e^{-i[p_1\cdot x_1 + p_2\cdot x_2 - p_1'\cdot x_3 - p_2'\cdot x_4]}$$

$$(\Box_{x_1} + m^2)\cdots(\Box_{x_4} + m^2) G^{(a)}(x_1, \cdots, x_4)$$

- The only dependence of $G^{(a)}$ in the coordinates, $x_i (i = 1, \cdots 4)$, is in the exponential, therefore,

$$(\Box_{x_i} + m^2) \to (-q_i^2 + m^2)$$

and using

$$(-q_i^2 + m^2)\Delta_F(q_i) = -i$$

we eliminate the propagators in the external legs of the diagram

Feynman rules for $\lambda\varphi^4$

- We then get

$$
\begin{aligned}
S_{fi}^{(a)} = & \frac{(-i\lambda)^2}{2!}\frac{1}{2}\int d^4x_1\cdots d^4x_4 \int \frac{d^4q_1}{(2\pi)^4}\cdots\frac{d^4q_5}{(2\pi)^4}(2\pi)^4\delta^4(q_1+q_2-q_3-q_4) \\
& e^{-i[x_1\cdot(p_1-q_1)+x_2\cdot(p_2-q_2)-x_3\cdot(p'_1-q_3)-x_4\cdot(p'_2-q_4)]}\Delta_F(q_5)\Delta_F(q_1+q_2-q_5) \\
= & \frac{(-i\lambda)^2}{2!}\frac{1}{2}\int \frac{d^4q_1}{(2\pi)^4}\cdots\frac{d^4q_5}{(2\pi)^4}(2\pi)^4\delta^4(q_1+q_2-q_3-q_4)(2\pi)^4\delta^4(p_1-q_1) \\
& (2\pi)^4\delta^4(p_2-q_2)(2\pi)^4\delta^4(p'_1-q_3)(2\pi)^4\delta^4(p'_2-q_4)\Delta_F(q_5)\Delta_F(q_1+q_2-q_5) \\
= & \frac{(-i\lambda)^2}{2!}\frac{1}{2}\int \frac{d^4q_5}{(2\pi)^4}(2\pi)^4\delta^4(p_1+p_2-p'_1-p'_2)\Delta_F(q_5)\Delta_F(p_1+p_2-q_5)
\end{aligned}
$$

- This expression can be written in the form

$$
S_{fi}^{(a)} = (2\pi)^4\delta^4(p_1+p_2-p'_1-p'_2)\frac{(-i\lambda)^2}{2!}\frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\Delta_F(q)\Delta_F(p_1+p_2-q)
$$

Feynman rules for $\lambda\varphi^4$

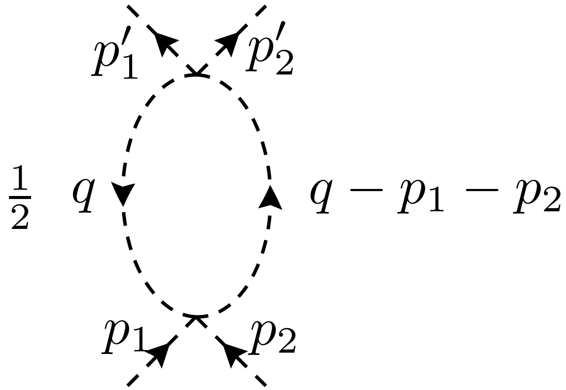
- If we denote by a') the diagram a) with the interchange $z_1 \leftrightarrow z_2$ and redo the calculation we get exactly the same result
- Therefore,

$$S_{fi}^{(a+a')} = (2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')(-i\lambda)^2\frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\Delta_F(q)\Delta_F(p_1+p_2-q)$$

or in terms of the T_{fi} matrix,

$$-iT_{fi}^{(a+a')} = (-i\lambda)^2\frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\Delta_F(q)\Delta_F(p_1+p_2-q)$$

- To encode this result we draw the Feynman diagram



Feynman rules for $\lambda\varphi^4$

- We find that in order to evaluate the $-iT$ matrix, we associate to each vertex a factor $(-i\lambda)$, to each internal line a propagator Δ_F and for each loop the integral $\int \frac{d^4q}{(2\pi)^4}$.
- Besides that we have 4-momentum conservation at each vertex.
- Finally there is a symmetry factor (see below) which takes the value $\frac{1}{2}$ for this diagram.
- If we repeat the calculations for diagrams $b) + b')$ and $c) + c')$ it is easy to see that we get,

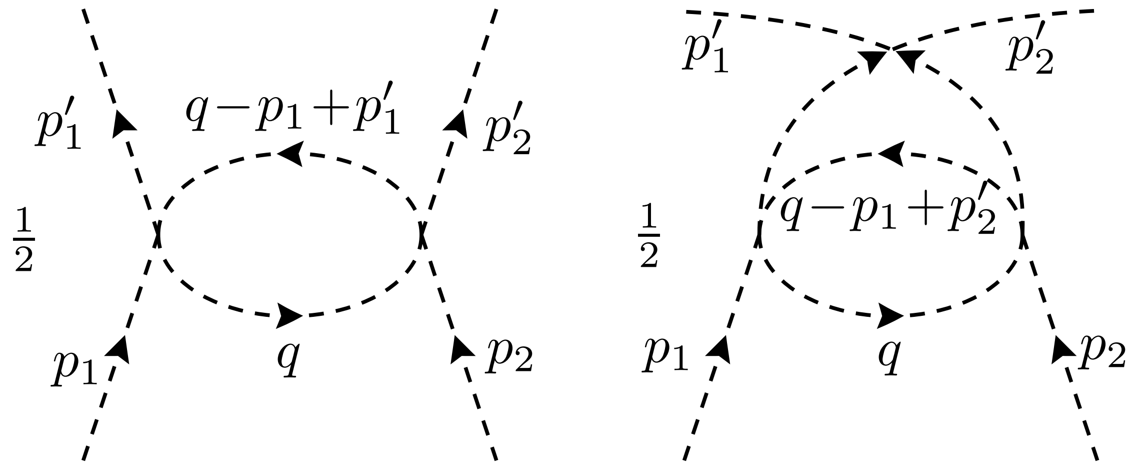
$$-iT_{fi}^{(b+b')} = (-i\lambda)^2 \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \Delta_F(q)\Delta_F(q - p_1 + p_1')$$

and

$$-iT_{fi}^{(c+c')} = (-i\lambda)^2 \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \Delta_F(q)\Delta_F(q - p_1 + p_2')$$

Feynman rules for $\lambda\varphi^4$

- These correspond to the diagrams



- After this exercise we are in position to state the Feynman rules with all generality for the $\lambda\varphi^4$ theory. These are rules for the $-iT$ matrix, that is, after we factorize $(2\pi)^4\delta^4(\cdots)$.

