arXiv:1701.07375v3 [hep-ph] 5 Apr 2017

# Polarization effects in light-by-light scattering: Euler-Heisenberg versus Born-Infeld*

Anton Rebhan and Günther Turk

*Institut für Theoretische Physik, Technische Universität Wien,*
*Wiedner Hauptstraße 8-10, A-1040 Vienna, Austria*
*anton.rebhan@tuwien.ac.at*

The angular dependence of the differential cross section of unpolarized light-by-light scattering summed over final polarizations is the same in any low-energy effective theory of quantum electrodynamics and also in Born-Infeld electrodynamics. In this paper we derive general expressions for polarization-dependent low-energy scattering amplitudes, including a hypothetical parity-violating situation. These are evaluated for quantum electrodynamics with charged scalar or spinor particles, which give strikingly different polarization effects. Ordinary quantum electrodynamics is found to exhibit rather intricate polarization patterns for linear polarizations, whereas supersymmetric quantum electrodynamics and Born-Infeld electrodynamics give particularly simple forms.

*Keywords*: QED; Born-Infeld electrodynamics; light-by-light scattering.

## 1. Introduction

In 1935, long before quantum electrodynamics (QED) was in place as the fundamental theory of electromagnetic interactions, Euler and Kockel[1,2] evaluated its implications on the nonlinear phenomenon of the scattering of light by light at energies below the electron-positron pair creating threshold.[a] The underlying effective action quartic in the electromagnetic field strength tensor that Euler and Kockel had obtained was then generalized to all orders in the famous paper by Heisenberg and Euler,[6] and extended to the case of charged scalar particles by Weisskopf,[7] all in 1936.[b]

A rather different action for nonlinear electrodynamics was proposed in 1934 by Born and Infeld,[10] whose aim was to eliminate the infinite self-energy of charged

---

[a]The ultrarelativistic limit was derived immediately thereafter, in 1936, by Akhiezer, Landau, and Pomeranchuk;[3,4] the complete leading-order result was worked out finally by Karplus and Neuman.[5]

[b]See Ref. 8, 9 for a review of further developments.

particles in classical electrodynamics and which for some time also carried the hope of taming the infinities of quantum field theory. Born-Infeld (BI) electrodynamics leads to light-by-light scattering already at the classical level, which was studied by Schrödinger in the 1940's.[11,12] Remarkably, this theory surfaced again in string theory as the effective action of Abelian vector fields in open bosonic strings.[13c] In fact, many of its curious properties can be understood from a string theoretic point of view.[18,19]

In this paper we revisit the polarization effects in low-energy light-by-light scattering that have been worked out previously in ordinary (spinor) QED[5] for circular polarizations, and we generalize to the most generic low-energy effective action quartic in field strengths, including also a parity (and CP) violating term.

With circular polarizations the various differential cross sections have a rather simple form, where only the magnitude, but not the angular dependence, depends on the parameters of the low-energy effective action, *i.e.*, on the matter content of the fundamental theory. However, with linear polarizations one obtains also widely different angular dependences. Moreover, P and CP odd effects are separated from the other contributions when linearly polarized states are considered.

We admit that our study is mostly of mere academic interest. We are not aware of any concrete theory in the current literature that would lead to the P and CP odd term in the effective action for low-energy light-by-light scattering that we are considering.[d] However, light-by-light scattering is one of the current research topics in high intensity laser physics[20] and polarization effects are of great relevance there, see *e.g.* Refs. 21, 22 where it has been proposed that the effect of vacuum birefringence[23–25] may be tested in counter-propagating laser beams (see also Ref. 26 for more general tests of nonlinear electrodynamics).

## 2. Low-energy effective actions for light-by-light scattering

In the limit of photon energies much smaller than the masses of charged particles, the latter can be integrated out, yielding a gauge and Lorentz invariant effective action that is constructed from the field strength tensor and where the leading terms involve the latter without further derivatives. In an Abelian theory, it is well known that there are only two independent Lorentz (pseudo-)scalars, which we define as

$$\mathcal{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \equiv -\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = -\frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2),$$
$$\mathcal{G} = \frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} = -\mathbf{E} \cdot \mathbf{B}, \tag{1}$$

---

[c]There are supersymmetric extensions of the BI Lagrangian which differ in terms beyond quartic order in the field strength,[14] however the full supersymmetry of ten-dimensional superstrings again singles out the original form.[15–17]

[d]One way to produce such a term would be a coupling of photons to axions and dilatons in a CP-breaking background.

Table 1. Coefficients $c_{1,2}/C$ with $C = \alpha^2/m^4$. (In the BI case we have $c_1 = c_2 = 1/(2b^2)$.)

|  | $c_1/C$ | $c_2/C$ |
|---|---|---|
| scalar QED | 7/90 | 1/90 |
| spinor QED | 8/45 | 14/45 |
| supersymmetric QED | 1/3 | 1/3 |

with $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ in the conventions of a mostly-minus metric and $\varepsilon^{0123} = +1$. More complicated Lorentz scalars such as *e.g.* $F^{\mu\nu}F_{\nu\lambda}F^{\lambda\rho}F_{\rho\mu}$ can always be reduced to combinations of $\mathcal{F}$ and $\mathcal{G}$. (This is most easily understood by the fact that rotational invariance already restricts to three possible invariants, namely $\mathbf{E}^2$, $\mathbf{B}^2$, and $\mathbf{E} \cdot \mathbf{B}$. Boost invariance reduces these to two only.)

The most general low-energy effective action for elastic light-by-light scattering therefore has the form

$$\mathcal{L}^{(4)}_{\text{low en.}} = c_1\mathcal{F}^2 + c_2\mathcal{G}^2 + c_3\mathcal{F}\mathcal{G}. \tag{2}$$

If one furthermore demands invariance under P and CP transformations, the third term is forbidden. It is kept here for generality and in order to see what features in the scattering cross section it would give rise to.

The one-loop contributions to $c_1$ and $c_2$ in spinor and scalar QED have been first obtained by Euler and Kockel[1,2] and Weisskopf,[7] respectively, and are reproduced in Table 1.

