# Reduction schemes for one-loop tensor integrals 

A. Denner ${ }^{\text {a, } *}$, S. Dittmaier ${ }^{\text {b }}$<br>${ }^{\text {a }}$ Paul Scherrer Institut, Würenlingen und Villigen, CH-5232 Villigen PSI, Switzerland<br>${ }^{\mathrm{b}}$ Max-Planck-Institut für Physik (Werner-Heisenberg-Institut), D-80805 München, Germany

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#### Abstract

We present new methods for the evaluation of one-loop tensor integrals which have been used in the calculation of the complete electroweak one-loop corrections to $\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow 4$ fermions. The described methods for 3-point and 4-point integrals are, in particular, applicable in the case where the conventional PassarinoVeltman reduction breaks down owing to the appearance of Gram determinants in the denominator. One method consists of different variants for expanding tensor coefficients about limits of vanishing Gram determinants or other kinematical determinants, thereby reducing all tensor coefficients to the usual scalar integrals. In a second method a specific tensor coefficient with a logarithmic integrand is evaluated numerically, and the remaining coefficients as well as the standard scalar integral are algebraically derived from this coefficient. For 5-point tensor integrals, we give explicit formulas that reduce the corresponding tensor coefficients to coefficients of 4-point integrals with tensor rank reduced by one. Similar formulas are provided for 6 -point functions, and the generalization to functions with more internal propagators is straightforward. All the presented methods are also applicable if infrared (soft or collinear) divergences are treated in dimensional regularization or if mass parameters (for unstable particles) become complex.


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## 1. Introduction

Future high-energy colliders, such as the LHC and the ILC, will allow us to search for new physics and to test the Standard Model of the electroweak and strong interaction with high precision. Various interesting processes naturally involve many particles in the final state, where "many" means three, four, or more particles. Such processes often proceed via one or more res-

[^0]onances that subsequently decay, or they represent an irreducible background to such resonance processes. In order to exhaust the potential of future colliders, precise theoretical predictions including strong and electroweak corrections to many-particle processes are mandatory.

The calculation of radiative corrections to complicated processes poses a number of problems. Besides the huge amount of algebra, the appearance of unstable particles, and the integration of the multi-dimensional phase space, a numerically stable evaluation of the loop integrals is an important ingredient. In this paper we are concerned with the calculation of one-loop integrals, including those with five and six external legs. The generalization from six to more external legs is straightforward.

Pioneering work in the calculation of one-loop integrals was performed by Veltman and collaborators. Together with ' $t$ Hooft, he provided compact explicit expressions for the basic one-loop integrals, the scalar 1-point, 2-point, 3-point, and 4-point integrals [1], which have been completed later by other authors [2]. Elaborating on an idea of Brown and Feynman [3], together with Passarino he provided systematic formulas that allow to reduce all tensor integrals with up to four internal propagators to the basic scalar integrals [4]. These methods are basically sufficient for the calculation of radiative corrections to processes with four external particles for non-exceptional configurations. Nevertheless, in the sequel some improvements and additions have been worked out. Van Oldenborgh and Vermaseren constructed a different tensor basis that allows to concentrate some of the numerical instabilities into a number of determinants [5]. Ezawa et al. performed the reduction using an orthonormal tensor basis [6]. A reduction in Feynman-parameter space, which is equivalent to the Passarino-Veltman scheme, is used in the GRACE package [7].

The main drawback of the Passarino-Veltman reduction and variants thereof is the appearance of Gram determinants in the denominator, which spoil the numerical stability if they become small. In processes with up to four external particles this happens usually only near the edge of phase space, e.g. for forward scattering or on thresholds. For the special cases where a Gram determinant is identically zero, alternative reduction procedures have been devised by Stuart and collaborators [8,9] (see also Ref. [10]). However, in processes with more than four external particles, Gram determinants also vanish within phase space, and methods for the calculation of tensor integrals are needed where Gram determinants are small but not exactly zero. In Ref. [11] such a method has been devised by constructing combinations of $N$-point and ( $N-1$ )-point scalar integrals that are finite in the limit of vanishing Gram determinants and using this limit if the Gram determinant becomes small.

On the other hand, alternative tensor reduction schemes have been developed using different sets of master integrals. Davydychev could relate the coefficients of one-loop tensor integrals to scalar integrals in a different number of space-time dimensions [12], and Tarasov found recursion relations between these integrals [13]. These methods have been further elaborated by different groups [14-18]. In this approach all one-loop tensor integrals can be reduced to finite 4-point integrals in $(D+2)$ dimensions and divergent 3-point and 2-point integrals in $D$ dimensions. Numerical instabilities in this reduction, which are also due to small Gram or other kinematical determinants, have been investigated in Ref. [18] for the massless case, and a systematic improvement by an iteration technique has been proposed. While numerically stable analytic expressions for the basic integrals are available for the massless case, these turn out to be hard to construct for the massive case. Therefore, one typically reduces these basic integrals to the usual scalar integrals or, in particular for vanishing Gram determinants, calculates them by numerical integration [19].

Other algorithms, which are based on recursion relations similar to Passarino-Veltman reduction and applicable irrespective of the number of external legs, have been presented in Refs. [20,21]. These algorithms do not completely avoid the appearance of inverse Gram determinants.

It was realized already in the sixties by Melrose that scalar integrals with more than four lines in the loop, i.e., 5-point and higher-point scalar integrals, can be reduced to scalar integrals with less internal propagators in four dimensions [22]. These methods were subsequently extended and improved by several authors [5,11,12,14-16,23-27] and generalized to dimensional regularization in Refs. [14,28,29]. In Ref. [26], a method for the reduction of 5-point integrals that completely avoids inverse leading Gram determinants has been worked out. Recently, a similar reduction has been found that even reduces 5-point tensor integrals to 4-point integrals with rank reduced by one [19]. In all these approaches 5- and higher-point tensor integrals are reduced to tensor integrals with less internal propagators.

Various approaches have been proposed that use numerical integration of loop integrals and are, thus, complementary to most of the methods mentioned so far. In the approach of Ref. [30], which has been elaborated for general one-loop integrals with up to six external legs, the Feynman-parameter integrals are rewritten in such a way that they can be numerically integrated in a stable way. A fully numerical approach to calculate loop integrals by contour integration was proposed in Ref. [31]. A semi-numerical approach that relies on the subtraction of UV and infrared divergences has been advocated in Ref. [32]. A different semi-numerical method makes use of the fact that all tensor one-loop integrals can be expressed in terms of one- and twodimensional parameter integrals which are suitable for numerical integration [33]. A numerical method based on multi-dimensional contour deformation has been proposed in Ref. [19]. Finally, Feynman-parameter integrals have been numerically performed with a small but finite "i $\epsilon$ " from the propagator denominators and a subsequent extrapolation $\epsilon \rightarrow 0$ in Ref. [34]. So far, none of these methods has proven their performance in calculations of higher-order corrections for processes with more than four external particles. In practice, one can still expect problems with the numerical stability of the algebraic reduction to standard forms in specific regions of phase space and with the speed of the underlying numerical integration of the basic loop integrals.

In this paper we describe methods that have actually been used in the calculation of the electroweak corrections to $\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow 4$ fermions [35], i.e., in the first established calculation of the complete one-loop electroweak corrections to a process with six external particles. ${ }^{1}$ In this approach, 6 -point integrals are directly expressed in terms of six 5 -point functions, and the 5 -point integrals are written in terms of five 4-point functions. While we used the methods described in Refs. [22,24] and Ref. [26] in the original calculation [35], in this paper we describe improved methods for the reduction of 6-point and 5-point integrals which have meanwhile been implemented in the code for the electroweak corrections to $\mathrm{e}^{+} \mathrm{e}^{+} \rightarrow 4$ fermions and which further improve its performance in numerical stability and CPU time. The 3-point and 4-point tensor integrals are algebraically reduced to the (standard) scalar 1-point, 2-point, 3-point, and 4-point functions as described below. For 1-point and 2-point integrals explicit numerically stable results are used.

In more detail, the 3-point and 4-point functions are reduced to scalar integrals according to the Passarino-Veltman algorithm if no small Gram determinants appear. This is the case for

[^1]most points in parameter space. If a small Gram determinant appears, the reduction of 4-point to 3-point or 3-point to 2-point functions is done differently. Here we have worked out two alternative calculational methods (referred to as "rescue systems" in Ref. [35]). One method makes use of suitable expansions of the tensor coefficients about the limit of vanishing Gram determinants. This is achieved in an iterative way and requires to calculate ( $N-1$ )-point functions of higher degree compared to the usual Passarino-Veltman reduction. ${ }^{2}$ Finally, again all tensor coefficients can be expressed in terms of the standard scalar 1-point, 2-point, 3-point, and 4-point functions. In practice, we use the first two to three terms in the expansions and we have to introduce different expansions for different regions of parameter space. In the second, alternative method we evaluate a specific tensor coefficient, the integrand of which is logarithmic in Feynman parametrization, by numerical integration. Then the remaining coefficients as well as the standard scalar integral are algebraically derived from this coefficient. This reduction again involves no inverse Gram determinants; instead inverse modified Cayley determinants appear. In this approach, the set of master integrals is not given by the standard scalar integrals anymore. For some specific 3-point integrals, where the modified Cayley determinant vanishes exactly, analytical results have been worked out that allow for a stable numerical evaluation.

The paper is organized as follows. We summarize our conventions and useful definitions in Section 2. The evaluation of 1-point and 2-point tensor integrals is summarized in Sections 3 and 4, respectively. In Section 5, we provide several methods for the reduction of 3-point and 4 -point tensor coefficients and describe their actual application to $\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow 4 f$ in Section 5.7. In Section 5.8 we consider UV and infrared divergences in detail and conclude that the proposed methods are valid independent of the method for infrared regularization. The reduction of 5-point and 6-point tensor integrals to integrals with smaller rank and smaller number of propagators is detailed in Sections 6 and 7, respectively. In Appendix A, we list the UV-divergent parts of oneloop integrals that enter the reduction formulas. Appendix B describes a treatment of singular 3-point integrals based on analytical methods. Finally, we discuss alternative reductions of 5- and 6-point tensor integrals in Appendices C and D, respectively.