Feynman rules for $\lambda\varphi^4$

1. Draw all topologically distinct diagrams with n external legs.
2. At each vertex multiply by the factor $(-i\lambda)$.
3. To each internal line associate a propagator $\Delta_F(q)$.
4. For each loop include an integral

$$\int \frac{d^4q}{(2\pi)^4}$$

   The direction of this momentum is irrelevant, but we have to respect 4-momentum conservation ate each vertex.

5. Multiply by the symmetry factor of the diagram. This is defined by,

$$S = \frac{\#\text{ of distinct ways of connecting the vertices to the external legs}}{\text{Permutations of each vertex} \times \text{Permutations of equal vertices}}$$

6. Add the contributions of all the topologically distinct diagrams. The result is the $-iT$ matrix amplitude that enters the formula for the cross section.

Lecture 5

Feynman rules for QED

- We now turn to the case of QED. Like $\lambda\phi^4$, it is a theory without derivatives and therefore,

$$\mathcal{L}_I = -\mathcal{H}_I = -e\, Q\overline{\psi}_{in}\gamma^\mu\psi_{in}A_\mu^{in}$$

where e is the absolute value of the electron charge, or the proton charge. For the electron the sign enters explicitly in $Q=-1$. This way of writing allows for obvious generalizations for particles with other charges, like for instance the quarks. For QED we have then,

$$\mathcal{L}_I^{\text{QED}} = e\, \overline{\psi}_{in}\gamma^\mu\psi_{in}A_\mu^{in}$$

- Due to the electric charge conservation, the Green functions that we have to deal with have an equal number of ψ and $\overline{\psi}$ fields. In general we have,

$$G(x_1\cdots x_n x_{n+1}\cdots x_{2n}; y_1\cdots y_p) =$$

$$= \langle 0|\, T(\psi(x_1)\cdots\psi(x_n)\overline{\psi}(x_{n+1})\cdots\overline{\psi}(x_{2n})A^{\mu_1}(y_1)\cdots A^{\mu_p}(y_p))\,|0\rangle$$

where, for simplicity, we omit the spinorial indices in the fermion fields. This equation is written in terms of the physical fields.

Feynman rules for QED

- Following a similar procedure to the scalar field case, we can obtain an expression for G in terms of the in fields.
- This will be,

$$G(x_1 \cdots x_{2n}; y_1 \cdots y_p) =$$

$$\frac{\langle 0 | T \psi_{in}(x_1) \cdots \overline{\psi}_{in}(x_{2n}) A_{in}^{\mu_1}(y_1) \cdots A_{in}^{\mu_p}(y_p) \; e^{[i \int d^4 z \mathcal{L}_I(z)]} |0\rangle}{\langle 0 | T \exp[i \int d^4 z \mathcal{L}_I(z)] |0\rangle}$$

$$= \langle 0 | T \psi_{in}(x_1) \cdots \overline{\psi}_{in}(x_{2n}) A_{in}^{\mu_1}(y_1) \cdots A_{in}^{\mu_p}(y_p) \; e^{[i \int d^4 t \mathcal{L}_I(z)]} |0\rangle_c$$

where the fields in $\mathcal{L}_I$ are normal ordered, and $\langle 0 | \cdots |0\rangle_c$ means that we only consider the connected diagrams.

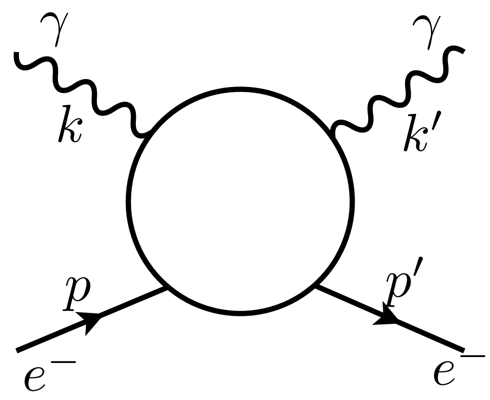
- To get the Feynman rules we will evaluate a few simple processes.

Compton Scattering

❒ Compton scattering corresponds to the following process,

$$e^- + \gamma \to e^- + \gamma$$

with the kinematics shown in the figure



❒ The S matrix element to evaluate is therefore,

$$S_{fi} = \langle (p', s'), k'; out | (p, s), k; in \rangle$$

❒ Using the LSZ reduction formula we can write,

$$S_{fi} = \int d^4x d^4x' \int d^4y d^4y' e^{-i[p\cdot x + k\cdot y - p'\cdot x' - k'\cdot y']} \varepsilon^{\mu}(k) \varepsilon^{*\mu'}(k')$$
$$\overline{u}(p', s')_{\alpha'} (i\overrightarrow{\not\partial}_{x'} - m)_{\alpha'\beta'} \vec{\Box}_y \vec{\Box}_{y'}$$
$$\langle 0 | T(\psi_{\beta'}(x') \overline{\psi}_{\beta}(x) A_{\mu}(y) A_{\mu'}(y')) | 0 \rangle (-i\overleftarrow{\not\partial}_x - m)_{\beta\alpha} u_{\alpha}(p, s)$$

Compton Scattering

- Our task is therefore to evaluate the Green function

$$G(x',x,y,y') \equiv \langle 0|\, T(\psi_{\beta'}(x')\overline{\psi}_{\beta}(x)A_{\mu}(y)A_{\mu'}(y'))\,|0\rangle$$