The case of low-energy light-by-light scattering in supersymmetric QED was discussed in Ref. 27 as an illustration of a connection between self-duality, helicity, and supersymmetry discovered initially in the context of supergravity.[28] In supersymmetric QED the matter content is given by two charged scalar particles in addition to the charged Dirac fermion. As shown in Table 1, adding twice the contributions of scalar QED to spinor QED leads to $c_1 = c_2$. This corresponds to self-duality of the quartic term,[27] since then one has

$$\mathcal{L}^{(4)}_{\text{low en.,susy}} \propto \mathcal{F}^2 + \mathcal{G}^2 = (\mathcal{F} + i\mathcal{G})(\mathcal{F} - i\mathcal{G}) \tag{3}$$

with

$$\mathcal{F} \pm i\mathcal{G} = \frac{1}{2}(F^{\pm}_{\mu\nu})^2, \quad F^{\pm}_{\mu\nu} := \frac{1}{2}(F_{\mu\nu} \pm i\tilde{F}_{\mu\nu}). \tag{4}$$

The same self-dual form at quartic order in the field strength is found in Born-Infeld electrodynamics, which is given by

$$\mathcal{L}^{\text{BI}} = -b^2\sqrt{-\det\left(g_{\mu\nu} + \frac{1}{b}F_{\mu\nu}\right)}$$

$$= -b^2\left(1 + 2b^{-2}\mathcal{F} - b^{-4}\mathcal{G}^2\right)^{1/2} = -b^2 - \mathcal{F} + \frac{1}{2b^2}(\mathcal{F}^2 + \mathcal{G}^2) + O(b^{-4}), \tag{5}$$

where the parameter $b$ has the meaning of a limiting field strength (in static situations). In fact, Born-Infeld electrodynamics features a nonlinear generalization of Hodge duality invariance that was pointed out already in 1935 by Schrödinger,[29] namely an invariance under the transformations $(\mathbf{E}+i\mathbf{H}) \to e^{i\alpha}(\mathbf{E}+i\mathbf{H})$, $(\mathbf{D}+i\mathbf{B}) \to e^{i\alpha}(\mathbf{D}+i\mathbf{B})$ with $\mathbf{D} = \partial\mathcal{L}/\partial\mathbf{E}$, $\mathbf{H} = -\partial\mathcal{L}/\partial\mathbf{B}$. (See Refs. 30–32 for further discussions.)

Note, however, that supersymmetric Euler-Heisenberg Lagrangians[33] are in general different from Born-Infeld Lagrangians and their supersymmetric generalizations[14] beyond the quartic term in the electromagnetic field strength.

## 3. Scattering Amplitudes

The amplitude for elastic photon scattering with given photon momenta $k_1^\mu, \ldots, k_4^\mu$ (with $\sum_i k_i = 0$) and polarizations $\epsilon_1, \ldots, \epsilon_4$ is obtained from (2) by

$$\mathcal{M}_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}(k_1, k_2, k_3, k_4) = \left[\prod_{j=1}^{4} i(k_j^\rho \epsilon_j^\sigma - k_j^\sigma \epsilon_j^\rho)\frac{\partial}{\partial F_{\rho\sigma}}\right] i\mathcal{L}^{(4)}. \tag{6}$$

This produces 24 terms for each of the terms in (2).[e]

For linear polarizations, we can write $\epsilon = (0, \mathbf{e})$ with a real unit vector $\mathbf{e}$ orthogonal to $\mathbf{k}$. We denote $\mathbf{e}_i$ and $\mathbf{e}_o$ for the directions in and out of the plane of the scattering, respectively, such that $\mathbf{e}_i$, $\mathbf{e}_o$ and $\mathbf{k}/|\mathbf{k}|$ form a right-handed orthogonal basis of unit vectors. For circular polarizations, we introduce the complex unit vectors

$$\mathbf{e}_\pm = \frac{1}{\sqrt{2}}(\mathbf{e}_i \pm i\mathbf{e}_o), \tag{7}$$

where the index $+/-$ denotes positive/negative helicities.[f] Note that $\mathbf{e}_\pm$ are orthonormal in the sense $\mathbf{e}_\pm^* \cdot \mathbf{e}_\pm = \mathbf{e}_\mp \cdot \mathbf{e}_\pm = 1$, $\mathbf{e}_\mp^* \cdot \mathbf{e}_\pm = \mathbf{e}_\pm \cdot \mathbf{e}_\pm = 0$.

The scattering amplitudes, being Lorentz scalars, can be expressed in terms of the Mandelstam variables $s, t, u$. In the center-of-mass system, the only variables are $\omega = |\mathbf{k}|$ and one polar angle $\theta$ (see Fig. 1), which are related to the Mandelstam variables by

$$\begin{aligned}
s &= (k_1 + k_2)^2 = 4\omega^2, \\
t &= (k_1 - k_1')^2 = -2\omega^2(1 - \cos\theta) = -4\omega^2 \sin^2 \tfrac{\theta}{2}, \\
u &= (k_1 - k_2')^3 = -2\omega^2(1 + \cos\theta) = -4\omega^2 \cos^2 \tfrac{\theta}{2},
\end{aligned} \tag{8}$$

---

[e]As already noted in Ref. 34, this immediately shows that the prescription given in the textbook by Itzykson and Zuber[35] has an error in the combinatorics. However, while the formula for $\mathcal{M}$ in Eq. (7-97) of Ref. 35 misses a factor 24, the final result for $d\sigma/d\Omega$ given therein is correct (but the resulting total cross section $\sigma$ contains a typo, see below for the correct value).

[f]In optics, positive helicity is often denoted as left-handed circular polarization, which is at variance with particle physics as well as IEEE conventions.
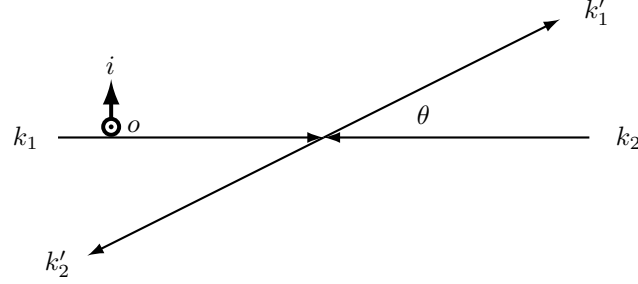
Fig. 1.   Kinematics of photon-photon collisions in the center-of-mass system.

where $k_1' = -k_3$ and $k_2' = -k_4$. (Note that in the case of complex polarization vectors the final polarizations in $\gamma\gamma \to \gamma\gamma$ are given by $\epsilon_1' = \epsilon_3^*$, $\epsilon_2' = \epsilon_4^*$.)