## 2. Conventions and notation

One-loop tensor $N$-point integrals have the general form

$$
\begin{equation*}
T^{N, \mu_{1} \ldots \mu_{P}}\left(p_{1}, \ldots, p_{N-1}, m_{0}, \ldots, m_{N-1}\right)=\frac{(2 \pi \mu)^{4-D}}{\mathrm{i} \pi^{2}} \int \mathrm{~d}^{D} q \frac{q^{\mu_{1}} \cdots q^{\mu_{P}}}{N_{0} N_{1} \cdots N_{N-1}} \tag{2.1}
\end{equation*}
$$

with the denominator factors

$$
\begin{equation*}
N_{k}=\left(q+p_{k}\right)^{2}-m_{k}^{2}+\mathrm{i} \epsilon, \quad k=0, \ldots, N-1, \quad p_{0}=0, \tag{2.2}
\end{equation*}
$$

where $\mathrm{i} \epsilon(\epsilon>0)$ is an infinitesimally small imaginary part. For $P=0$, i.e., no integration momenta in the numerator of the loop integral, (2.1) defines the scalar $N$-point integral $T_{0}^{N}$. Following the notation of Ref. [1] we set $T^{1}=A, T^{2}=B, T^{3}=C, T^{4}=D, T^{5}=E, T^{6}=F$. Throughout we use the conventions of Refs. [24,26] to decompose the tensor integrals into Lorentz-covariant structures.

In order to be able to write down the tensor decompositions in a concise way we use a notation (similar to the one of Ref. [4]) in which curly braces denote symmetrization with respect to

[^2]Lorentz indices in such a way that all non-equivalent permutations of the Lorentz indices on metric tensors $g$ and a generic momentum $p$ contribute with weight one and that in covariants with $n_{p}$ momenta $p_{i_{j}}^{\mu_{j}}\left(j=1, \ldots, n_{p}\right)$ only one representative out of the $n_{p}$ ! permutations of the indices $i_{j}$ is kept. Thus, we have for example

$$
\begin{align*}
& \{p \ldots p\}_{i_{1} \ldots i_{P}}^{\mu_{1} \ldots \mu_{P}}=p_{i_{1}}^{\mu_{1}} \cdots p_{i_{P}}^{\mu_{P}}, \\
& \{g p\}_{i_{1}}^{\mu \nu \rho}=g^{\mu v} p_{i_{1}}^{\rho}+g^{v \rho} p_{i_{1}}^{\mu}+g^{\rho \mu} p_{i_{1}}^{v}, \\
& \{g p p\}_{i_{1} i_{2}}^{\mu \nu \rho \sigma}=g^{\mu \nu} p_{i_{1}}^{\rho} p_{i_{2}}^{\sigma}+g^{\mu \rho} p_{i_{1}}^{\sigma} p_{i_{2}}^{v}+g^{\mu \sigma} p_{i_{1}}^{v} p_{i_{2}}^{\rho}+g^{v \rho} p_{i_{1}}^{\sigma} p_{i_{2}}^{\mu}+g^{\rho \sigma} p_{i_{1}}^{v} p_{i_{2}}^{\mu}+g^{\sigma v} p_{i_{1}}^{\rho} p_{i_{2}}^{\mu}, \\
& \{g g\}^{\mu \nu \rho \sigma}=g^{\mu \nu} g^{\rho \sigma}+g^{\nu \rho} g^{\mu \sigma}+g^{\rho \mu} g^{v \sigma} . \tag{2.3}
\end{align*}
$$

This definition is unique up to the selection of the representative permutations of the momenta. For our calculation this does not matter, since the covariants are always contracted with quantities that are totally symmetric in the indices $i_{j}$. In fact in our calculation the definition is equivalent to a normalization of the sum of the $n_{p}$ ! covariants with a factor $1 / n_{p}!$; in this case the third line of (2.3) would contain 12 instead of 6 terms on the r.h.s.

We decompose the general tensor integral into Lorentz-covariant structures as

$$
\begin{align*}
T^{N, \mu_{1} \ldots \mu_{P}}= & \sum_{n=0}^{\left[\frac{P}{2}\right]} \sum_{i_{2 n+1}, \ldots, i_{P}=1}^{N-1}\{\underbrace{g \ldots g}_{n} p \ldots p\}_{i_{2 n+1} \ldots i_{P}}^{\mu_{1} \ldots \mu_{P}} T_{\underbrace{}_{2 n}}^{N} \underbrace{N}_{i_{2 n}} i_{2 n+1} \ldots i_{P} \\
= & \sum_{i_{1}, \ldots, i_{P}=1}^{N-1} p_{i_{1}}^{\mu_{1}} \ldots p_{i_{P}}^{\mu_{P}} T_{i_{1} \ldots i_{P}}^{N}+\sum_{i_{3}, \ldots, i_{P}=1}^{N-1}\{g p \ldots p\}_{i_{3} \ldots i_{P}}^{\mu_{1} \ldots \mu_{P}} T_{00 i_{3} \ldots i_{P}}^{N} \\
& +\sum_{i_{5}, \ldots, i_{P}=1}^{N-1}\{g g p \ldots p\}_{i_{5} \ldots i_{P}}^{\mu_{1} \ldots \mu_{P}} T_{0000 i_{5} \ldots i_{P}}^{N}+\ldots \\
& + \begin{cases}\sum_{i_{P}=1}^{N-1}\{g \ldots g p\}_{i_{P}}^{\mu_{1} \ldots \mu_{P}} \underbrace{T_{0 \ldots 0}^{N} i_{P}}_{\underbrace{}_{P-1}}, & \text { for } P \text { odd, } \\
\{g \ldots g\}^{\mu_{1} \ldots \mu_{P}} \underbrace{N}_{P_{P}}, & \text { for } P \text { even, },\end{cases} \tag{2.4}
\end{align*}
$$

where [ $P / 2$ ] is the largest natural number smaller or equal to $P / 2$. For each metric tensor in the Lorentz covariant the corresponding coefficient carries an index pair " 00 " and for each momentum $p_{i_{r}}$ it carries the corresponding index $i_{r}$.

For tensor integrals up to rank five the decompositions more explicitly read

$$
\begin{aligned}
& T^{N, \mu}=\sum_{i_{1}=1}^{N-1} p_{i_{1}}^{\mu} T_{i_{1}}^{N}, \quad T^{N, \mu \nu}=\sum_{i_{1}, i_{2}=1}^{N-1} p_{i_{1}}^{\mu} p_{i_{2}}^{\nu} T_{i_{1} i_{2}}^{N}+g^{\mu \nu} T_{00}^{N}, \\
& T^{N, \mu \nu \rho}=\sum_{i_{1}, i_{2}, i_{3}=1}^{N-1} p_{i_{1}}^{\mu} p_{i_{2}}^{\nu} p_{i_{3}}^{\rho} T_{i_{1} i_{2} i_{3}}^{N}+\sum_{i_{1}=1}^{N-1}\{g p\}_{i_{1}}^{\mu \nu \rho} T_{00 i_{1}}^{N}, \\
& T^{N, \mu \nu \rho \sigma}=\sum_{i_{1}, i_{2}, i_{3}, i_{4}=1}^{N-1} p_{i_{1}}^{\mu} p_{i_{2}}^{\nu} p_{i_{3}}^{\rho} p_{i_{4}}^{\sigma} T_{i_{1} i_{2} i_{3} i_{4}}^{N}+\sum_{i_{1}, i_{2}=1}^{N-1}\{g p p\}_{i_{1} i_{2}}^{\mu \nu \rho \sigma} T_{00 i_{1} i_{2}}^{N}+\{g g\}^{\mu \nu \rho \sigma} T_{0000}^{N},
\end{aligned}
$$

$$
\begin{align*}
T^{N, \mu \nu \rho \sigma \tau}= & \sum_{i_{1}, i_{2}, i_{3}, i_{4}, i_{5}=1}^{N-1} p_{i_{1}}^{\mu} p_{i_{2}}^{v} p_{i_{3}}^{\rho} p_{i_{4}}^{\sigma} p_{i_{5}}^{\tau} T_{i_{1} i_{2} i_{3} i_{4} i_{5}}^{N}+\sum_{i_{1}, i_{2}, i_{3}=1}^{N-1}\{g p p p\}_{i_{1} i_{2} i_{3}}^{\mu \nu \rho \sigma \tau} T_{00 i_{1} i_{2} i_{3}}^{N} \\
& +\sum_{i_{1}=1}^{N-1}\{g g p\}_{i_{1}}^{\mu \nu \rho \sigma \tau} T_{0000 i_{1}}^{N} . \tag{2.5}
\end{align*}
$$

Because of the symmetry of the tensor $T_{\mu_{1} \ldots \mu_{P}}^{N}$ all coefficients $T_{i_{1} \ldots i_{P}}^{N}$ are symmetric under permutation of all indices. For convenience we assume this symmetry also for indices that are zero.

When reducing a tensor integral $T_{\mu_{1} \ldots \mu_{P}}^{N+1}$, one encounters tensor integrals that are obtained by omitting the $k$ th denominator $N_{k}$; we denote such integrals $T_{\mu_{1} \ldots \mu_{P}}^{N}(k)$. In the decomposition of $T_{\mu_{1} \ldots \mu_{P}}^{N}(k), k=1, \ldots, N$, shifted indices appear which we denote as

$$
\left(i_{r}\right)_{k}= \begin{cases}i_{r}, & \text { for } k>i_{r},  \tag{2.6}\\ i_{r}-1, & \text { for } k<i_{r} .\end{cases}
$$

After cancelling the denominator $N_{0}$ the resulting tensor integrals are not in the standard form but can be expressed in terms of standard integrals by shifting the integration momentum. We choose to perform the shift $q \rightarrow q-p_{1}$, so that the following $N$-point integrals appear:

$$
\begin{gather*}
\tilde{T}^{N, \mu_{1} \ldots \mu_{P}}(0)=\frac{(2 \pi \mu)^{(4-D)}}{\mathrm{i} \pi^{2}} \int \mathrm{~d}^{D} q \frac{q^{\mu_{1}} \cdots q^{\mu_{P}}}{\tilde{N}_{1} \cdots \tilde{N}_{N}} \\
\tilde{N}_{k}=\left(q+p_{k}-p_{1}\right)^{2}-m_{k}^{2}+\mathrm{i} \epsilon, \quad k=1, \ldots, N . \tag{2.7}
\end{gather*}
$$