- The lowest contribution is quadratic in the interaction. We get

$$G(x,x',y,y') =$$

$$= \frac{(ie)^2}{2!}\int d^4z_1 d^4z_2\, \langle 0|\, T(\psi^{in}_{\beta'}(x')\overline{\psi}^{in}_{\beta}(x)A^{in}_{\mu}(y)A^{in}_{\mu'}(y')$$

$$:\overline{\psi}_{in}(z_1)\gamma^{\sigma}\psi_{in}(z_1)A^{in}_{\sigma}(z_1) :: \overline{\psi}^{in}(z_2)\gamma^{\rho}\psi^{in}(z_2)A^{in}_{\rho}(z_2) :)\,|0\rangle$$

$$= \frac{(ie)^2}{2!}(\gamma^{\sigma})_{\gamma\delta}\;(\gamma^{\rho})_{\gamma'\delta'}\;\int d^4z_1 d^4z_2\, \langle 0|\, T(\psi^{in}_{\beta'}(x')\overline{\psi}^{in}_{\beta}(x)A^{in}_{\mu}(y)A^{in}_{\mu'}(y')$$

$$:\overline{\psi}^{in}_{\gamma}(z_1)\psi^{in}_{\delta}(z_1)A^{in}_{\sigma}(z_1) :: \overline{\psi}^{in}_{\gamma'}(z_2)\psi^{in}_{\delta'}(z_2)A^{in}_{\rho}(z_2) :)\,|0\rangle$$

Compton Scattering

- Now we use Wick's theorem to write $\langle 0| T(\cdots) |0\rangle$ in terms of the propagators. We get,

$$\langle 0| T(\psi^{in}_{\beta'}(x')\overline{\psi}^{in}_{\beta}(x)A^{in}_{\mu}(y)A^{in}_{\mu'}(y') :\overline{\psi}^{in}_{\gamma}(z_1)\psi^{in}_{\delta}(z_1)A^{in}_{\sigma}(z_1)::\overline{\psi}^{in}_{\gamma'}(z_2)\psi^{in}_{\delta'}(z_2)A^{in}_{\rho}(z_2)):) |0\rangle$$

$$\begin{aligned}
= \quad & \langle 0| T\psi^{in}_{\beta'}(x')\overline{\psi}^{in}_{\gamma}(z_1) |0\rangle \langle 0| T\psi^{in}_{\delta'}(z_2)\overline{\psi}^{in}_{\beta}(x) |0\rangle \langle 0| T\psi^{in}_{\delta}(z_1)\overline{\psi}_{\gamma'}(z_2) |0\rangle \\
& \quad \langle 0| T(A^{in}_{\mu}(y)A^{in}_{\sigma}(z_1)) |0\rangle \langle 0| TA^{in}_{\mu'}(y')A^{in}_{\rho}(z_2) |0\rangle \\
& + \langle 0| T\psi^{in}_{\beta'}(x')\overline{\psi}^{in}_{\gamma}(z_1) |0\rangle \langle 0| T\psi^{in}_{\delta'}(z_2)\overline{\psi}^{in}_{\beta}(x) |0\rangle \langle 0| T\psi^{in}_{\delta}(z_1)\overline{\psi}_{\gamma'}(z_2)|0 \\
& \quad \langle 0| TA^{in}_{\mu}(y)A^{in}_{\rho}(z_2) |0\rangle \langle 0| TA^{in}_{\mu'}(y')A^{in}_{\sigma}(z_1) |0\rangle \\
& + \langle 0| T\psi^{in}_{\beta'}(x')\overline{\psi}^{in}_{\gamma'}(z_2) |0\rangle \langle 0| T\psi^{in}_{\delta}(z_1)\overline{\psi}^{in}_{\beta}(x) |0\rangle \langle 0| T\psi^{in}_{\delta'}(z_2)\overline{\psi}^{in}_{\gamma}(z_1) |0\rangle \\
& \quad \langle 0| TA^{in}_{\mu}(y)A^{in}_{\sigma}(z_1) |0\rangle \langle 0| TA^{in}_{\mu'}(y')A^{in}_{\rho}(z_2) |0\rangle \\
& + \langle 0| T\psi^{in}_{\beta'}(x')\overline{\psi}^{in}_{\gamma'}(z_2) |0\rangle \langle 0| T\psi^{in}_{\delta}(z_1)\overline{\psi}^{in}_{\beta}(x) |0\rangle \langle 0| T\psi^{in}_{\delta'}(z_2)\overline{\psi}^{in}_{\gamma}(z_1) |0\rangle \\
& \quad \langle 0| TA^{in}_{\mu}(y)A^{in}_{\rho}(z_2) |0\rangle \langle 0| TA^{in}_{\mu'}(y')A^{in}_{\sigma}(z_1) |0\rangle
\end{aligned}$$

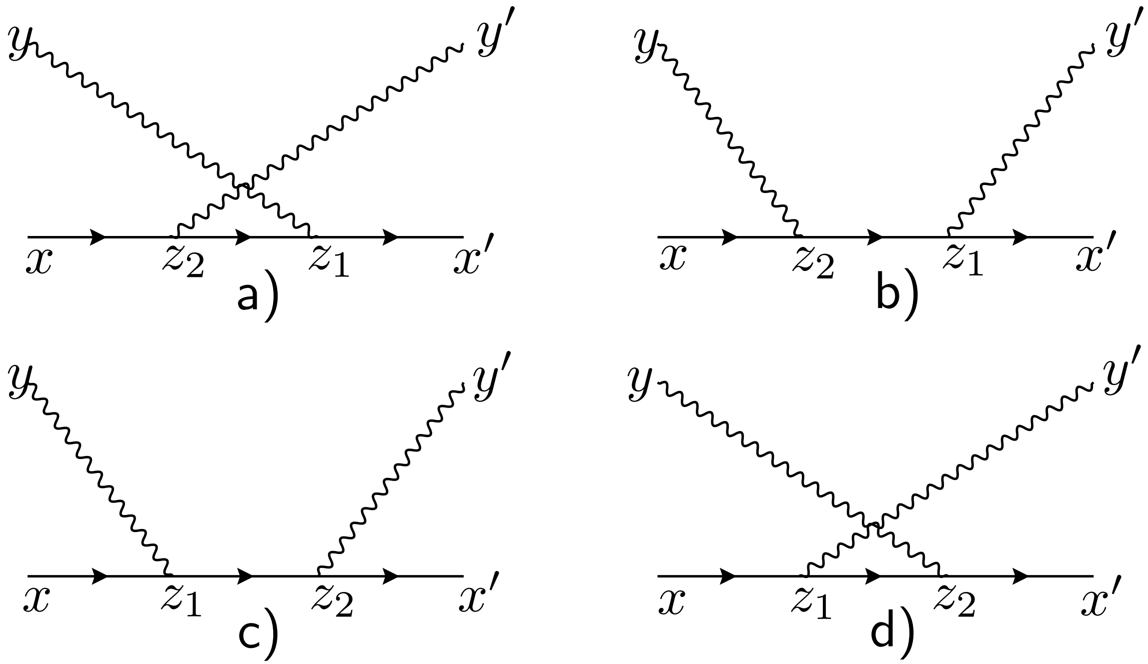
Compton Scattering

❒ This can be written as

$$\langle 0|\, T(\psi^{in}_{\beta'}(x')\overline{\psi}^{in}_{\beta}(x)A^{in}_{\mu}(y)A^{in}_{\mu'}(y') :\overline{\psi}^{in}_{\gamma}(z_1)\psi^{in}_{\delta}(z_1)A^{in}_{\sigma}(z_1) :: \overline{\psi}^{in}_{\gamma'}(z_2)\psi^{in}_{\delta'}(z_2)A^{in}_{\rho}(z_2)) :) \,|0\rangle$$