Evaluating (6) for circular polarizations we obtain

$$
\begin{aligned}
-i\mathcal{M}_{++++} &= \tfrac{1}{2}(c_1 - c_2 + ic_3)(s^2 + t^2 + u^2) \\
&= 4(c_1 - c_2 + ic_3)\omega^4(3 + \cos^2\theta), \quad\quad (9) \\
\mathcal{M}_{+++-} = \mathcal{M}_{++-+} &= \mathcal{M}_{+-++} = \mathcal{M}_{-+++} = 0, \quad\quad (10) \\
-i\mathcal{M}_{++--} &= \tfrac{1}{2}(c_1 + c_2)s^2 = 8(c_1 + c_2)\omega^4, \quad\quad (11) \\
-i\mathcal{M}_{+-+-} &= \tfrac{1}{2}(c_1 + c_2)t^2 = 8(c_1 + c_2)\omega^4\sin^4(\theta/2), \quad\quad (12) \\
-i\mathcal{M}_{+--+} &= \tfrac{1}{2}(c_1 + c_2)u^2 = 8(c_1 + c_2)\omega^4\cos^4(\theta/2), \quad\quad (13)
\end{aligned}
$$

and all other amplitudes are obtained by complex conjugation which flips all helicities, e.g. $\mathcal{M}_{----} = \mathcal{M}_{++++}^*$.

For the coefficients $c_i$ corresponding to spinor QED (see Table 1), this reproduces the low-energy result given in Refs. 5, 36 (as shown in the latter, amplitudes with an odd number of $+$ or $-$ helicities start to contribute at order $\omega^6/m^6$).

Notice that the P and CP odd contribution proportional to $c_3$ shows up only in the amplitude $M_{++++} = \mathcal{M}_{++++}^*$, corresponding to scattering with polarizations $++ \to --$ and $-- \to ++$, where it introduces a phase in the otherwise purely imaginary expression.

6  *A. Rebhan, G. Turk*

The amplitudes for the linear polarizations in and out of the collision plane read

$$-i\mathcal{M}_{iiii} = \tfrac{1}{2}c_1(s^2 + t^2 + u^2) = 4c_1\omega^4(3 + \cos^2\theta), \tag{14}$$

$$-i\mathcal{M}_{iiio} = -i\mathcal{M}_{iioi} = -i\mathcal{M}_{ioii} = -i\mathcal{M}_{oiii} = -\tfrac{1}{4}c_3(s^2 + t^2 + u^2)$$
$$= -2c_3\omega^4(3 + \cos^2\theta), \tag{15}$$

$$-i\mathcal{M}_{iioo} = -\tfrac{1}{2}c_1 s^2 + \tfrac{1}{2}c_2(t^2 + u^2) = -8c_1\omega^4 + 4c_2\omega^4(1 + \cos^2\theta), \tag{16}$$

$$-i\mathcal{M}_{ioio} = -\tfrac{1}{2}(c_1 + c_2)su - \tfrac{1}{4}(c_1 - c_2)(s^2 + t^2 + u^2)$$
$$= [4(c_1 + c_2)(1 + \cos\theta) + 2(c_2 - c_1)(3 + \cos^2\theta)]\omega^4,$$
$$= [11c_2 - 3c_1 + 4(c_1 + c_2)\cos\theta + (c_2 - c_1)\cos 2\theta]\omega^4 \tag{17}$$

$$-i\mathcal{M}_{iooi} = -\tfrac{1}{2}(c_1 + c_2)st - \tfrac{1}{4}(c_1 - c_2)(s^2 + t^2 + u^2)$$
$$= [4(c_1 + c_2)(1 - \cos\theta) + 2(c_2 - c_1)(3 + \cos^2\theta)]\omega^4$$
$$= [11c_2 - 3c_1 - 4(c_1 + c_2)\cos\theta + (c_2 - c_1)\cos 2\theta]\omega^4. \tag{18}$$

All amplitudes are invariant under flipping all linear polarizations $i \leftrightarrow o$, which fixes those not explicitly given. (Note that also the amplitudes with linear polarizations can be expressed solely in terms of squares of Mandelstam variables by rewriting $su = (t^2 - s^2 - u^2)/2$ and $st = (u^2 - s^2 - t^2)/2$.)

In contrast to the case of circular polarizations, all amplitudes for linear polarizations are purely imaginary and the P and CP odd contribution is separated in the amplitudes with an odd number of $i$ or $o$ polarizations.

In supersymmetric QED and in Born-Infeld electrodynamics, where $c_3 = 0$ and $c_1 = c_2 = 1/(2b^2)$, the scattering amplitudes simplify in that $\mathcal{M}_{++++}^{\mathrm{BI/susy}} = \mathcal{M}_{----}^{\mathrm{BI/susy}} = 0$, because $\mathcal{L}^{(4)} \propto (F_{\mu\nu}^+)^2(F_{\mu'\nu'}^-)^2$ requires an equal number of $+$ and $-$ helicities. The amplitudes with mixed linear polarizations also simplify and take the special forms

$$-i\mathcal{M}_{iioo}^{\mathrm{BI/susy}} = -2b^{-2}\omega^4\sin^2\theta,$$
$$-i\mathcal{M}_{ioio}^{\mathrm{BI/susy}} = 8b^{-2}\omega^4\cos^2(\theta/2),$$
$$-i\mathcal{M}_{iooi}^{\mathrm{BI/susy}} = 8b^{-2}\omega^4\sin^2(\theta/2). \tag{19}$$

## 4. Differential Cross Sections

The final expression for the differential cross section reads

$$\frac{d\sigma}{d\Omega} = \frac{1}{(16\pi)^2\omega^2}|\mathcal{M}_{\epsilon_1\epsilon_2\epsilon_1'^*\epsilon_2'^*}(k_1, k_2, -k_1', -k_2')|^2 \tag{20}$$

in the center-of-mass system, to which we will stick in what follows.