Note that the scalar integral $T_{0}^{N} \equiv T^{N}$ and the tensor coefficients $T_{00}^{N}, T_{0000}^{N}, \ldots$ are invariant under this shift. The other coefficients of $T_{\mu_{1} \ldots \mu_{P}}^{N}(k)$ can be recursively obtained as

$$
\begin{align*}
& T_{\underbrace{N}_{2 n} \ldots 0 i_{2 n+1} \ldots i_{P}}(0)=\tilde{T}_{\underbrace{N}_{2 n}}^{\tilde{0}_{0} 0} i_{2 n+1}-1, \ldots, i_{P}-1(0), \quad i_{2 n+1}, \ldots, i_{P}>1, \\
& T_{\underbrace{N}_{2 n}}^{0 \ldots 0} 1 i_{2 n+2 \ldots i_{P}}(0)=-T_{\underbrace{N}_{2 n}}^{0 \ldots 0} i_{2 n+2 \ldots i_{P}}(0)-\sum_{r=2}^{N} \underbrace{T_{0 \ldots 0}^{N} r i_{2 n+2 \ldots i_{P}}(0), \quad i_{2 n+2}, \ldots, i_{P}>0}_{2 n} 0 \tag{2.8}
\end{align*}
$$

The recursion is solved by

$$
\begin{align*}
& T_{\underbrace{N \ldots}_{2 n}}^{\substack{n \\
i_{2 n+k+1}}} \underbrace{1 \ldots 1}_{k} i_{2 n+k+1} \ldots i_{P}(0)=(-1)^{k} \sum_{l=0}^{k}\binom{k}{l} \sum_{i_{1}, \ldots, i_{l}=1}^{N-1} \underbrace{\tilde{T}_{0}^{N}}_{2 n} .
\end{align*}
$$

We also use the notation $\bar{\delta}_{i j}=1-\delta_{i j}$, i.e., $\sum_{i} \bar{\delta}_{i j}(\cdots)=\sum_{i \neq j}(\cdots)$, and employ the caret "^", to indicate indices that are omitted, i.e.,

$$
\begin{equation*}
T_{i_{1} \ldots \hat{i}_{r} \ldots i_{P}}^{N} \equiv T_{i_{1} \ldots i_{r-1} i_{r+1} \ldots i_{P}}^{N} . \tag{2.10}
\end{equation*}
$$

In the reduction formulas for the $(N+1)$-point functions the Gram matrix

$$
Z^{(N)}=\left(\begin{array}{ccc}
2 p_{1} p_{1} & \cdots & 2 p_{1} p_{N}  \tag{2.11}\\
\vdots & \ddots & \vdots \\
2 p_{N} p_{1} & \cdots & 2 p_{N} p_{N}
\end{array}\right)
$$

appears. Its determinant, the Gram determinant, is denoted by

$$
\begin{equation*}
\Delta^{(N)}=\operatorname{det} Z^{(N)}, \tag{2.12}
\end{equation*}
$$

and its inverse can be written as

$$
\begin{equation*}
\left(Z^{(N)}\right)_{i j}^{-1}=\frac{1}{\Delta^{(N)}} \tilde{Z}_{j i}^{(N)}, \tag{2.13}
\end{equation*}
$$

where $\tilde{Z}_{i j}^{(N)}$ is the adjoint of $Z_{i j}^{(N)}$, which can be calculated as

$$
\tilde{Z}_{i j}^{(N)}=(-1)^{i+j}\left|\begin{array}{cccccc}
2 p_{1} p_{1} & \cdots & 2 p_{1} p_{j-1} & 2 p_{1} p_{j+1} & \cdots & 2 p_{1} p_{N}  \tag{2.14}\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
2 p_{i-1} p_{1} & \cdots & 2 p_{i-1} p_{j-1} & 2 p_{i-1} p_{j+1} & \cdots & 2 p_{i-1} p_{N} \\
2 p_{i+1} p_{1} & \cdots & 2 p_{i+1} p_{j-1} & 2 p_{i+1} p_{j+1} & \cdots & 2 p_{i+1} p_{N} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
2 p_{N} p_{1} & \cdots & 2 p_{N} p_{j-1} & 2 p_{N} p_{j+1} & \cdots & 2 p_{N} p_{N}
\end{array}\right|,
$$

i.e., from a reduced determinant of $Z^{(N)}$ where the $i$ th row and the $j$ th column have been discarded.

We introduce a generalization of the adjoint by

$$
\begin{align*}
& \tilde{\tilde{Z}}_{(i k)(j l)}^{(N)} \\
& \quad=(-1)^{i+j+k+l} \operatorname{sgn}(i-k) \operatorname{sgn}(l-j) \\
& \quad\left|\begin{array}{ccccccccc}
2 p_{1} p_{1} & \cdots & 2 p_{1} p_{j-1} & 2 p_{1} p_{j+1} & \cdots & 2 p_{1} p_{l-1} & 2 p_{1} p_{l+1} & \cdots & 2 p_{1} p_{N} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
2 p_{i-1} p_{1} & \cdots & 2 p_{i-1} p_{j-1} & 2 p_{i-1} p_{j+1} & \cdots & 2 p_{i-1} p_{l-1} & 2 p_{i-1} p_{l+1} & \cdots & 2 p_{i-1} p_{N} \\
2 p_{i+1} p_{1} & \cdots & 2 p_{i+1} p_{j-1} & 2 p_{i+1} p_{j+1} & \cdots & 2 p_{i+1} p_{l-1} & 2 p_{i+1} p_{l+1} & \cdots & 2 p_{i+1} p_{N} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
2 p_{k-1} p_{1} & \cdots & 2 p_{k-1} p_{j-1} & 2 p_{k-1} p_{j+1} & \cdots & 2 p_{k-1} p_{l-1} & 2 p_{k-1} p_{l+1} & \cdots & 2 p_{k-1} p_{N} \\
2 p_{k+1} p_{1} & \cdots & 2 p_{k+1} p_{j-1} & 2 p_{k+1} p_{j+1} & \cdots & 2 p_{k+1} p_{l-1} & 2 p_{k+1} p_{l+1} & \cdots & 2 p_{k+1} p_{N} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
2 p_{N} p_{1} & \cdots & 2 p_{N} p_{j-1} & 2 p_{N} p_{j+1} & \cdots & 2 p_{N} p_{l-1} & 2 p_{N} p_{l+1} & \cdots & 2 p_{N} p_{N}
\end{array}\right|, \tag{2.15}
\end{align*}
$$

i.e., it is defined from a reduced determinant of $Z^{(N)}$ where the $i$ th and $k$ th rows and the $j$ th and $l$ th columns have been discarded. Moreover, it is defined to vanish for $i=k$ or $j=l$. For the case $N=2$, it is given by

$$
\begin{equation*}
\tilde{\tilde{Z}}_{(i k)(j l)}^{(2)}=\delta_{i l} \delta_{k j}-\delta_{i j} \delta_{k l} . \tag{2.16}
\end{equation*}
$$

Expanding the determinant of $Z^{(N)}$ along the $k$ th row or the $l$ th column, respectively, it can be written as

$$
\begin{equation*}
\Delta^{(N)}=\sum_{m=1}^{N} Z_{k m}^{(N)} \tilde{Z}_{k m}^{(N)}=\sum_{m=1}^{N} Z_{m l}^{(N)} \tilde{Z}_{m l}^{(N)}, \tag{2.17}
\end{equation*}
$$

where $k$ and $l$ are not summed. Expanding the determinant in (2.14) with the $l$ th column replaced by $\left(Z_{1 k}, \ldots, Z_{N k}\right)^{\mathrm{T}}$ along the $l$ th column yields the relation

$$
\begin{equation*}
\sum_{m=1}^{N} \tilde{\tilde{Z}}_{(i m)(j l)}^{(N)} Z_{m k}^{(N)}=\tilde{Z}_{i l}^{(N)} \delta_{j k}-\tilde{Z}_{i j}^{(N)} \delta_{l k} \tag{2.18}
\end{equation*}
$$

and analogously

$$
\begin{equation*}
\sum_{m=1}^{N} \tilde{Z}_{(i k)(j m)}^{(N)} Z_{l m}^{(N)}=\tilde{Z}_{k j}^{(N)} \delta_{i l}-\tilde{Z}_{i j}^{(N)} \delta_{k l} \tag{2.19}
\end{equation*}
$$

These imply the equations

$$
\begin{equation*}
\tilde{\tilde{Z}}_{(i k)(j l)}^{(N)}=\left(Z^{(N)}\right)_{j k}^{-1} \tilde{Z}_{i l}^{(N)}-\left(Z^{(N)}\right)_{l k}^{-1} \tilde{Z}_{i j}^{(N)}=\left[\tilde{Z}_{i l}^{(N)} \tilde{Z}_{k j}^{(N)}-\tilde{Z}_{i j}^{(N)} \tilde{Z}_{k l}^{(N)}\right] / \Delta^{(N)} \tag{2.20}
\end{equation*}
$$

and

$$
\begin{align*}
& \sum_{m, n=1}^{N} \tilde{\tilde{Z}}_{(i m)(j n)}^{(N)} Z_{m n}^{(N)}=\tilde{Z}_{i j}^{(N)}(1-N), \\
& \sum_{m, n=1}^{N} \tilde{\tilde{Z}}_{(i m)(j n)}^{(N)} Z_{m k}^{(N)} Z_{l n}^{(N)}=\Delta^{(N)} \delta_{i l} \delta_{j k}-\tilde{Z}_{i j}^{(N)} Z_{l k}^{(N)} \tag{2.21}
\end{align*}
$$

An important special case of the last relation is

$$
\begin{equation*}
\Delta^{(N)}=Z_{l k}^{(N)} \tilde{Z}_{l k}^{(N)}+\sum_{m, n=1}^{N} \tilde{Z}_{(l m)(k n)}^{(N)} Z_{m k}^{(N)} Z_{l n}^{(N)} \tag{2.22}
\end{equation*}
$$

The relations (2.13)-(2.22) are valid for any (not necessarily symmetric) $N \times N$ matrix $Z^{(N)}$ with determinant $\Delta^{(N)}$.