$$\begin{aligned} = \quad & S_{F\beta'\gamma}(x'-z_1)S_{F\delta'\beta}(z_2-x)S_{F\delta\gamma'}(z_1-z_2)D_{F\mu\sigma}(y-z_1)D_{F\mu'\rho}(y'-z_2) \\ & + S_{F\beta'\gamma}(x'-z_1)S_{F\delta'\beta}(z_2-x)S_{F\delta\gamma'}(z_1-z_2)D_{F\mu\rho}(y-z_2)D_{F\mu'\sigma}(y'-z_1) \\ & + S_{F\beta'\gamma'}(x'-z_2)S_{F\delta\beta}(z_1-x)S_{F\delta'\gamma}(z_2-z_1)D_{F\mu\sigma}(y-z_1)D_{F\mu'\rho}(y'-z_2) \\ & + S_{F\beta'\gamma'}(x'-z_2)S_{F\delta\beta}(z_1-x)S_{F\delta'\gamma}(z_2-z_1)D_{F\mu\rho}(y-z_2)D_{F\mu'\sigma}(y'-z_1) \end{aligned}$$

❒ To better understand this it is useful to draw the corresponding diagrams



Compton Scattering

- From this figure it is clear that $b) \equiv c)$ and $a) \equiv d)$ because z_1 and z_2 are irrelevant labels. From this we get a factor of 2 that is going to cancel the $\frac{1}{2!}$
- In fact this result is general, for n vertices we have $n!$ that cancels against the factor $\frac{1}{n!}$ from the expansion of the exponential
- We have then only two distinct diagrams that we take as c) and d). Then, including already the factor of 2, we get for diagrama c)

$$G^{(c)}(x,x',y,y')=(ie)^2(\gamma^\sigma)_{\gamma\delta}(\gamma^\rho)_{\gamma'\delta'}\int d^4z_1 d^4z_2\ S_{F\beta'\gamma'}(x'-z_2)S_{F\delta\beta}(z_1-x)$$
$$S_{F\delta'\gamma}(z_2-z_1)D_{F\mu\sigma}(y-z_1)D_{F\mu'\rho}(y'-z_2)$$

- To proceed we could, like in the case of $\lambda\varphi^4$, introduce the Fourier transform of the propagators. However, it is easier to get rid of the external legs using

$$(i\partial\!\!\!/_x - m)_{\alpha\lambda} S_{F\lambda\beta}(x-y) = i\delta_{\alpha\beta}\delta^4(x-y)$$

$$S_{F\alpha\lambda}(x-y)(-i\overleftarrow{\partial\!\!\!/}_y - m)_{\lambda\beta} = i\delta_{\alpha\beta}\delta^4(x-y)$$

$$\Box_x D_{F\mu\nu}(x-y) = i g_{\mu\nu}\delta^4(x-y)$$

Compton Scattering

- We get therefore,

$$S_{fi}^{(c)} = (ie)^2 \int d^4x d^4x' d^4y d^4y' e^{-i(p\cdot x + k\cdot y - p'\cdot x' - k'\cdot y')} \varepsilon^{\mu}(k) g_{\mu\sigma} \varepsilon^{*\mu'}(k') g_{\mu'\rho}$$

$$(\gamma^{\sigma})_{\gamma\delta} (\gamma^{\rho})_{\gamma'\delta'} \ \overline{u}(p', s')_{\alpha'} \delta_{\alpha'\gamma'} u_{\alpha}(p, s) \ \delta_{\delta\alpha}$$

$$\int d^4z_1 d^4z_2 \delta^4(x' - z_2) \delta^4(x - z_1) \delta^4(y - z_1) \delta^4(y' - z_2) \ S_{F\delta'\gamma}(z_2 - z_1)$$

$$= (ie)^2 \int d^4z_1 d^4z_2 e^{-i(p\cdot z_1 + k\cdot z_1 - p'\cdot z_2 - k'\cdot z_2)} \varepsilon^{\mu}(k) \varepsilon^{*\mu'}(k')$$

$$\overline{u}(p', s')_{\alpha'} (\gamma_{\mu'})_{\alpha'\delta'} S_{F\delta'\gamma}(z_2 - z_1) (\gamma_{\mu})_{\gamma\alpha} u_{\alpha}(p, s)$$

- Finally we use

$$S_F(z_2 - z_1) = \int \frac{d^4q}{(2\pi)^4} \frac{i(\not{q} + m)}{q^2 - m^2 + i\epsilon} e^{-iq\cdot(z_2 - z_1)}$$

$$\equiv \int \frac{d^4q}{(2\pi)^4} S_F(q) e^{-iq\cdot(z_2 - z_1)}$$

Compton Scattering

- And we get

$$S_{fi}^{(c)} = \int \frac{d^4q}{(2\pi)^4} d^4z_1 d^4z_2 e^{-iz_1\cdot(p+k-q)+iz_2\cdot(p'+k'-q)}$$

$$\varepsilon^{\mu}(k)\varepsilon^{\mu'*}(k')\overline{u}(p',s')(ie\gamma_{\mu'})S_F(q)(ie\gamma_{\mu})u(p,s)$$

$$=(2\pi)^4\delta^{(4)}(p+k-p'-k)\cdot$$

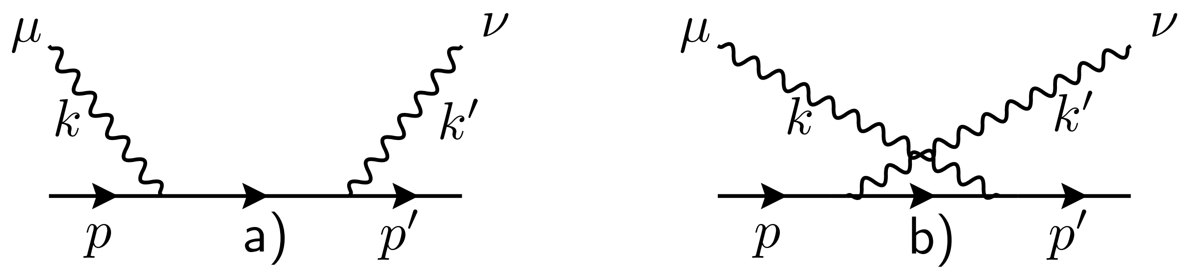
$$\varepsilon^{\mu}(k)\varepsilon^{\mu'*}(k')\overline{u}(p',s')(ie\gamma_{\mu'})S_F(p+k)(ie\gamma_{\mu})u(p,s)$$