Let us just point out that with the results given above in terms of Mandelstam variables, an equivalent, frame-independent expression for light-by-light scattering is given by

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}|\mathcal{M}|^2, \tag{21}$$

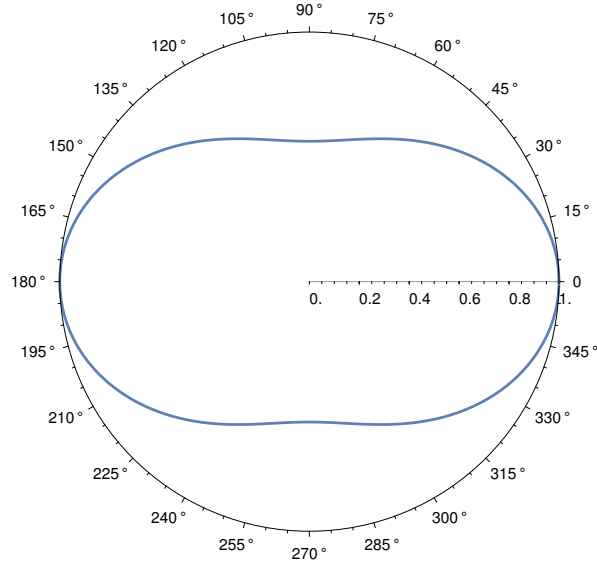which would be useful for describing the scattering of photons with unequal energies.

Fig. 2.   Polar plot of the universal form of $d\sigma/d\Omega \propto (3 + \cos^2\theta)^2$ for unpolarized light-by-light scattering at leading order in $\alpha$ and $\omega/m$. The same angular dependence of $d\sigma/d\Omega$ appears for photon polarizations $++ \rightarrow --$ and $-- \rightarrow ++$ and also in all parity-violating contributions involving $c_3$ such as $d\sigma/d\Omega$ for $ii \rightarrow io$.

### 4.1.  *Unpolarized inital states with summation over final polarizations*

The unpolarized differential cross section for low-energy light-by-light scattering, averaged over inital polarizations and summed over final polarizations, reads

$$\frac{d\sigma^{\text{unpol.}}}{d\Omega} = \frac{\omega^6}{64\pi^2} \left(3c_1^2 - 2c_1 c_2 + 3c_2^2 + 2c_3^2\right)(3 + \cos^2\theta)^2. \tag{22}$$

Evidently, this result has a universal dependence on the scattering angle, which is displayed as a polar plot in Fig. 2.

In ordinary spinor QED (see Table 1), this gives the well-known result[2, 5, 35, 37]

$$\frac{d\sigma^{\text{unpol.}}_{\text{QED}}}{d\Omega} = \frac{139\alpha^4\omega^6}{(180\pi)^2 m^8}(3 + \cos^2\theta)^2. \tag{23}$$

Replacing electrons by two charged scalar fields of the same mass as electrons would amount to replacing the factor 139 by 34. Scalar QED, even with the same number of degrees of freedom as ordinary QED, thus turns out to be much less efficient in scattering light by light in the low-energy region. Finally, supersymmetric QED would have a factor 225 in place of 139.

The total cross section is given by

$$\sigma = \frac{1}{2} \int d\Omega \frac{d\sigma}{d\Omega}, \tag{24}$$

where the factor $1/2$ is due to having identical particles in the final state. (Alternatively, one could do without this symmetry factor and integrate over only one hemisphere.[35]) This yields

$$\sigma(\gamma\gamma \to \gamma\gamma)^{\text{unpol.}} = \frac{7(3c_1^2 - 2c_1c_2 + 3c_2^2 + 2c_3^2)\omega^6}{20\pi}. \tag{25}$$

In ordinary QED one obtains

$$\sigma(\gamma\gamma \to \gamma\gamma)_{\text{QED}}^{\text{unpol.}} = \frac{973\alpha^4\omega^6}{10125\pi m^8} \tag{26}$$

in agreement with Refs. 2, 5, 37.[g]

### 4.2. *Final polarization with initial unpolarized photons*

When the polarizations of the photons after the scattering of initially unpolarized photons are measured, the angular dependence of the differential cross section is in general different from (22).

Separating the contributions of equal and opposite circular polarizations in the final state, we obtain

$$\frac{d\sigma^{\text{unpol.}\to++}}{d\Omega} = \frac{\omega^6}{2(16\pi)^2} \left( 131(c_1^2 + c_2^2) - 134c_1c_2 + 99c_3^2 \right.$$
$$\left. + ((c_1 - c_2)^2 + c_3^2) \left[ 28\cos 2\theta + \cos 4\theta \right] \right) \tag{27}$$

and

$$\frac{d\sigma^{\text{unpol.}\to+-}}{d\Omega} = \frac{\omega^6}{4(16\pi)^2}(c_1 + c_2)^2 \left[ 35 + 28\cos 2\theta + \cos 4\theta \right]. \tag{28}$$

(Twice the sum of (27) and (28) reproduces (22), as it should.)

The results for the three QED theories of Table 1 are compared in Fig. 3. (In the case of scalar QED, we have doubled the matter content and considered two charged scalar fields, because the supersymmetric case corresponds to the combination of one Dirac fermion and two scalars as charged matter fields.)

A noteworthy feature appears in the supersymmetric/Born-Infeld case in that the differential cross section for unpol. $\to ++$ or $--$ is completely isotropic, while ordinary QED shows (a rather moderate amount of) anisotropy. On the other hand, the result for unpol. $\to +-$ or $-+$ has a universal angular dependence.

---

[g]Ref. 35 contains a typo here: the factor $\frac{56}{11}$ in Eq. (7-101) should read $\frac{56}{5}$.

Turning next to the case of linear polarizations, we obtain

$$\frac{d\sigma^{\text{unpol.}\to\text{ii}}}{d\Omega} = \frac{\omega^6}{4(16\pi)^2}\Big(262c_1^2 - 96c_1c_2 + 38c_2^2 + 99c_3^2$$
$$+4(14c_1^2 - 8c_1c_2 + 6c_2^2 + 7c_3^2)\cos 2\theta + (2(c_1^2 + c_2^2) + c_3^2)\cos 4\theta\Big) \qquad (29)$$

and

$$\frac{d\sigma^{\text{unpol.}\to\text{io}}}{d\Omega} = \frac{\omega^6}{4(16\pi)^2}\Big(35c_1^2 - 102c_1c_2 + 259c_2^2 + 99c_3^2$$
$$+4(7c_1^2 - 6c_1c_2 + 15c_2^2 + 7c_3^2)\cos 2\theta + ((c_1 - c_2)^2 + c_3^2)\cos 4\theta\Big), \qquad (30)$$

which are displayed for the three versions of QED in Fig. 4.