We further introduce the $(N+1) \times(N+1)$ matrix

$$
X^{(N)}=\left(\begin{array}{cccc}
2 m_{0}^{2} & f_{1} & \cdots & f_{N}  \tag{2.23}\\
f_{1} & 2 p_{1} p_{1} & \cdots & 2 p_{1} p_{N} \\
\vdots & \vdots & \ddots & \vdots \\
f_{N} & 2 p_{N} p_{1} & \cdots & 2 p_{N} p_{N}
\end{array}\right)
$$

with

$$
\begin{equation*}
f_{k}=p_{k}^{2}-m_{k}^{2}+m_{0}^{2}, \quad k=1, \ldots, N \tag{2.24}
\end{equation*}
$$

Its determinant is given by

$$
\operatorname{det}\left(X^{(N)}\right)=2 m_{0}^{2} \Delta^{(N)}-\sum_{n, m=1}^{N} f_{n} f_{m} \tilde{Z}_{n m}^{(N)}=\left|\begin{array}{cccc}
Y_{00} & Y_{01} & \cdots & Y_{0 N}  \tag{2.25}\\
Y_{10} & Y_{11} & \cdots & Y_{1 N} \\
\vdots & \vdots & \ddots & \vdots \\
Y_{N 0} & Y_{N 1} & \cdots & Y_{N N}
\end{array}\right|=\operatorname{det}(Y)
$$

where

$$
\begin{equation*}
Y_{i j}=m_{i}^{2}+m_{j}^{2}-\left(p_{i}-p_{j}\right)^{2}, \quad i, j=0, \ldots, N \tag{2.26}
\end{equation*}
$$

The matrix $Y=\left(Y_{i j}\right)$ is sometimes called modified Cayley matrix and its determinant the modified Cayley determinant [22]. Its elements are related to those of the Gram matrix via

$$
\begin{equation*}
Y_{i j}=Z_{i j}^{(N)}-f_{i}-f_{j}+2 m_{0}^{2}, \quad Y_{0 i}=Y_{i 0}=-f_{i}+2 m_{0}^{2}, \quad i, j=1, \ldots, N \tag{2.27}
\end{equation*}
$$

The vanishing of $\operatorname{det}\left(X^{(N)}\right)$ is a necessary condition for the appearance of leading Landau singularities [38]. The adjoint of $X_{i j}^{(N)}, i, j=0, \ldots, N$, can be expressed as

$$
\begin{align*}
& \tilde{X}_{00}^{(N)}=\Delta^{(N)}, \\
& \tilde{X}_{0 i}^{(N)}=\tilde{X}_{i 0}^{(N)}=-\sum_{n=1}^{N} \tilde{Z}_{i n}^{(N)} f_{n}, \\
& \tilde{X}_{i j}^{(N)}=2 m_{0}^{2} \tilde{Z}_{i j}^{(N)}+\sum_{n, m=1}^{N} \tilde{\tilde{Z}}_{(i n)(j m)}^{(N)} f_{n} f_{m}, \quad i, j=1, \ldots, N . \tag{2.28}
\end{align*}
$$

For later use we also consider the generalized adjoint of $X^{(N)}$. The relevant part of it is given by

$$
\begin{align*}
& \tilde{\tilde{X}}_{(0 i)(0 j)}^{(N)}=-\tilde{Z}_{i j}^{(N)}, \quad i, j=1, \ldots, N, \\
& \tilde{\tilde{X}}_{(0 i)(j k)}^{(N)}=\tilde{\tilde{X}}_{(j k)(0 i)}^{(N)}=-\sum_{n=1}^{N} f_{n} \tilde{\tilde{Z}}_{(n i)(j k)}^{(N)}, \quad i, j, k=1, \ldots, N . \tag{2.29}
\end{align*}
$$

These relations together with (2.20) imply

$$
\begin{align*}
& \operatorname{det}\left(X^{(N)}\right) \tilde{Z}_{i j}^{(N)}=\Delta^{(N)} \tilde{X}_{i j}^{(N)}-\tilde{X}_{i 0}^{(N)} \tilde{X}_{0 j}^{(N)} \\
& \operatorname{det}\left(X^{(N)}\right) \tilde{\tilde{X}}_{(0 i)(j k)}^{(N)}=\tilde{X}_{0 k}^{(N)} \tilde{X}_{i j}^{(N)}-\tilde{X}_{i k}^{(N)} \tilde{X}_{0 j}^{(N)} \tag{2.30}
\end{align*}
$$

## 3. Evaluation of 1-point functions

The scalar 1-point integral for an arbitrary complex mass $m_{0}$ is given by

$$
\begin{equation*}
A_{0}\left(m_{0}\right)=m_{0}^{2}\left[\Delta+\ln \left(\frac{\mu^{2}}{m_{0}^{2}}\right)+1\right] \tag{3.1}
\end{equation*}
$$

where $\Delta$ is the standard one-loop divergence

$$
\begin{equation*}
\Delta=\frac{2}{4-D}-\gamma_{\mathrm{E}}+\ln (4 \pi) \tag{3.2}
\end{equation*}
$$

in $D$ space-time dimensions with $\gamma_{\mathrm{E}}$ denoting Euler's constant. The tensor integrals of rank $2 n$ ( $n=1,2, \ldots$ ) are given by

$$
\begin{equation*}
A^{\mu_{1} \ldots \mu_{2 n}}=\{\underbrace{g \ldots g}_{n}\}^{\mu_{1} \ldots \mu_{2 n}} A_{\underbrace{0 \ldots 0}_{2 n}}, \tag{3.3}
\end{equation*}
$$

where the tensor coefficients are easily evaluated to

$$
\begin{equation*}
A_{\underbrace{0 \ldots 0}_{2 n}}=\frac{m_{0}^{2 n}}{2^{n}(n+1)!}\left[A_{0}\left(m_{0}\right)+m_{0}^{2} \sum_{k=1}^{n} \frac{1}{k+1}\right] . \tag{3.4}
\end{equation*}
$$

Because of Lorentz invariance obviously all tensors of odd rank vanish.

## 4. Evaluation of 2-point functions

In the following we assume that at least one of the parameters $p_{1}^{2}, m_{0}, m_{1}$ is different from zero; otherwise the 2-point integrals identically vanish in dimensional regularization,

$$
\begin{equation*}
B_{\ldots . .}(0,0,0) \equiv 0, \tag{4.1}
\end{equation*}
$$

where the dots stand for any Lorentz index or any index of a tensor coefficient.
Up to rank 3 the 2-point tensor integrals are decomposed as

$$
\begin{align*}
& B^{\mu}=p_{1}^{\mu} B_{1}, \quad B^{\mu \nu}=p_{1}^{\mu} p_{1}^{\nu} B_{11}+g^{\mu \nu} B_{00}, \\
& B^{\mu \nu \rho}=p_{1}^{\mu} p_{1}^{\nu} p_{1}^{\rho} B_{111}+\{g p\}_{1}^{\mu \nu \rho} B_{001} . \tag{4.2}
\end{align*}
$$

The tensor coefficients can be algebraically reduced to scalar 1- and 2-point integrals, $A_{0}$ and $B_{0}$, with the Passarino-Veltman algorithm [4] as more generally described in the next section. The corresponding results for tensors up to rank 3 are, e.g., given in the appendix of Ref. [26]. The algebraic reduction for the coefficients $B_{00 i_{3} i_{4} \ldots}$, which correspond to covariants involving the metric tensor,

$$
\begin{align*}
& B_{00}=\frac{1}{6}\left[A_{0}(0)+f_{1} B_{1}+2 m_{0}^{2} B_{0}+m_{0}^{2}+m_{1}^{2}-\frac{1}{3} p_{1}^{2}\right], \\
& B_{001}=\frac{1}{8}\left[-A_{0}(0)+f_{1} B_{11}+2 m_{0}^{2} B_{1}-\frac{1}{6}\left(2 m_{0}^{2}+4 m_{1}^{2}-p_{1}^{2}\right)\right], \quad \text { etc. } \tag{4.3}
\end{align*}
$$

are numerically well behaved. However, the reduction formulas for the coefficients $B_{1 \ldots .1}$ corresponding to the covariant $p_{1}^{\mu_{1}} \cdots p_{1}^{\mu_{P}}$ involve a factor $1 / p_{1}^{2}$ in each reduction step, so that these reduction formulas become numerically unstable for small $p_{1}^{2}$. Owing to the simplicity of 2-point integrals it is, however, possible to derive closed expressions for these coefficients that are numerically stable for all values of $p_{1}^{2}$. Such a derivation is described below. Assuming the knowledge of the coefficients $B_{1 \ldots 1}$, the remaining coefficients $B_{0 \ldots 01 \ldots 1}$ can be obtained from the recurrence relations

$$
\begin{align*}
& B_{\underbrace{0 \ldots 0}_{2 n+2} \underbrace{1 \ldots 1}_{P-2 n-2}}^{1+\ldots 1}=-\frac{1}{2(P-2 n-1)}[A_{2 n}^{0 \ldots 0} \underbrace{1 \ldots 1}_{P-2 n-1}(0)+f_{1} B_{\underbrace{0 \ldots 0}_{2 n} \underbrace{1 \ldots 1}_{P-2 n-1}}^{1 \ldots}+2 p_{1}^{2} B_{\underbrace{0 \ldots 0}_{2 n} \underbrace{1 \ldots 1}_{P-2 n}}^{1 \ldots}], \\
& \quad n=0, \ldots,\left[\frac{P-2}{2}\right], \tag{4.4}
\end{align*}
$$

or

$$
\begin{align*}
& B_{\underbrace{0 \ldots 0}_{2 n+2}}^{\underbrace{1 \ldots 1}_{P-2 n-2}}=\frac{1}{2(P+1)}[A_{2 n}^{0 \ldots 0} \underbrace{1 \ldots 1}_{P-2 n-2}(0)+2 m_{0}^{2} B_{\underbrace{0 \ldots 0}_{2 n}}^{\underbrace{1 \ldots 1}_{P-2 n-2}}+f_{1} B_{\underbrace{0 \ldots 0}_{2 n} \underbrace{1 \ldots 1}_{P-2 n-1}}^{1 \ldots 1} \\
& -2(D-4) B_{\underbrace{}_{2 n+2}}^{\underbrace{}_{0-2 n-2}} \underbrace{1 \ldots 1}_{P-21}], \quad n=0, \ldots,\left[\frac{P-2}{2}\right] \text {. } \tag{4.5}
\end{align*}
$$

The coefficients $A_{0 \ldots 01 \ldots 1}(0)$ are given by

$$
\begin{equation*}
A_{\underbrace{0 \ldots 0}_{2 n}}^{\underbrace{1 \ldots 1}_{P-2 n-1}}(0)=(-1)^{P-2 n-1} \tilde{A}_{\underbrace{0 \ldots 0}_{2 n}}(0), \tag{4.6}
\end{equation*}
$$

where $\tilde{A}_{0 \ldots 0}(0)$ can be obtained from (3.4). The finite polynomial quantities $(D-4) B_{00 \ldots}$ can easily be derived by exploiting (4.5) for the UV-singular parts; explicit results for tensors up to rank 5 are summarized in Appendix A.