- Therefore, the T matrix transition amplitude is,

$$-iT_{fi}^{(c)} = \varepsilon^{\mu}(k)\varepsilon^{\mu'*}(k')\overline{u}(p',s')(ie\gamma_{\mu'})S_F(p+k)(ie\gamma_{\mu})u(p,s)$$

Compton Scattering

- This corersponds to the diagram on the left



- We factor out the quantity $(ie\gamma_\mu)$, because it will be clear that this quantity will be the Feynman rule for the vertex. The arrows in these diagrams correspond to the flow of electric charge. Notice that to an electron in the initial state we associate a spinor $u(p,s)$ and for an electron in the final state we associate the spinor $\overline{u}(p',s')$. Since the electron line as to be a c-number, we start writing the line in the reverse order of that of the arrows.

- In a similar way for diagram d) we will get the diagram represented on the right that corresponds to the following expression,

$$-iT^{(d)}_{fi} = \varepsilon^{\mu}(k)\varepsilon^{*\mu'}(k')\overline{u}(p',s')(ie\gamma_\mu)S_F(p-k')(ie\gamma_{\mu'})u(p,s)$$

- We are almost in a position to state the Feynman rules for QED. Before that we will look at a case where we have positrons.

Electron–positron elastic scattering (Bhabha scattering)

- We will consider electron-positron elastic scattering, the so-called *Bhabha scattering*,

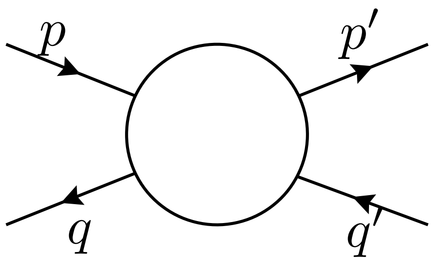
$$e^-(p) + e^+(q) \to e^-(p') + e^+(q')$$

- This example will teach us two things. First, how positrons (that is the anti-particles) enter in the amplitudes. Secondly we will learn that, sometimes, due the anti-commutation rules of the fermions, we will get relative minus signs between different diagrams.

- We have,

$$S_{fi} = \langle (p', s'), (q', \overline{s'}); out | (p, s), (q, \overline{s}); in \rangle$$

corresponding to the kinematics in the figure



Electron–positron elastic scattering (Bhabha scattering)

- Notice that the arrows are in the direction of flow of charge of the electron, but the momenta do correspond to the real momenta of the particles or antiparticles in that frame: p entering and p' exiting for the electron, and q entering and q' exiting for the positron.

- In the following we will not show the spin dependence in order to simplify the notation. Then we write,

$$S_{fi} = \int d^4x d^4y d^4x' d^4y' e^{-i[p\cdot x+q\cdot y-p'\cdot x'-q'\cdot y']}$$

$$\overline{u}(p')_\alpha (i\overrightarrow{\not\partial}_{x'} - m)_{\alpha\beta}\ \overline{v}_\gamma(q)(i\overrightarrow{\not\partial}_y - m)_{\gamma\delta}$$

$$\langle 0|\, T\overline{\psi}_{\delta'}(y')\psi_\beta(x')\overline{\psi}_{\beta'}(x)\psi_\delta(y)\,|0\rangle$$

$$(-i\overleftarrow{\not\partial}_x - m)_{\beta'\alpha'} u_{\alpha'}(p)(-i\overleftarrow{\not\partial}_{y'} - m)_{\delta'\gamma'} v_{\gamma'}(q')$$

- We have, therefore, to evaluate the Green function

$$G(y',x',x,y) \equiv \langle 0|\, T\overline{\psi}_{\delta'}(y')\psi_\beta(x')\overline{\psi}_{\beta'}(x)\psi_\delta(y)\,|0\rangle$$

Electron–positron elastic scattering (Bhabha scattering)

❒ The lowest order contribution is of second order in the coupling. We have (to simplify we omit the label in),

$$G(y',x',x,y) = \frac{(ie)^2}{2}(\gamma^\mu)_{\epsilon\epsilon'}(\gamma^\nu)_{\varphi\varphi'}\int d^4z_1 d^4z_2$$

$$\langle 0|\, T\overline{\psi}_{\delta'}(y')\psi_\beta(x')\overline{\psi}_{\beta'}(x)\psi_\delta(y) : \overline{\psi}_\epsilon(z_1)\psi_{\epsilon'}(z_1)A_\mu(z_1) :: \overline{\psi}_\varphi(z_2)\psi_{\varphi'}(z_2)A_\nu(z_2) : |0\rangle$$

$$= \frac{(ie)^2}{2}(\gamma^\mu)_{\varepsilon\varepsilon'}(\gamma^\nu)_{\varphi\varphi'}\int d^4z_1 d^4z_2$$

$$\Big[- S_{F\beta\epsilon}(x'-z_1)S_{F\epsilon'\beta'}(z_1-x)S_{F\delta\varphi}(y-z_2)S_{F\varphi'\delta'}(z_2-y')D_{F\mu\nu}(z_1-z_2)$$