Now we find that for the same linear polarizations in the final state, scalar and spinor QED are rather similar in form as well as magnitude (when both have the same number of charged degrees of freedom), but this is completely different for opposite linear polarizations. In the latter case, the scalar QED result is extremely suppressed, in particular for right-angle scattering $|\theta| = \pi/2$.
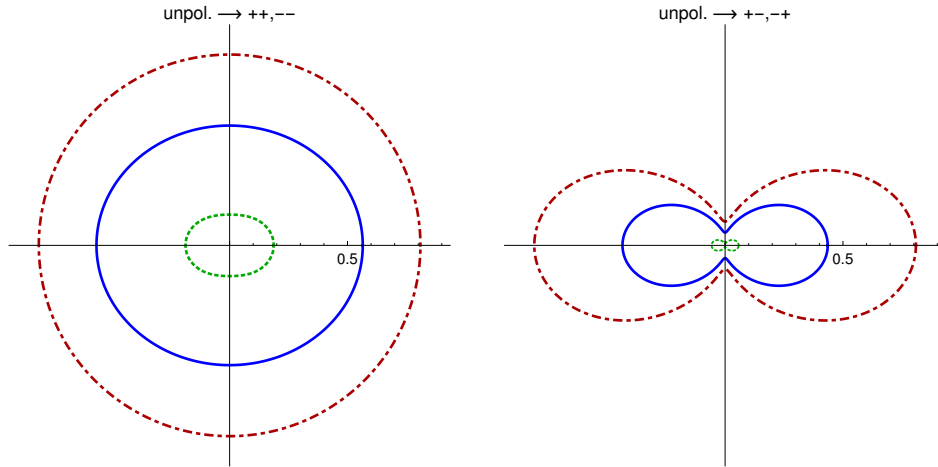


Fig. 3. Leading-order differential cross section for scattering of unpolarized photons into two photons of same (left) and opposite (right) circular polarizations, for ordinary QED (full lines), scalar QED with two charged scalar fields of mass equal to electrons (dotted lines), and supersymmetric QED (dashed-dotted lines), normalised to the maximal value of unpolarized scattering in ordinary QED. The supersymmetric result, which has the same form as in Born-Infeld theory, turns out to be completely isotropic in the case *unpol.* → ++, −−.
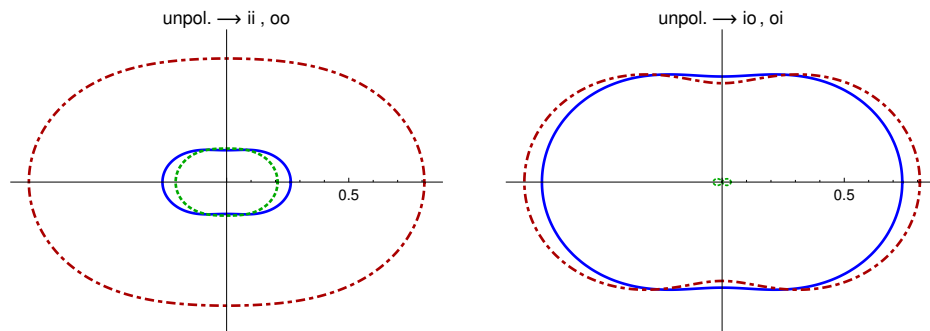
Fig. 4.   Same as Fig. 3, but for scattering into photons of same (left) and opposite (right) linear polarizations. For same polarizations, scalar QED and ordinary QED are rather similar; for opposite polarizations the scalar QED result is extremely suppressed, in particular at $|\theta| = \pi/2$.

### 4.3. *Polarised initial and final states*

The total scattering cross sections of initial polarized states is given by the above expressions through

$$\frac{d\sigma^{\epsilon\epsilon' \to any}}{d\Omega} = 4 \frac{d\sigma^{\text{unpol.} \to \epsilon\epsilon'}}{d\Omega}. \tag{31}$$

New features are brought about when both initial states have definite polarizations.

When all polarizations are circular, the angular dependence has universal form and only the magnitude varies between different theories. However, with linear polarizations, these differences become visible as different angular patterns, occasionally involving destructive interference in certain directions. The only exceptions are the case of all four linear polarizations being equal and the hypothetical P and CP violating contribution involving $c_3$, which have the same angular dependence as the unpolarized case. However, while the contribution involving $c_3$ gets buried in parity conserving contributions to $++ \to --$ and $-- \to ++$, with linear polarizations it would constitute the leading low-energy contribution to scattering with an odd number of $i$ or $o$ polarizations (if such P and CP violating vacuum polarization effects should exist).

In Figures 5–7 we juxtapose the different patterns for differential cross sections with processes involving linear polarizations that are not all equal.

In Fig. 5 the case of scattering with parallel linear polarizations is displayed, where the final state has parallel linear polarizations orthogonal to the initial ones. Here the three theories differ most conspicuously: ordinary spinor QED has maximal scattering in forward and backward directions, scalar QED is roughly isotropic, and supersymmetric QED (as well as BI electrodynamics) is maximal at right angle scattering. However, while the latter shows a perfect squared dipole pattern, the ordinary QED result is four lobes, with tiny lobes at right angles, made visible only in the greatly magnified Fig. 6. At right angles the differential cross section is a

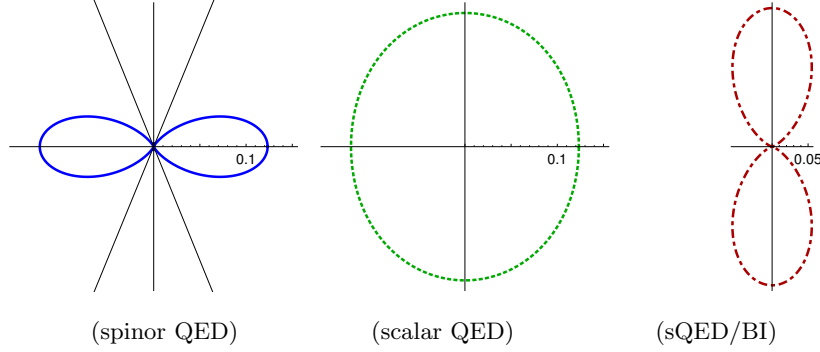(spinor QED)             (scalar QED)             (sQED/BI)