We derive the expressions for $B_{1 \ldots 1}$ by explicitly solving the Feynman-parameter integral

$$
\begin{equation*}
B_{\underbrace{}_{n}}^{\underbrace{}_{n 1}}=\int_{0}^{1} \mathrm{~d} x(-x)^{n}\left\{\Delta+\ln \mu^{2}-\ln \left[-p_{1}^{2} x(1-x)+m_{0}^{2}(1-x)+m_{1}^{2} x-\mathrm{i} \epsilon\right]\right\} . \tag{4.7}
\end{equation*}
$$

In the following result we support complex mass parameters; more precisely, the real parts of $m_{i}^{2}$ must be non-negative, the imaginary parts negative or zero. The final results are conveniently written as

$$
\begin{equation*}
B_{\underbrace{1 \ldots 1}_{n}}^{\underbrace{1 \ldots 1}}=\frac{(-1)^{n}}{n+1}\left\{\Delta+\ln \left(\frac{\mu^{2}}{m_{0}^{2}}\right)-\sum_{k=1}^{2} f_{n}\left(x_{k}\right)\right\}, \tag{4.8}
\end{equation*}
$$

with $x_{k}$ denoting the solutions of the quadratic equation

$$
\begin{equation*}
0=-p_{1}^{2} x(1-x)+m_{0}^{2}(1-x)+m_{1}^{2} x-\mathrm{i} \epsilon \tag{4.9}
\end{equation*}
$$

For $p_{1}^{2}=0$ one of the $x_{k}$ is formally $\infty$. The auxiliary functions

$$
\begin{equation*}
f_{n}(x) \equiv(n+1) \int_{0}^{1} \mathrm{~d} t t^{n} \ln \left(1-\frac{t}{x}\right) \tag{4.10}
\end{equation*}
$$

can be evaluated in a numerically stable way by choosing one of the two representations

$$
\begin{equation*}
f_{n}(x)=\left(1-x^{n+1}\right) \ln \left(\frac{x-1}{x}\right)-\sum_{l=0}^{n} \frac{x^{n-l}}{l+1}=\ln \left(1-\frac{1}{x}\right)+\sum_{l=n+1}^{\infty} \frac{x^{n-l}}{l+1} \tag{4.11}
\end{equation*}
$$

The first form is numerically stable for intermediate values of $|x| \neq 0$. For $x \rightarrow 0, f_{n}(x)$ develops a true logarithmic singularity; for $x \rightarrow 1$ the logarithm $\ln (1-1 / x)$ is suppressed because of its prefactor. The second equality in (4.11) yields numerically stable results for large $|x|$. In practice, we take the first form for $|x|<10$ and the second otherwise. The case where one of the $x_{k}$ is zero corresponds to $m_{0}=0$ and can be easily obtained via taking the limit $m_{0} \rightarrow 0$,

$$
\begin{equation*}
B_{\underbrace{1 \ldots 1}_{n}}\left(p_{1}^{2}, 0, m_{1}\right)=\frac{(-1)^{n}}{n+1}\left\{\Delta+\ln \left(\frac{\mu^{2}}{m_{1}^{2}-p_{1}^{2}-\mathrm{i} \epsilon}\right)+\frac{1}{n+1}-f_{n}\left(1-\frac{m_{1}^{2}-\mathrm{i} \epsilon}{p_{1}^{2}}\right)\right\} . \tag{4.12}
\end{equation*}
$$

For $p_{1}^{2}=m_{1}^{2}$ this further simplifies to

$$
\begin{equation*}
B_{\underbrace{1 \ldots 1}_{n}}^{1 \ldots}\left(m_{1}^{2}, 0, m_{1}\right)=\frac{(-1)^{n}}{n+1}\left\{\Delta+\ln \left(\frac{\mu^{2}}{m_{1}^{2}}\right)+\frac{2}{n+1}\right\} . \tag{4.13}
\end{equation*}
$$

In the vicinity of the last two special cases one of the $x_{k}$ becomes small, so that the leading (logarithmic) term in $f_{n}\left(x_{k}\right)$ cancels against the explicit logarithm in (4.8). Although this somewhat worsens the precision of the evaluation, we did not find problems with this approach in practice. Nevertheless we have additionally implemented a more sophisticated representation of $B_{1 \ldots 1}$ with more branches where such cancellations are avoided.

In the above derivation we essentially followed the approach described in the appendix of Ref. [4]; the results given there are, however, not applicable to the general case of complex masses.

## 5. Reduction of 3-point and 4-point functions

The tensor decompositions of 3-point tensor integrals up to rank 4 and 4-point tensor integrals up to rank 5 read explicitly

$$
\begin{align*}
& C^{\mu}=\sum_{i_{1}=1}^{2} p_{i_{1}}^{\mu} C_{i_{1}}, \quad C^{\mu \nu}=\sum_{i_{1}, i_{2}=1}^{2} p_{i_{1}}^{\mu} p_{i_{2}}^{\nu} C_{i_{1} i_{2}}+g^{\mu \nu} C_{00}, \\
& C^{\mu \nu \rho}=\sum_{i_{1}, i_{2}, i_{3}=1}^{2} p_{i_{1}}^{\mu} p_{i_{2}}^{v} p_{i_{3}}^{\rho} C_{i_{1} i_{2} i_{3}}+\sum_{i_{1}=1}^{2}\{g p\}_{i_{1}}^{\mu \nu \rho} C_{00 i_{1}}, \\
& C^{\mu \nu \rho \sigma}=\sum_{i_{1}, i_{2}, i_{3}, i_{4}=1}^{2} p_{i_{1}}^{\mu} p_{i_{2}}^{\nu} p_{i_{3}}^{\rho} p_{i_{4}}^{\sigma} C_{i_{1} i_{2} i_{3} i_{4}}+\sum_{i_{1}, i_{2}=1}^{2}\{g p p\}_{i_{1} i_{2}}^{\mu \nu \rho \sigma} C_{00 i_{1} i_{2}}+\{g g\}^{\mu \nu \rho \sigma} C_{0000},  \tag{5.1}\\
& D^{\mu}=\sum_{i_{1}=1}^{3} p_{i_{1}}^{\mu} D_{i_{1}}, \quad D^{\mu \nu}=\sum_{i_{1}, i_{2}=1}^{3} p_{i_{1}}^{\mu} p_{i_{2}}^{\nu} D_{i_{1} i_{2}}+g^{\mu \nu} D_{00}, \\
& D^{\mu \nu \rho}=\sum_{i_{1}, i_{2}, i_{3}=1}^{3} p_{i_{1}}^{\mu} p_{i_{2}}^{v} p_{i_{3}}^{\rho} D_{i_{1} i_{2} i_{3}}+\sum_{i_{1}=1}^{3}\{g p\}_{i_{1}}^{\mu \nu \rho} D_{00 i_{1}}, \\
& D^{\mu \nu \rho \sigma}=\sum_{i_{1}, i_{2}, i_{3}, i_{4}=1}^{3} p_{i_{1}}^{\mu} p_{i_{2}}^{v} p_{i_{3}}^{\rho} p_{i_{4}}^{\sigma} D_{i_{1} i_{2} i_{3} i_{4}}+\sum_{i_{1}, i_{2}=1}^{3}\{g p p\}_{i_{1} i_{2}}^{\mu \nu \rho \sigma} D_{00 i_{1} i_{2}}+\{g g\}^{\mu \nu \rho \sigma} D_{0000}, \\
& D^{\mu \nu \rho \sigma \tau}=\sum_{i_{1}, i_{2}, i_{3}, i_{4}, i_{5}=1}^{3} p_{i_{1}}^{\mu} p_{i_{2}}^{\nu} p_{i_{3}}^{\rho} p_{i_{4}}^{\sigma} p_{i_{5}}^{\tau} D_{i_{1} i_{2} i_{3} i_{4} i_{5}}+\sum_{i_{1}, i_{2}, i_{3}=1}^{3}\{g p p p\}_{i_{1} i_{2} i_{3}}^{\mu \nu \rho \tau} D_{00 i_{1} i_{2} i_{3}}^{3} \\
& +\sum_{i_{1}=1}^{3}\{g g p\}_{i_{1}}^{\mu \nu \rho \sigma \tau} D_{0000 i_{1} .} . \tag{5.2}
\end{align*}
$$

Because of the symmetry of the tensor $T_{\mu_{1} \ldots \mu_{P}}^{N}$ all coefficients $C_{i_{1} \ldots i_{P}}$, and $D_{i_{1} \ldots i_{P}}$ are symmetric under permutation of all indices. To be specific, in the following we give the reduction formulas for the 4-point functions, i.e. $N=4$. To obtain the corresponding results for 3-point functions one has to perform the substitutions

$$
\begin{array}{lrr}
C_{\ldots} \rightarrow B_{\ldots}, & D_{\ldots} \rightarrow C_{\ldots}, & Z^{(3)} \rightarrow Z^{(2)}, \\
\Delta^{(3)} \rightarrow \Delta^{(2)}, & X^{(3)} \rightarrow X^{(2)}, & N \rightarrow 3, \tag{5.3}
\end{array}
$$

and similar obvious substitutions.