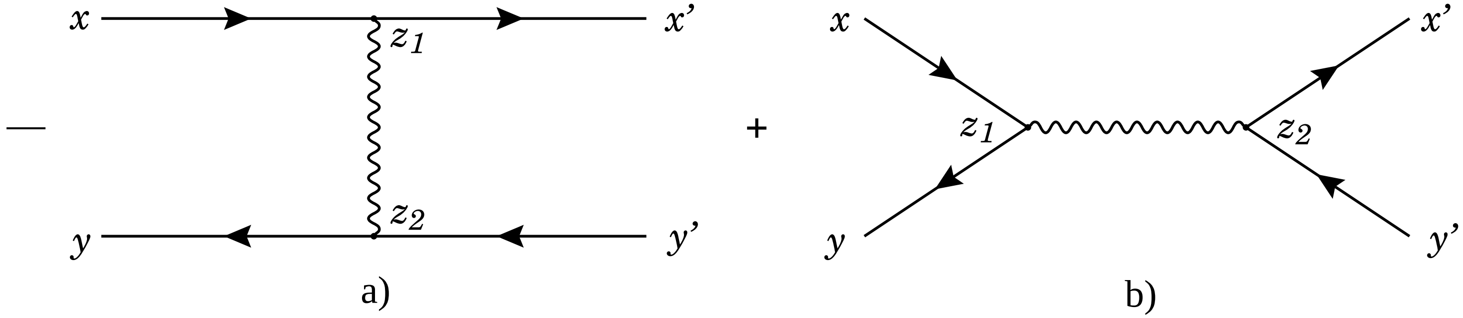
$$+ S_{F\delta\epsilon}(y-z_1)S_{F\epsilon'\beta'}(z_1-x)S_{F\beta\varphi}(x'-z_2)S_{F\varphi'\delta'}(z_2-y')D_{F\mu\nu}(z_1-z_2)$$

$$+ (z_1 \leftrightarrow z_2)\Big]$$

❒ Once more the exchange $(z_1 \leftrightarrow z_2)$ compensates for the factor $\frac{1}{2!}$ and we have two diagrams with a relative minus sign

Electron–positron elastic scattering (Bhabha scattering)

- The diagrams are



- Let us look at the contribution of diagram a),

$$S_{fi}^{(a)} = -\int d^4x d^4y d^4x' d^4y' d^4z_1 d^4z_2 (ie)^2 (\gamma^\mu)_{\varepsilon\varepsilon'} (\gamma^\nu)_{\varphi\varphi'} e^{-i[p\cdot x+q\cdot y-p'\cdot x'-q'\cdot y']}$$

$$\overline{u}(p')_\alpha (i\overrightarrow{\partial\!\!\!/}'_x - m)_{\alpha\beta} \overline{v}_\gamma(q) (i\overrightarrow{\partial\!\!\!/}_y - m)_{\gamma\delta}$$

$$S_{F\beta\varepsilon}(x'-z_1) S_{F\varepsilon'\beta'}(z_1-x) S_{F\delta\varphi}(y-z_2) S_{F\varphi'\delta'}(z_2-y')$$

$$(-i\overleftarrow{\partial\!\!\!/}_x - m)_{\beta'\alpha'} u_{\alpha'}(p) (-i\overleftarrow{\partial\!\!\!/}'_y - m)_{\delta'\gamma'} v_{\gamma'}(q') D_{F\mu\nu}(z_1-z_2)$$

$$= -\int d^4z_1 d^4z_2 e^{-i[p\cdot z_1+q\cdot z_2-p'\cdot z_1-q'\cdot z_2]}$$

$$\overline{u}(p')(ie\gamma^\mu) u(p) \overline{v}(q) (ie\gamma^\nu) v(q') D_{F\mu\nu}(z_1-z_2)$$

Electron–positron elastic scattering (Bhabha scattering)

- Using now the Fourier transform of the photon propagator,

$$D_{F\mu\nu}(z_1 - z_2) = \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\mu\nu}}{k^2 + i\varepsilon} e^{-ik\cdot(z_1 - z_2)}$$
$$\equiv \int \frac{d^4k}{(2\pi)^u} D_{F\mu\nu}(k) e^{-ik\cdot(z_1 - z_2)}$$

we get

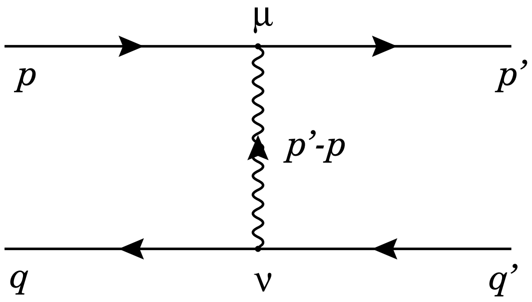
$$S_{fi}^{(a)} = -\overline{u}(p')(ie\gamma^\mu)u(\rho)\overline{v}(q)(ie\gamma^\nu)v(q')$$
$$\int d^4z_1 d^4z_2 \frac{d^4k}{(2\pi)^4} D_{F\mu\nu}(k) e^{-iz_1\cdot(p-p'+k)} e^{-iz_2\cdot(q-q'-k)}$$
$$= -(2\pi)^4\delta^4(p + q - p' - q')\overline{u}(p')(ie\gamma^\nu)u(p)\overline{v}(q)(ie\gamma^\mu)v(q')D_{F\mu\nu}(p' - p)$$

- Therefore the T matrix element is,

$$-iT_{fi}^{(a)} = -\overline{u}(p')(ie\gamma^\mu)u(p)D_{F\mu\nu}(p' - p)\overline{v}(q)(ie\gamma^\nu)v(q')$$

Electron–positron elastic scattering (Bhabha scattering)

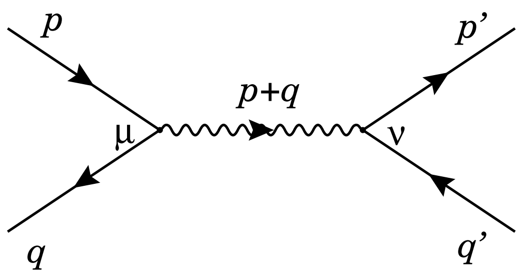
- This corresponds the Feynman diagram



- In a similar way we would get

$$-iT_{fi}^{(b)} = \overline{v}(q)(ie\gamma^{\mu})u(p)D_{F\mu\nu}(p+q)\overline{u}(p')(ie\gamma^{\nu})v(q')$$

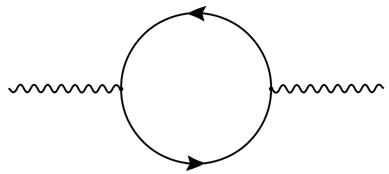
that corresponds to the diagram



- Which of the diagrams has the minus sign is irrelevant, because this is the lowest order diagram. It depends on the conventions determining how to build the in state. Only the relative sign is important. However, higher order terms have to respect the same conventions.