Fig. 5.   Leading-order differential cross section for scattering of two photons with parallel linear polarizations into two photons with polarizations orthogonal to the inital ones ($ii \to oo$ or $oo \to ii$), normalized to the QED result for $ii \to any$ at $\theta = 0$, for ordinary QED (left), scalar QED (middle), and supersymmetric QED (right). Here the ordinary QED result has a pronounced maximum at $\theta = 0, \pi$, a secondary tiny maximum at $|\theta| = \pi/2$ (cf. Fig. 6), and zeros at $|\cos\theta| = 1/\sqrt{7} \approx 0.378$ (denoted by thin straight lines); the scalar QED result is maximal at $|\theta| = \pi/2$ but close to isotropic; the supersymmetric QED (Born-Infeld) result is proportional to $\sin^4\theta$, the square of a perfect dipole.
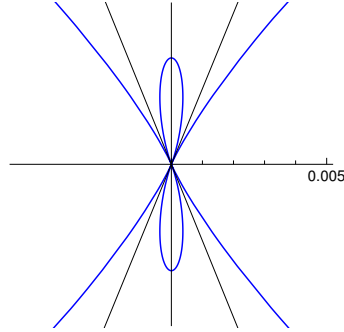


Fig. 6.   Magnified version of the QED result for scatterings with polarizations $ii \to oo$ or $oo \to ii$. The magnitude of the secondary maximum at $|\theta| = \pi/2$ is smaller than the primary maximum by a factor of $1/36 \approx 0.028$.

factor of $1/36 \approx 0.028$ smaller than the maximal values at $\theta = 0$ and $\pi$, and a zero occurs at $|\cos\theta| = 1/\sqrt{7} \approx 0.378$.

In Fig. 7 the differential cross section for scattering with orthogonal linear polarizations are given, $io \to io$ or $oi \to oi$. (The flipped cases $io \to oi$ or $oi \to io$ are given by the mirror images $\theta \to \theta + \pi$.) Here scalar QED not only exhibits a rather different pattern than spinor and supersymmetric QED, it is also greatly suppressed, by an order of magnitude, similarly to the case unpol. $\to$ io shown above. Scalar QED has zeros at $\cos\theta = (4 - \sqrt{13})/3 \approx 0.13$ and a maximum for back scattering ($\theta = \pi$). This maximum exactly equals the minimal value of the QED result at the same angle (with two charged scalars), leading to a destructive interference at $\theta = \pi$ in the supersymmetric result.
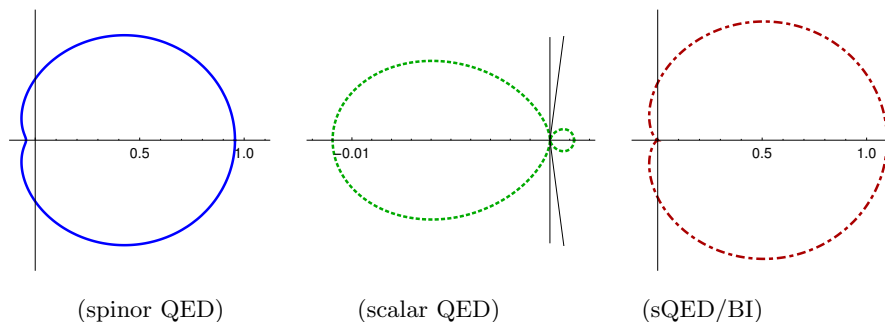
12   *A. Rebhan, G. Turk*



(spinor QED)            (scalar QED)            (sQED/BI)

Fig. 7.   Same as Fig. 5, but for scattering of two photons with orthogonal linear polarizations $io \to io$ or $oi \to oi$. The scalar QED result, which is an order of magnitude smaller than the spinor QED result, has zeros at $\cos\theta = (4 - \sqrt{13})/3 \approx 0.13$ (denoted by thin straight lines). Its maximum at $\theta = \pi$ is exactly equal to the minimal value of the QED result there and interferes destructively in the supersymmetric result at $\theta = \pi$.

## 5. Exceptional properties of Born-Infeld electrodynamics

The special feature of self-duality of BI electrodynamics and of supersymmetric QED which underlies the fact that $\mathcal{M}_{++++} = M^*_{----} = 0$ is also responsible for the absence of vacuum birefringence.[34h] Vacuum birefringence means different indices of refraction for different polarizations in the vacuum polarized by electromagnetic fields (either the field of another wave or a constant external field), which to leading order are determined by

$$\frac{n_1 - 1}{n_2 - 1} = \frac{c_1}{c_2} \tag{32}$$

when $c_3 = 0$,[i] and this equals unity for BI electrodynamics and of supersymmetric QED.

However, BI electrodynamics has further exceptional properties, some of which go beyond supersymmetric Euler-Heisenberg Lagrangians, but which can be understood from a string theoretic point of view.[19]

In general, a Lagrangian that is nonlinear in the two quadratic invariants $\mathcal{F}$ and $\mathcal{G}$ still has exact solutions in the form of monochromatic waves, for which $\mathcal{F} = \mathcal{G} = 0$, independent of the size of the amplitude and thus of the degree of nonlinearity. But it is no longer possible to superimpose different monochromatic plane waves such that they form stable localized wave packets. Instead, their time evolution permits singularities such as shock formation where the limit of applicability of the effective field theory is reached.[39]

BI electrodynamics is *exceptional* (in the sense of Lax) that no shocks are formed.[19,40] A plane wave with arbitrary polarization and arbitrary profile prop-

---

[h]In Ref. 38 the absence of vacuum birefringence has been shown to hold also for the Euler-Heisenberg Lagrangian of N=2 supersymmetric QCD at strong coupling as derived from gauge-gravity duality.
[i]See Appendix A for the generalization to $c_3 \neq 0$.

agating with the speed of light is an exact solution of the field equations of BI electrodynamics.[41] *E.g.,* the ansatz $A_y = f(t, x)$ leads to

$$\left[1 + b^{-2}(f')^2\right]\ddot{f} - 2b^{-2}f'\dot{f}\dot{f}' - \left[1 - b^{-2}(\dot{f})^2\right]f'' = 0, \tag{33}$$

which is solved by $f(t, x) = g(t - x)$ and $f(t, x) = g(t + x)$ with arbitrary function $g$.