### 5.1. Conventional Passarino-Veltman reduction

The one-loop tensor integrals can be reduced to scalar integrals recursively by inversion of systems of linear equations [4]. The inhomogeneity of these equations consists of coefficients of
lower rank. The equations of this system are obtained by contracting $T_{\mu_{1} \ldots \mu_{P}}^{N}$ with the $(N-1)$ external momenta $p_{k}^{\mu_{1}}$ and for $P \geqslant 2$ also by contraction with the metric $g^{\mu_{1} \mu_{2}}$. Contracting (2.1) with $p_{k}^{\mu_{1}}$ and using

$$
\begin{equation*}
2 p_{k} q=N_{k}-N_{0}-f_{k}, \tag{5.4}
\end{equation*}
$$

each of the first two terms on the r.h.s. of (5.4) cancels exactly one propagator denominator of $p_{k}^{\mu_{1}} T_{\mu_{1} \ldots \mu_{P}}^{N}$ and the third term is proportional to $T_{\mu_{2} \ldots \mu_{P}}^{N}$. Likewise the contraction of (2.1) with $g^{\mu_{1} \mu_{2}}$ yields a factor $q^{2}$ in the numerator of $g^{\mu_{1} \mu_{2}} T_{\mu_{1} \ldots \mu_{P}}^{N}$, which can be written as

$$
\begin{equation*}
q^{2}=N_{0}+m_{0}^{2} . \tag{5.5}
\end{equation*}
$$

The $N_{0}$ term cancels the first propagator, the second term leads to the tensor $T_{\mu_{3} \ldots \mu_{P}}^{N}$. This yields

$$
\begin{align*}
& 2 p_{k}^{\mu_{1}} T_{\mu_{1} \ldots \mu_{P}}^{N}=T_{\mu_{2} \ldots \mu_{P}}^{N-1}(k)-T_{\mu_{2} \ldots \mu_{P}}^{N-1}(0)-f_{k} T_{\mu_{2} \ldots \mu_{P}}^{N},  \tag{5.6}\\
& g^{\mu_{1} \mu_{2}} T_{\mu_{1} \mu_{2} \ldots \mu_{P}}^{N}=T_{\mu_{3} \ldots \mu_{P}}^{N-1}(0)+m_{0}^{2} T_{\mu_{3} \ldots \mu_{P}}^{N} . \tag{5.7}
\end{align*}
$$

Note that for $T_{\ldots}^{N-1}(0)$ a shift of the integration momentum $q^{\mu} \rightarrow q^{\mu}-p_{1}^{\mu}$ has to be done in order to achieve the standard form (2.1). The tensor integrals with shifted momenta $\tilde{T}_{\mu_{1} \ldots \mu_{P}}^{N-1}(0)$ are defined in (2.7). Inserting the Lorentz decompositions (5.2) into (5.6) and (5.7), the desired recurrence relations can be read off by comparing coefficients.

From (5.6) we obtain

$$
\begin{align*}
S_{k i_{2} \ldots i_{P}}^{P} & \equiv C_{\left(i_{2}\right)_{k} \ldots\left(i_{P}\right)_{k}}(k) \bar{\delta}_{k i_{2}} \ldots \bar{\delta}_{k i_{P}}-C_{i_{2} \ldots i_{P}}(0)-f_{k} D_{i_{2} \ldots i_{P}} \\
& =\sum_{m=1}^{N-1} Z_{k m}^{(3)} D_{m i_{2} \ldots i_{P}}+2 \sum_{r=2}^{P} \delta_{k i_{r}} D_{00 i_{2} \ldots \hat{i}_{r} \ldots i_{P}}, \\
k= & 1, \ldots, N-1, \quad i_{2}, \ldots, i_{P}=0, \ldots, N-1, \tag{5.8}
\end{align*}
$$

and from (5.7)

$$
\begin{align*}
S_{00 i_{3} \ldots i_{P}}^{P} & \equiv 2 C_{i_{3} \ldots i_{P}}(0)+2 m_{0}^{2} D_{i_{3} \ldots i_{P}} \\
& =\sum_{n, m=1}^{N-1} Z_{n m}^{(3)} D_{n m i_{3} \ldots i_{P}}+2\left(D+P-2+\sum_{r=3}^{P} \bar{\delta}_{i_{r} 0}\right) D_{00 i_{3} \ldots i_{P}}, \\
i_{3}, \ldots, i_{P} & =0, \ldots, N-1, \tag{5.9}
\end{align*}
$$

where the matrix $Z^{(3)}$ is defined in (2.11). Eqs. (5.8) and (5.9) can be solved for the coefficients of $D^{\mu_{1} \ldots \mu_{P}}$ as

$$
\begin{align*}
D_{00 i_{3} \ldots i_{P}}= & \frac{1}{2(3+P-N)}\left[-2(D-4) D_{00 i_{3} \ldots i_{P}}+C_{i_{3} \ldots i_{P}}(0)+2 m_{0}^{2} D_{i_{3} \ldots i_{P}}\right. \\
& \left.+\sum_{n=1}^{N-1} f_{n} D_{n i_{3} \ldots i_{P}}\right],  \tag{5.10}\\
D_{i_{1} \ldots i_{P}}= & \sum_{n=1}^{N-1}\left(Z^{(3)}\right)_{i_{1} n}^{-1}\left(S_{n i_{2} \ldots i_{P}}^{P}-2 \sum_{r=2}^{P} \delta_{n i_{r}} D_{00 i_{2} \ldots \hat{i}_{r} \ldots i_{P}}\right), \quad i_{1} \neq 0 . \tag{5.11}
\end{align*}
$$



Fig. 1. Schematic illustration of conventional Passarino-Veltman reduction.

The relations (5.10) and (5.11) determine $D_{i_{1} \ldots i_{P}}$ in terms of $D_{i_{1} \ldots i_{P-1}}$ and 3-point functions. Using these relations recursively, all coefficients of 4-point functions can be expressed in terms of 3-point functions and the scalar 4-point function $D_{0}$. The finite polynomial quantities $(D-4) D_{00 i_{3} \ldots i_{P}}$ can easily be derived by exploiting (5.10) for the UV-singular parts; explicit results for tensors up to rank 7 are summarized in Appendix A. As explained in Section 5.8, IR divergences do not occur in $D_{00 i_{3} \ldots i_{P}}$. More explicit formulas for all tensor functions up to rank 5 are given in the appendix of Ref. [26].

Fig. 1 illustrates the Passarino-Veltman reduction scheme for 4-point integrals in a plane of tensor coefficients where the rank of the tensor increases by going down in the rows and the number of index pairs " 00 " increases by going to the right in the columns. The steps in the algorithm are indicated by arrows that show which coefficient is deduced from previously calculated ones. The numbers close to the arrows correspond to the step number which is identical to the rank of the tensor coefficients to be calculated; the labels " $a$ ", " $b$ ", etc. give the order in which the coefficients within a step are calculated.

Eq. (5.11) becomes numerically unstable if $Z^{(3)}$ is nearly singular, i.e., if the Gram determinant $\Delta^{(3)}$ is close to zero. Reduction schemes for this case are described in Sections 5.3-5.6.

### 5.2. Alternative Passarino-Veltman-like reduction

An alternative to the conventional Passarino-Veltman reduction can be obtained as follows. Eqs. (5.8) and (5.10) can be written as

$$
\left(\begin{array}{cccc}
2 m_{0}^{2} & f_{1} & f_{2} & f_{3} \\
f_{1} & 2 p_{1} p_{1} & 2 p_{1} p_{2} & 2 p_{1} p_{3} \\
f_{2} & 2 p_{2} p_{1} & 2 p_{2} p_{2} & 2 p_{2} p_{3} \\
f_{3} & 2 p_{3} p_{1} & 2 p_{3} p_{2} & 2 p_{3} p_{3}
\end{array}\right)\left(\begin{array}{c}
D_{i 2 \ldots} \ldots i_{1} \\
D_{1 i_{2}} \ldots i_{p} \\
D_{2 i_{2} \ldots i_{p}} \\
D_{3 i_{2} \ldots i_{P}}
\end{array}\right)
$$

$$
\begin{align*}
& =X^{(3)}\left(\begin{array}{c}
D_{i_{2} \ldots i_{P}} \\
D_{1 i_{2} \ldots i_{P}} \\
D_{2 i_{2} \ldots i_{P}} \\
D_{3 i_{2} \ldots i_{P}}
\end{array}\right)=\left(\begin{array}{c}
2(D+P-N) D_{00 i_{2} \ldots i_{P}}-C_{i_{2} \ldots i_{P}}(0) \\
\hat{S}_{1 i_{2} \ldots i_{P}}^{P}-2 \sum_{r=2}^{P} \delta_{1 i_{r}} D_{00 i_{2} \ldots \hat{i}_{r} \ldots i_{P}} \\
\hat{S}_{2 i_{2} \ldots i_{P}}^{P}-2 \sum_{r=2}^{P} \delta_{2 i_{r}} D_{00 i_{2} \ldots \hat{i}_{r} \ldots i_{P}} \\
\hat{S}_{3 i_{2} \ldots i_{P}}^{P}-2 \sum_{r=2}^{P} \delta_{3 i_{r}} D_{00 i_{2} \ldots \hat{i}_{r} \ldots i_{P}}
\end{array}\right), \\
& i_{2}, \ldots, i_{P}=0, \ldots, N-1, \tag{5.12}
\end{align*}
$$

where on the r.h.s. the matrix $X^{(3)}$ defined in (2.23) appears and the following abbreviations are introduced,

$$
\begin{equation*}
\hat{S}_{k i_{2} \ldots i_{P}}^{P}=C_{\left(i_{2}\right)_{k} \ldots\left(i_{P}\right)_{k}}(k) \bar{\delta}_{k i_{2}} \ldots \bar{\delta}_{k i_{P}}-C_{i_{2} \ldots i_{P}}(0)=S_{k i_{2} \ldots i_{P}}^{P}+f_{k} D_{i_{2} \ldots i_{P}} \tag{5.13}
\end{equation*}
$$