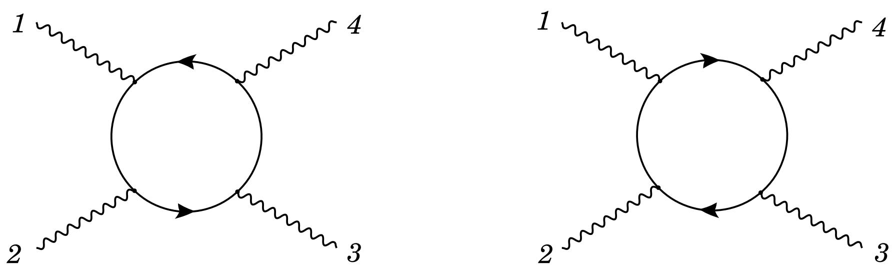
Fermion Loops

- Before we summarize the Feynman rules for QED let us look at what happens with fermion loops. One such example is the second order correction to the photon propagator



- First of all, the loop orientation it is only relevant if leads to topologically different diagrams. Therefore the following diagrams are topologically equivalent and only one should be considered.

- However the diagrams below are topologically distinct and both should be considered.



Fermion Loops

- The second aspect that is relevant is a possible sign coming from the anti-commutation of the fermion fields, that should affect some diagrams, and in particular the fermion loop.

- To understand this sign we should note that by definition of loop, the internal lines are not connected to external fermion lines, they should originate only in the interaction. Therefore they should come from terms of the form

$$\langle 0 | T \cdots : \overline{\psi}(z_1) \not{A}(z_1) \psi(z_1) : \cdots : \overline{\psi}(z_n) \not{A}(z_n) \psi(z_n) : \cdots | 0 \rangle \ .$$

- Now it is clear that in order to make the appropriate contractions of the fermion fields to bring them to the form of the Feynman propagator, $\langle 0 | T \psi(z_1) \overline{\psi}(z_2) | 0 \rangle$, it is necessary to make an odd number of permutations of the fermion fields, and therefore we get a $(-)$ sign for the loops.

- This sign is physically relevant because there is a lowest order diagram where the photons do not interact, corersponding to the free propagator. So the minus sign is defined in relation to this lowest order diagram and therefore it is not arbitrary (see the difference with respect to the discussion of the Bahbha scattering).

Feynman rules for QED

We are now in position to state the Feynman rules for QED

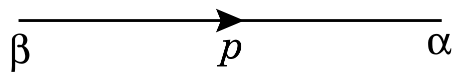
1. For a given process, draw all topologically distinct diagrams.

2. For each electron entering a diagram a factor $u(p,s)$. If it leaves the diagram a factor $\overline{u}(p,s)$.

3. For each positron leaving the diagram (final state) a factor $v(p,s)$. It it enters the diagram (initial state) then we have a factor $\overline{v}(p,s)$.

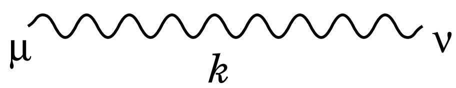
4. For each photon in the initial state we have the vector $\varepsilon^{\mu}(k)$ and in the final state $\varepsilon^{*\mu}(k)$.

5. For each internal fermionic line the propagator



$$S_{F_{\alpha\beta}}(p) = i\frac{(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\varepsilon}$$

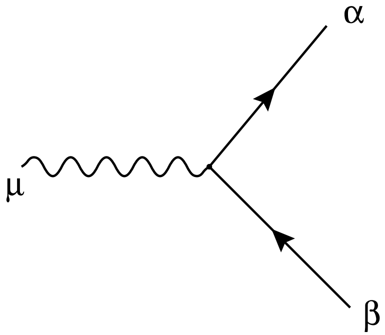
6. For each virtual photon the propagator (Feynman gauge)



$$D_{F\mu\nu}(k) = -i\frac{g_{\mu\nu}}{k^2}$$

Feynman rules for QED

7. For each vertex the factor



$$ie\gamma^\mu$$

8. For each internal momentum, not fixed by conservation of momenta, as in the case of loops, a factor

$$\int \frac{d^4q}{(2\pi)^4}$$

9. For each loop of fermions a -1 sign.

10. A factor of -1 between diagrams that differ by exchange of fermionic lines. In doubt, revert to Wick's theorem.

Feynman rules for QED: Comments

- In QED there are no symmetry factors, that is, they are always equal to 1.
- In our discussion we did not consider the Z factors that come in the reduction formulas. This is true in lowest order in perturbation theory. They can be calculated also in perturbation theory.
- Their definition is (for instance for the electron),

$$\lim_{\not{p}\to m} S'_F(p) = Z_2 S_F(p)$$

where $S'_F(p)$ is the propagator of the theory with interactions. Then we can obtain, in perturbation theory,

$$Z_2 = 1 + O(\alpha) + \cdots$$

- In higher orders it is necessary to correct the external lines with these $\sqrt{Z}$ factors.

General formalism for getting the Feynman rules

- After showing how to obtain the Feynman rules for $\lambda\phi^4$ and QED, we are going to present here, without proof, a general method to obtain the Feynman rules of any theory, including the case when the interactions have derivatives, that we have excluded up to now, and that is very important for the Standard Model.

- This method can only be fully justified with the methods of Chapter 5. For simplicity we will consider only scalar fields

- The starting point is the action taken as a functional of the fields,

$$\Gamma_0[\varphi] \equiv \int d^4x \mathcal{L}[\varphi] \cdot$$

- In fact, $\Gamma_0[\varphi]$ is the generating functional of the one particle irreducible Green functions in lowest order, as we will see in Chapter 5

General formalism for getting the Feynman rules: Propagators

1. Start by evaluating $\Gamma_0^{(2)}(x_i, x_j) \equiv \dfrac{\delta^2\Gamma_0[\varphi]}{\delta\varphi(x_i)\delta\varphi(x_j)}$

2. Then evaluate the Fourier Transform (FT) to get $\Gamma_0^{(2)}(p_i, p_j)$ defined by the relation

$$(2\pi)^4\delta^4(p_i + p_j)\Gamma_0^{(2)}(p_i, p_j) \equiv \int d^4x_i d^4x_j e^{-i(p_i\cdot x_i + p_j\cdot x_j)}\Gamma_0^{(2)}(x_i, x_j)$$

   where all the momenta are *incoming*.