In 1943, Schrödinger[12] moreover found that two counter-propagating circularly polarized monochromatic plane waves form an exact solution where the phase velocity $v$ of the two waves in the center-of-mass system is reduced with $v^{-2} - 1$ being proportional to the energy density. Each of the two monochromatic plane waves therefore represents a medium with a certain index of refraction for the other one.

A necessary condition for this to be possible can in fact be seen to hold in the above results for polarized differential cross sections. In BI electrodynamics we have $\mathcal{M}_{++++} = 0$, which actually means that there is no scattering of two right-handed photons into left-handed ones, $\sigma(++ \rightarrow --) = 0$; BI electrodynamics preserves helicity.[42] The so-called[43j] MHV amplitude $\mathcal{M}_{++--} \neq 0$ corresponds (here slightly confusingly) to $++ \rightarrow ++$, and this is isotropic. The situation is therefore similar to electrodynamics in an isotropic medium with a polarization-independent refractive index. The homogeneous isotropic scattering superimposed on the individual plane wave traveling originally with the speed of light reduces its phase velocity uniformly. In the case of light-by-light scattering a counter-propagating wave of same helicity effectively provides such an isotropic medium.

## 6. Conclusion

In this paper we have studied the general form of polarization-dependent differential cross sections of light-by-light scattering at low energies, even including a parity violating contribution. While the angular dependence of all amplitudes with given helicity states has a universal form such that only their magnitude varies between different effective field theories (with a zero for $\mathcal{M}_{++++} = \mathcal{M}^*_{----}$ in the case of supersymmetric QED and Born-Infeld electrodynamics), linear polarizations lead to interesting patterns for different theories. Moreover, a parity violating contribution, which in the case of circular polarizations is buried in the parity conserving contributions of (non-supersymmetric) QED to polarizations $++ \rightarrow --$ and $-- \rightarrow ++$, appears as a leading contribution in scattering with an odd number of parallel linear polarizations.

---

[j]Amplitudes with two helicities of one type and all the others of the other type are called maximally helicity-violating (MHV) because the even more helicity-violating ones with all helicity indices equal, or only one unequal, vanish in a *supersymmetric* gauge theory, and have no cuts for general gauge theories.

14   *A. Rebhan, G. Turk*

## Acknowledgments

## Appendix A. Generalized formulae for vacuum birefringence

The continuous scattering of light in a background provided by extended and strong coherent electromagnetic fields can be described by refractive indices, which in general depend on polarization. This gives rise to the phenomenon of vacuum birefringence,[23–25] for which recently direct evidence has been claimed in the optical polarimetry observation of the isolated neutron star RX J1856.5−3754.[44]

In this Appendix we extend the analysis of Ref. 21 to the general form of the Lagrangian for nonlinear electrodynamics (2), including the P and CP violating term proportional to $c_3$.

Linearizing the field equations following from (2) in the presence of electromagnetic background fields, $F_{\mu\nu} \to F_{\mu\nu} + (\partial_\mu A_\nu - \partial_\nu A_\mu)$, one obtains the following fluctuation equation for the potential $A^\mu(k)$ in momentum space,

$$(1 - 2c_1\mathcal{F} - c_3\mathcal{G})(g_{\mu\nu}k^2 - k_\mu k_\nu)A^\nu = \left[2c_1 b_\mu b_\nu + 2c_2 \tilde{b}_\mu \tilde{b}_\nu + c_3(b_\mu \tilde{b}_\nu + \tilde{b}_\mu b_\nu)\right] A^\nu \tag{A.1}$$

with

$$b_\mu = F_{\mu\nu}k^\nu, \quad \tilde{b}_\mu = \tilde{F}_{\mu\nu}k^\nu, \tag{A.2}$$

satisfying $b \cdot k = 0 = \tilde{b} \cdot k$. Nontrivial background fields lead to nonzero $k^2 \sim \max(c_1, c_2, c_3)$ ($\propto \alpha^2$ in QED). To first order[k] in the $c$'s, one can drop the terms involving $\mathcal{F}$ and $\mathcal{G}$ on the left-hand side of (A.1). Furthermore, since $k^\mu$ is approximately light-like, we also have $b \cdot \tilde{b} = 0$ and $b^2 = \tilde{b}^2 = -\omega^2 Q^2$ with a nonnegative quantity[21] $Q^2$. The latter equals $(\mathbf{e}_k \times \mathbf{B})^2$ in a constant magnetic background field; in a counter-propagating plane wave one has $Q^2 = 4I$, with $I$ denoting the background energy density.

With $k^\mu = \omega(1, n\mathbf{e}_k)$, (A.1) has eigenvector solutions $A^\nu \propto \beta b^\nu + \tilde{\beta}\tilde{b}^\nu$ with two possible values for the refractive index $n$,

$$n_{1,2}^2 - 1 = \left[c_1 + c_2 \pm \sqrt{(c_1 - c_2)^2 + c_3^2}\right] Q^2. \tag{A.3}$$

With $c_3 = 0$ this reduces to $n_{1,2} \approx 1 + c_{1,2}Q^2$ and $A_1^\nu \propto b^\nu$, $A_2^\nu \propto \tilde{b}^\nu$. When $c_3 \neq 0$, $b^\nu$ and $\tilde{b}^\nu$ get mixed with an angle $\delta$ given by

$$\tan\delta = \frac{\beta_2}{\tilde{\beta}_2} = -\frac{\tilde{\beta}_1}{\beta_1} = \frac{c_3}{c_1 - c_2 + \sqrt{(c_1 - c_2)^2 + c_3^2}} \,. \tag{A.4}$$

---

[k]A more general analysis of birefringence effects in nonlinear electrodynamics (likewise permitting parity violating terms) can be found in Ref. 45.