Multiplying (5.12) with the matrix $\tilde{X}^{(3)}$ from the left, we obtain

$$
\begin{align*}
\operatorname{det}\left(X^{(3)}\right) D_{i_{2} \ldots i_{P}}= & \Delta^{(3)}\left[2(4+P-N) D_{00 i_{2} \ldots i_{P}}+2(D-4) D_{00 i_{2} \ldots i_{P}}-C_{i_{2} \ldots i_{P}}(0)\right] \\
& +\sum_{n=1}^{N-1} \tilde{X}_{0 n}^{(3)}\left[\hat{S}_{n i_{2} \ldots i_{P}}^{P}-2 \sum_{r=2}^{P} \delta_{n i_{r}} D_{00 i_{2} \ldots \hat{i}_{r} \ldots i_{P}}\right] \tag{5.14}
\end{align*}
$$

and

$$
\begin{align*}
\operatorname{det}\left(X^{(3)}\right) D_{i_{1} i_{2} \ldots i_{P}}= & \tilde{X}_{i_{1} 0}^{(3)}\left[2(4+P-N) D_{00 i_{2} \ldots i_{P}}+2(D-4) D_{00 i_{2} \ldots i_{P}}-C_{i_{2} \ldots i_{P}}(0)\right] \\
& +\sum_{n=1}^{N-1} \tilde{X}_{i_{1} n}^{(3)}\left[\hat{S}_{n i_{2} \ldots i_{P}}^{P}-2 \sum_{r=2}^{P} \delta_{n i_{r}} D_{00 i_{2} \ldots \hat{i}_{r} \ldots i_{P}}\right], \quad i_{1} \neq 0 . \tag{5.15}
\end{align*}
$$

Eq. (5.14) yields $D_{00 i_{2} \ldots i_{P}}$ in terms of $D_{00 i_{2} \ldots \hat{i}_{r} \ldots i_{P}}, D_{i_{2} \ldots i_{P}}$, and 3-point functions,

$$
\begin{align*}
& 2(4+P-N) \Delta^{(3)} D_{00 i_{2} \ldots i_{P}} \\
& =-2 \Delta^{(3)}(D-4) D_{00 i_{2} \ldots i_{P}}+\Delta^{(3)} C_{i_{2} \ldots i_{P}}(0)+\operatorname{det}\left(X^{(3)}\right) D_{i_{2} \ldots i_{P}} \\
& \quad-\sum_{n=1}^{N-1} \tilde{X}_{0 n}^{(3)}\left[\hat{S}_{n i_{2} \ldots i_{P}}^{P}-2 \sum_{r=2}^{P} \delta_{n i_{r}} D_{00 i_{2} \ldots \hat{i}_{r} \ldots i_{P}}\right], \tag{5.16}
\end{align*}
$$

and thereafter (5.15) yields $D_{i_{1} \ldots i_{P}}$. Using these relations recursively, all coefficients of 4-point functions can be expressed in terms of 3-point functions and the scalar 4-point function $D_{0}$. While the final results are of course identical to those of the usual Passarino-Veltman reduction, the order in which the different coefficients are calculated is different. As a consequence, this recursion can, in some cases, be numerically more stable than the conventional Passarino-Veltman reduction, in particular, if all the quantities $\Delta^{(3)}, \tilde{X}_{k 0}^{(3)}$, and $\tilde{X}_{k l}^{(3)}$ become small.

For the tensor coefficients up to rank 3 the reduction formulas explicitly read

$$
\begin{align*}
& 2(5-N) \Delta^{(3)} D_{00}=-2 \Delta^{(3)}(D-4) D_{00}+\Delta^{(3)} C_{0}(0)+\operatorname{det}\left(X^{(3)}\right) D_{0}-\sum_{n=1}^{N-1} \tilde{X}_{0 n}^{(3)} \hat{S}_{n}^{1} \\
& \operatorname{det}\left(X^{(3)}\right) D_{i_{1}}=\tilde{X}_{i_{1} 0}^{(3)}\left[2(5-N) D_{00}+2(D-4) D_{00}-C_{0}(0)\right]+\sum_{n=1}^{N-1} \tilde{X}_{i_{1} n}^{(3)} \hat{S}_{n}^{1}, \tag{5.17}
\end{align*}
$$

$$
\begin{align*}
& \begin{array}{c}
2(6-N) \Delta^{(3)} D_{00 i_{2}}=-2 \Delta^{(3)}(D-4) D_{00 i_{2}}+\Delta^{(3)} C_{i_{2}}(0)+\operatorname{det}\left(X^{(3)}\right) D_{i_{2}} \\
\quad-\sum_{n=1}^{N-1} \tilde{X}_{0 n}^{(3)}\left[\hat{S}_{n i_{2}}^{2}-2 \delta_{n i_{2}} D_{00}\right], \\
\begin{aligned}
& \operatorname{det}\left(X^{(3)}\right) D_{i_{1} i_{2}}= \tilde{X}_{i_{1} 0}^{(3)}\left[2(6-N) D_{00 i_{2}}+2(D-4) D_{00 i_{2}}-C_{i_{2}}(0)\right] \\
&+\sum_{n=1}^{N-1} \tilde{X}_{i_{1} n}^{(3)}\left[\hat{S}_{n i_{2}}^{2}-2 \delta_{n i_{2}} D_{00}\right], \quad i_{1}, i_{2} \neq 0, \\
& 2(7-N) \Delta^{(3)} D_{00 i_{2} i_{3}}=-2 \Delta^{(3)}(D-4) D_{00 i_{2} i_{3}}+\Delta^{(3)} C_{i_{2} i_{3}}(0)+\operatorname{det}\left(X^{(3)}\right) D_{i_{2} i_{3}} \\
& \quad-\sum_{n=1}^{N-1} \tilde{X}_{0 n}^{(3)}\left[\hat{S}_{n i_{2} i_{3}}^{3}-2 \delta_{n i_{2}} D_{00 i_{3}}-2 \delta_{n i_{3}} D_{00 i_{2}}\right],
\end{aligned} \\
\begin{aligned}
\operatorname{det}\left(X^{(3)}\right) D_{i_{1} i_{2} i_{3}}= & \tilde{X}_{i_{1} 0}^{(3)}\left[2(7-N) D_{00 i_{2} i_{3}}+2(D-4) D_{0 i_{2} i_{3}}-C_{i_{2} i_{3}}(0)\right] \\
& +\sum_{n=1}^{N-1} \tilde{X}_{i_{1} n}^{(3)}\left[\hat{S}_{n i_{2} i_{3}}^{3}-2 \delta_{n i_{2}} D_{00 i_{3}}-2 \delta_{n i_{3}} D_{00 i_{2}}\right], \quad i_{1}, i_{2}, i_{3} \neq 0 .
\end{aligned}
\end{array} .
\end{align*}
$$

Note that (5.21) holds also for $i_{2}=i_{3}=0$.
The 3-point tensor coefficients that result from omitting $N_{0}$ in the 4-point integrals are defined according to (2.8) or more explicitly

$$
\begin{align*}
& C_{i_{1}}(0)=\tilde{C}_{i_{1}-1}(0), \quad i_{1}=2, \ldots, N-1, \\
& C_{1}(0)=-\sum_{n=2}^{N-1} C_{n}(0)-C_{0}(0),  \tag{5.23}\\
& C_{i_{1} i_{2}}(0)=\tilde{C}_{i_{1}-1, i_{2}-1}(0), \quad i_{1}, i_{2}=2, \ldots, N-1, \\
& C_{1 i_{1}}(0)=-\sum_{n=2}^{N-1} C_{n i_{1}}(0)-C_{i_{1}}(0), \quad i_{1}=1, \ldots, N-1,  \tag{5.24}\\
& C_{i_{1} i_{2} i_{3}}(0)=\tilde{C}_{i_{1}-1, i_{2}-1, i_{3}-1}(0), \quad i_{1}, i_{2}, i_{3}=2, \ldots, N-1, \\
& C_{1 i_{1} i_{2}}(0)=-\sum_{n=2}^{N-1} C_{n i_{1} i_{2}}(0)-C_{i_{1} i_{2}}(0), \quad i_{1}, i_{2}=1, \ldots, N-1 . \tag{5.25}
\end{align*}
$$

Fig. 2 illustrates the alternative Passarino-Veltman reduction scheme for 4-point integrals in the plane of tensor coefficients similarly to Fig. 1 of the previous section for the conventional variant.

### 5.3. Reduction with modified Cayley determinants

Eq. (5.12) can also be exploited directly to calculate tensor coefficients of lower-rank from higher-rank tensors. Specifically, the coefficients $D_{i_{1} \ldots i_{P}}$ with $i_{1} \neq 0$ for tensors of rank $P$ are expressed in terms of the coefficients $D_{00 i_{2} \ldots i_{P}}$ for tensors of rank $(P+1)$. This means, (5.12) recursively expresses tensor coefficients $D_{i_{1} \ldots i_{P}}$ in terms of $C$ functions and of a single coefficient $D_{0 \ldots 0}$ which results from $D_{i_{1} \ldots i_{P}}$ upon replacing all non-zero indices $i_{k}$ by " 00 ". For sufficiently


Fig. 2. Schematic illustration of alternative Passarino-Veltman reduction.
high tensor rank $P$, viz. $P \geqslant 2 N-4$, the integrand of the Feynman parameter integral of $D_{0 \ldots 0}$ involves only polynomials and logarithms of the integration parameters $x_{l}$. Such integrals are numerically well behaved, because singularities appearing in logarithms can be safely treated numerically. The explicit form of the Feynman-parameter integral for the general coefficient $T_{0 \ldots 0}^{N}$ with $P \geqslant 2 N-4$ is given below.

In summary, Eq. (5.12) provides a method for deducing all tensor coefficients $D_{i_{1} \ldots i_{P}}$ (including the standard scalar integral $D_{0}$ ) from $C$ functions and the numerically evaluated coefficient $D_{0 \ldots 0}$ of tensor rank $2 P$. This procedure does not involve the inverse of the Gram determinant $\Delta^{(3)}$, as it is the case in the two versions of Passarino-Veltman reduction described in the previous sections. However, the method involves the inverse of the modified Cayley determinant $\operatorname{det}\left(X^{(3)}\right)$, so that it becomes unstable if $\operatorname{det}\left(X^{(3)}\right)$ becomes small. It is also interesting to note that the numerically evaluated coefficient $D_{0 \ldots 0}$ enters this reduction with a prefactor $\Delta^{(3)}$. Thus, this method becomes particularly precise if $\Delta^{(3)}$ is small, where Passarino-Veltman reduction is unstable, because the error in the numerical calculation of $D_{0 \ldots .}$ is suppressed in this case. Note, however, that both the reduction of this section and Passarino-Veltman reduction become problematic if both $\Delta^{(3)}$ and $\operatorname{det}\left(X^{(3)}\right)$ are small.