3. The Feynman propagator is then

$$G^{(0)}_{F\,ij} = i[\Gamma_0^{(2)}(p_i, p_j)]^{-1} \ .$$

4. Do not forget that $p_i = -p_j$.

General formalism for getting the Feynman rules: Vertices

1. Evaluate $\Gamma_0^{(n)}(x_1 \cdots x_n) = \dfrac{\delta^n \Gamma_0[\varphi]}{\delta\varphi(x_1)\cdots\delta\varphi(x_n)}$

2. Then take the Fourier Transform to obtain

$$(2\pi)^4 \delta^4(p_1 + p_2 + \cdots + p_n)\Gamma_0^{(n)}(p_1 \cdots p_n)$$

$$\equiv \int d^4x_1 \cdots d^4x_n e^{-i(p_1\cdot x_1 + \cdots p_n \cdot x_n)}\Gamma_0^{(n)}(x_1 \cdots x_n)$$

3. The vertex in momenta space is then given by the rule

$$i\Gamma_0^{(n)}(p_1, \cdots p_n)$$

General formalism for getting the Feynman rules: Comments

- For fermionic fields it is necessary to take care with the order of the derivation. The convention that we take is

$$\frac{\delta^2}{\delta\psi_\alpha(x)\delta\overline{\psi}_\beta(y)}\left(\overline{\psi}(z)\Gamma\psi(z)\right) \equiv \Gamma_{\beta\alpha}\delta^4(z-x)\delta^4(z-y)$$

  $\psi_\alpha(x)$ e $\psi_\beta(x)$ are here taken as classical anti-commuting fields (Grassmann variables, see Chapter 5)

- The functional derivatives are defined by

$$\frac{\delta\varphi_i(x)}{\delta\varphi_k(y)} \equiv \delta_{ik}\delta^4(x-y)$$

Example: Scalar Electrodynamics

- The Lagrangian is

$$\mathcal{L} = (\partial_\mu - ieQA_\mu)\varphi^*(\partial^\mu + ieQA^\mu)\varphi - m\varphi^*\varphi + \mathcal{L}_{\text{Maxwell}} - \frac{\lambda}{4}(\varphi^*\varphi)^2$$

- Therefore

$$\mathcal{L}_{int} = -ieQ\varphi^* \overleftrightarrow{\partial}_\mu \varphi A^\mu + e^2Q^2\varphi^*\varphi A_\mu A^\mu$$

- The propagators are the usual ones, let us consider only the vertices. There are two vertices. The cubic vertex is

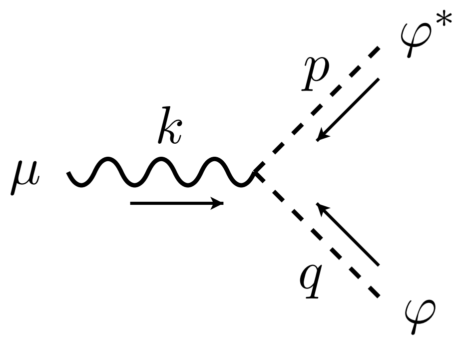


Figure 5: Cubic vertex in scalar QED.

Example: Scalar Electrodynamics

- We calculate

$$\Gamma^{(3)}_{\mu}(x_1,x_2,x_3) = -ieQ \int d^4z \delta^4(z-x_1)(\overrightarrow{\partial}_{\mu} - \overleftarrow{\partial}_{\mu})\delta^4(z-x_2)\delta^4(z-x_3)$$

- Therefore

$$(2\pi)^4\delta^4(p+k+q)\Gamma^{(3)}_{\mu}(p,q,k) \equiv$$

$$\equiv -ieQ \int d^4z d^4x_1 d^4x_2 d^4x_3 e^{-i(x_1\cdot p + x_2\cdot q + x_3\cdot k)}$$

$$\delta^4(z-x_1)\ (\overrightarrow{\partial}_{\mu} - \overleftarrow{\partial}_{\mu})\delta^4(z-x_2)\delta^4(z-x_3)$$

$$= -ieQ \int d^4z d^4x_2 e^{-i[(p+k)\cdot z + q\cdot x_2]}\partial_{\mu}\delta^4(z-x_2)$$

$$+ ieQ \int d^4z d^4x_1 e^{-i[p\cdot x_1 + (q+k)\cdot z]}\partial_{\mu}\delta^4(z-x_1)$$

$$= -ieQ(ip_{\mu} - iq_{\mu})(2\pi)^4\delta^4(p+q+k)$$

- Therefore we obtain for this vertex

$$i\Gamma_\mu(p,q,k) = ieQ\ (p_\mu - q_\mu) = -ieQ\ (q_\mu - p_\mu)$$

- The other vertex is

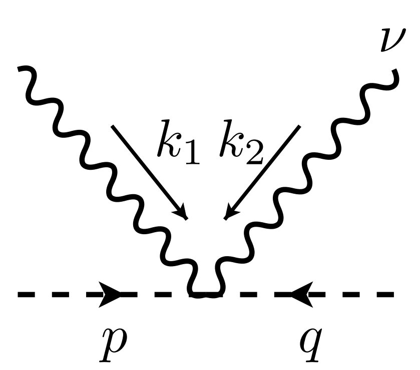


Figure 6: Quartic vertex in scalar QED (seagull).

- We obtain,

$$\Gamma^{(4)}_{\mu\nu}(x_1,x_2,x_3,x_4) = 2e^2Q^2\delta^4(x_1-x_2)\delta^4(x_1-x_3)\delta^4(x_1-x_4)g_{\mu\nu}$$

Example: Scalar Electrodynamics

- Doing the Fourier transform we get

$$\Gamma^{(4)}_{\mu\nu}(p,q,k_1,k_2) = 2(eQ)^2 g_{\mu\nu}$$

- We finally get for the Feynman rule

$$i2e^2Q^2\ g_{\mu\nu}$$

Example: Scalar Electrodynamics

- From the above results we can enunciate a simple rule for interactions that have derivatives of fields.

  Consider that we have one field in the Lagrangian that has a derivative, say $\partial_\mu \phi$. Then the rule is

  $$\partial_\mu \phi \to -i \; (\textit{incoming momentum})_\mu$$

  In the end do not forget to multiply the result by i.

- As an example consider the following term in the Lagrangian for scalar electrodynamics

  $$\mathcal{L} = ieQ\partial_\mu \varphi^* \varphi A^\mu + \cdots$$

- If p is the incoming momentum of the line associated with the field φ^*, we have

  $$\text{Vertex} = i \times (ieQ) \times (-ip_\mu) = i\, e\, Q\, p_\mu$$

  in agreement with what we got before