## References

1. H. Euler and B. Kockel, *Naturwiss.* **23**, 246 (1935).
2. H. Euler, *Ann. d. Phys.* **26**, 398 (1936).
3. A. Akhieser, L. Landau and I. Pomeranchook, *Nature* **138**, 206 (1936).
4. A. Achieser, *Physik Z. Sowjetunion* **11**, 263 (1937).
5. R. Karplus and M. Neuman, *Phys. Rev.* **83**, 776 (1951).
6. W. Heisenberg and H. Euler, *Z. Phys.* **98**, 714 (1936).
7. V. Weisskopf, *Kong. Dan. Vid. Selsk. Mat.-fys. Medd.* **XIV/6**, 1 (1936).
8. W. Dittrich and H. Gies, *Springer Tracts Mod. Phys.* **166**, 1 (2000).
9. G. V. Dunne, Heisenberg-Euler effective lagrangians: Basics and extensions, in *From Fields to Strings: Circumnavigating Theoretical Physics - Ian Kogan Memorial Collection*, eds. M. Shifman, A. Vainshtein and J. Wheater (World Scientific, Singapore, 2004) pp. 445–522, `arXiv:hep-th/0406216`.
10. M. Born and L. Infeld, *Proc. Roy. Soc. Lond.* **A144**, 425 (1934).
11. E. Schrödinger, *Proc. Roy. Irish Acad. (A)* **47**, 77 (1942).
12. E. Schrödinger, *Proc. Roy. Irish Acad. (A)* **49**, 59 (1943).
13. E. S. Fradkin and A. A. Tseytlin, *Phys. Lett.* **B163**, 123 (1985).
14. S. Cecotti and S. Ferrara, *Phys. Lett.* **B187**, 335 (1987).
15. A. A. Tseytlin, *Nucl. Phys.* **B276**, 391 (1986), [Erratum: Nucl. Phys.B291, 876 (1987)].
16. R. R. Metsaev, M. Rakhmanov and A. A. Tseytlin, *Phys. Lett.* **B193**, 207 (1987).
17. A. A. Tseytlin, Born-Infeld action, supersymmetry and string theory, in *The Many Faces of the Superworld - Yuri Golfand Memorial Volume*, ed. M. A. Shifman (World Scientific, Singapore, 1999), pp. 417–452, `arXiv:hep-th/9908105`.
18. G. W. Gibbons, *Nucl. Phys.* **B514**, 603 (1998), `arXiv:hep-th/9709027`.
19. G. W. Gibbons and C. A. R. Herdeiro, *Phys. Rev.* **D63**, 064006 (2001), `arXiv:hep-th/0008052`.
20. M. Marklund and J. Lundin, *Eur. Phys. J.* **D55**, 319 (2009), `arXiv:0812.3087 [hep-th]`.
21. T. Heinzl, B. Liesfeld, K.-U. Amthor, H. Schwoerer, R. Sauerbrey and A. Wipf, *Opt. Commun.* **267**, 318 (2006), `arXiv:hep-ph/0601076`.
22. F. Karbstein, H. Gies, M. Reuter and M. Zepf, *Phys. Rev.* **D92**, 071301 (2015), `arXiv:1507.01084 [hep-ph]`.
23. J. J. Klein and B. P. Nigam, *Phys. Rev.* **135**, B1279 (1964).
24. R. Baier and P. Breitenlohner, *Nuovo Cim.* **47**, 117 (1967).
25. E. Brezin and C. Itzykson, *Phys. Rev.* **D3**, 618 (1971).
26. M. Fouché, R. Battesti and C. Rizzo, *Phys. Rev.* **D93**, 093020 (2016), `arXiv:1605.04102 [physics.optics]`.
27. M. J. Duff and C. J. Isham, *Phys. Lett.* **B86**, 157 (1979).
28. M. T. Grisaru, H. N. Pendleton and P. van Nieuwenhuizen, *Phys. Rev.* **D15**, 996 (1977).
29. E. Schrödinger, *Proc. Roy. Soc. Lond.* **A150**, 465 (1935).
30. I. Białynicki-Birula, Nonlinear electrodynamics: Variations on a theme by Born and Infeld, in *Quantum theory of particles and fields - Birthday volume dedicated to Jan Lopuszański*, eds. B. Jancewicz and J. Lukierski (World Scientific, Singapore, 1983), pp. 31–48.
31. G. W. Gibbons and D. A. Rasheed, *Nucl. Phys.* **B454**, 185 (1995), `arXiv:hep-th/9506035`.
32. M. K. Gaillard and B. Zumino, Selfduality in nonlinear electromagnetism, in *Supersymmetry and quantum field theory - Proceedings of the D. Volkov Memorial Seminar, Kharkov, Ukraine, January 5-7, 1997*, Lect. Notes Phys. Vol. 509 (1998), pp. 121–129,

16   *A. Rebhan, G. Turk*

arXiv:hep-th/9705226.

33. S. M. Kuzenko and S. J. Tyler, *JHEP* **05**, 081 (2007), arXiv:hep-th/0703269.
34. J. M. Dávila, C. Schubert and M. A. Trejo, *Int. J. Mod. Phys. A* **29**, 1450174 (2014), arXiv:1310.8410 [hep-ph].
35. C. Itzykson and J.-B. Zuber, *Quantum field theory* (McGraw-Hill, Singapore, 1985).
36. Y. Liang and A. Czarnecki, *Can. J. Phys.* **90**, 11 (2012), arXiv:1111.6126 [hep-ph].
37. E. Lifshitz, V. Berestetskii and L. Pitaevskii, *Quantum electrodynamics* (Butterworth-Heinemann, Oxford, 1982).
38. K. Hashimoto, T. Oka and A. Sonoda, *JHEP* **06**, 085 (2014), arXiv:1403.6336 [hep-th].
39. M. Lutzky and J. S. Toll, *Phys. Rev.* **113**, 1649 (1959).
40. G. Boillat, *J. Math. Phys.* **11**, 941 (1970).
41. G. B. Whitham, *Linear and nonlinear waves* (Wiley, New York, 1974).
42. A. A. Rosly and K. G. Selivanov,  Helicity conservation in Born-Infeld theory, in *Quarks. Proceedings, 12th International Seminar on High Energy Physics, Quarks'2002, Novgorod, Russia, June 1-7, 2002*, (2002), arXiv:hep-th/0204229.
43. Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, *Nucl. Phys.* **B425**, 217 (1994), arXiv:hep-ph/9403226.
44. R. P. Mignani, V. Testa, D. G. Caniulef, R. Taverna, R. Turolla, S. Zane and K. Wu, *Mon. Not. R. Astron. Soc.* **465**, 492 (2016), arXiv:1610.08323 [astro-ph.HE].
45. Y. N. Obukhov and G. F. Rubilar, *Phys. Rev.* **D66**, 024042 (2002), arXiv:gr-qc/0204028.