For tensor coefficients up to rank 3 the reduction formulas explicitly read

$$
\begin{align*}
\operatorname{det}\left(X^{(3)}\right) D_{0000}= & \Delta^{(3)}\left[2(9-N) D_{000000}+2(D-4) D_{000000}-C_{0000}(0)\right] \\
& +\sum_{n=1}^{N-1} \tilde{X}_{0 n}^{(3)} \hat{S}_{n 0000}^{5},  \tag{5.26}\\
\operatorname{det}\left(X^{(3)}\right) D_{00}= & \Delta^{(3)}\left[2(7-N) D_{0000}+2(D-4) D_{0000}-C_{00}(0)\right]+\sum_{n=1}^{N-1} \tilde{X}_{0 n}^{(3)} \hat{S}_{n 00}^{3}, \tag{5.27}
\end{align*}
$$

$$
\begin{align*}
& \operatorname{det}\left(X^{(3)}\right) D_{0}=\Delta^{(3)}\left[2(5-N) D_{00}+2(D-4) D_{00}-C_{0}(0)\right]+\sum_{n=1}^{N-1} \tilde{X}_{0 n}^{(3)} \hat{S}_{n}^{1},  \tag{5.28}\\
& \operatorname{det}\left(X^{(3)}\right) D_{0000 i_{1}}= \tilde{X}_{i_{1} 0}^{(3)}\left[2(9-N) D_{000000}+2(D-4) D_{000000}-C_{0000}(0)\right] \\
&+\sum_{n=1}^{N-1} \tilde{X}_{i_{1} n}^{(3)} \hat{S}_{n 0000}^{5},  \tag{5.29}\\
& \operatorname{det}\left(X^{(3)}\right) D_{00 i_{1}}= \tilde{X}_{i_{1} 0}^{(3)}\left[2(7-N) D_{0000}+2(D-4) D_{0000}-C_{00}(0)\right] \\
&+\sum_{n=1}^{N-1} \tilde{X}_{i_{1} n}^{(3)} \hat{S}_{n 00}^{3},  \tag{5.30}\\
& \operatorname{det}\left(X^{(3)}\right) D_{00 i_{1} i_{2}}= \tilde{X}_{i_{1} 0}^{(3)}\left[2(8-N) D_{0000 i_{2}}+2(D-4) D_{0000 i_{2}}-C_{00 i_{2}}(0)\right] \\
&+\sum_{n=1}^{N-1} \tilde{X}_{i_{1} n}^{(3)}\left[\hat{S}_{n 00 i_{2}}^{4}-2 \delta_{n i_{2}} D_{0000}\right], \quad i_{1}, i_{2} \neq 0 . \tag{5.31}
\end{align*}
$$

Finally, $D_{i_{1}}, D_{i_{1} i_{2}}$, and $D_{i_{1} i_{2} i_{3}}$ are obtained from (5.18), (5.20), and (5.22), respectively. Thus, all 4-point tensor coefficients up to tensor rank 3 can be recursively deduced from $D_{000000}$ and 3-point coefficients.

Fig. 3 illustrates the reduction scheme for 4-point integrals up to rank 3 in the plane of tensor coefficients similar to the previous sections. The steps of the reduction now proceed from right to left, starting with a basis integral $D_{0 \ldots 0}$ with as many index pairs " 00 " as the finally aimed tensor rank, i.e., for rank 3 with $D_{000000}$. In each step we get all coefficients of at least one rank lower with one index pair " 00 " less than in the previous steps.

Generically the Feynman-parameter integral for $T_{0 \ldots 0}^{N}$ reads

$$
\begin{align*}
T_{2 k}^{N}=\ldots & \frac{1}{2^{k}(2+k-N)!}\left(\prod_{j=0}^{N-1} \int_{0}^{\infty} \mathrm{d} x_{j}\right) \delta\left(1-\sum_{l=0}^{N-1} \alpha_{l} x_{l}\right)\left(\sum_{m=0}^{N-1} x_{m}\right)^{N-4-2 k} A^{2+k-N} \\
& \times\left[\Delta+\sum_{n=1}^{2+k-N} \frac{1}{n}-\ln \left(\frac{A-\mathrm{i} \epsilon}{\mu^{2}}\right)+2 \ln \left(\sum_{m=0}^{N-1} x_{m}\right)\right], \quad k \geqslant N-2 \tag{5.32}
\end{align*}
$$

with the shorthand

$$
\begin{equation*}
A=A\left(x_{0}, \ldots, x_{N-1}\right)=\left(\sum_{l=0}^{N-1} x_{l} p_{l}\right)^{2}-\left(\sum_{m=0}^{N-1} x_{m}\right)\left(\sum_{n=0}^{N-1} x_{n}\left(p_{n}^{2}-m_{n}^{2}\right)\right) . \tag{5.33}
\end{equation*}
$$

The real parameters $\alpha_{l}$ appearing in (5.32) are widely arbitrary; they only have to fulfil the constraints $\alpha_{l} \geqslant 0$ and $\sum_{l=0}^{N-1} \alpha_{l}>0$. For the numerical evaluation of the Feynman-parameter integral it is convenient to take the uniform choice $\alpha_{l}=1$, in which case the integral runs over the $(N-1)$-dimensional unit simplex $\sigma_{N-1}$,

$$
\begin{equation*}
x_{0}=1-\sum_{l=1}^{N-1} x_{l}, \quad 0<x_{j}<1-\sum_{l=1}^{j-1} x_{l}, \quad j=1, \ldots, N-1 . \tag{5.34}
\end{equation*}
$$

The integral representation (5.32) is valid both for real and complex masses.


Fig. 3. Schematic illustration of the reduction with modified Cayley determinants.

Specifically, the integrals for $C_{000000}$ and $D_{000000}$, which are needed for tensors of rank 3, are given by

$$
\begin{align*}
C_{000000}= & \frac{1}{2880}\left(\Delta+\frac{3}{2}\right)\left[s_{12}^{2}+p_{1}^{4}+p_{2}^{4}+s_{12}\left(p_{1}^{2}+p_{2}^{2}\right)+p_{1}^{2} p_{2}^{2}\right. \\
& -3\left(m_{0}^{2} s_{12}+p_{1}^{2} m_{2}^{2}+p_{2}^{2} m_{1}^{2}\right) \\
& -6\left[s_{12}\left(m_{1}^{2}+m_{2}^{2}\right)+p_{1}^{2}\left(m_{0}^{2}+m_{1}^{2}\right)+p_{2}^{2}\left(m_{0}^{2}+m_{2}^{2}\right)\right] \\
& \left.+15\left[m_{0}^{4}+m_{1}^{4}+m_{2}^{4}+m_{0}^{2} m_{1}^{2}+m_{0}^{2} m_{2}^{2}+m_{1}^{2} m_{2}^{2}\right]\right] \\
& -\frac{1}{16} \int_{\sigma_{2}} \mathrm{~d}^{2} x A^{2} \ln \left(\frac{A-\mathrm{i} \epsilon}{\mu^{2}}\right),  \tag{5.35}\\
D_{000000}= & (\Delta+1)\left[-\frac{1}{960}\left(s_{12}+s_{13}+s_{23}+p_{1}^{2}+p_{2}^{2}+p_{3}^{2}\right)\right. \\
& \left.+\frac{1}{192}\left(m_{0}^{2}+m_{1}^{2}+m_{2}^{2}+m_{3}^{2}\right)\right]-\frac{1}{8} \int_{\sigma_{3}} \mathrm{~d}^{3} x A \ln \left(\frac{A-\mathrm{i} \epsilon}{\mu^{2}}\right), \tag{5.36}
\end{align*}
$$

with the shorthands

$$
\begin{equation*}
s_{12}=\left(p_{1}-p_{2}\right)^{2}, \quad s_{13}=\left(p_{1}-p_{3}\right)^{2}, \quad s_{23}=\left(p_{2}-p_{3}\right)^{2} . \tag{5.37}
\end{equation*}
$$

For an efficient numerical integration of these integrals we use a fortran code based on the DCUHRE algorithm [39], as included in the CUBA library [40]. The UV-divergent parts are integrated analytically in order to ensure exact cancellation of the singularities.

As mentioned above, the procedure described in this section becomes unstable if $\operatorname{det}\left(X^{(N-1)}\right)$ becomes small. The basis integrals $T_{0 \ldots 0}^{N}$ are still safely calculated via the numerical integration, but using the described relations to deduce the remaining coefficients accumulates an instability in each step that turns an index pair " 00 " into a non-zero tensor index or that eliminates an index pair " 00 ". This accumulation of an instability can be suppressed by extending the set of basis integrals. For instance, the 3-point tensor coefficients $C_{i_{1} i_{2} i_{3}}$ can be deduced from the coefficients $C_{00}, C_{0000}$, and $C_{000000}$, which all have logarithmic integrands in their Feynman parametrizations, upon using the above relations only once. ${ }^{3}$ If $\operatorname{det}\left(X^{(N-1)}\right)$ is not small, we prefer to deduce all tensor coefficients from one basis integral (e.g., $D_{000000}$ for $D_{i_{1} i_{2} i_{3}}$ ), because no instabilities accumulate and the recursion preserves relations among the tensor coefficients, which are less accurately valid if several coefficients are calculated numerically.

If $\operatorname{det}\left(X^{(N-1)}\right)=0$, the described procedure is not applicable. For instance, this is the case for 3-point functions that are either soft or collinear singular. Such cases are much simpler than the case with general kinematics, so that they can be treated more directly. For processes with light external fermions only, $\operatorname{det}\left(X^{(N-1)}\right)$ is zero only for 3-point functions $(N=3)$ where a photon or a gluon is attached to an external fermion. A fully analytic treatment of these cases, which admits a numerically stable evaluation, is described in Appendix B; this method can be extended to similar cases that appear in other processes.

Finally, we remark that the method of this section is somewhat related to the fully numerical procedure advocated in Ref. [30]. There, a method is described how the Feynman-parameter representation of one-loop integrals is, upon partial integrations, transformed into integrals with logarithmic integrands, which are then treated numerically. The occurring algebraic coefficients that express the original integral in terms of logarithmic integrals are related to the coefficients of the inverse of the matrix $X^{(N)}$ introduced in this paper. In fact we have verified that the reduction of the scalar integral $C_{0}$ to logarithmic integrals leads to the same results as our equation (5.28) for $N=3$ [see (5.3)]. Therefore, like in our approach, also in the approach of Ref. [30], the cases with small or vanishing modified Cayley determinant $\operatorname{det}\left(X^{(N-1)}\right)$ require a special treatment. Moreover, we emphasize that we treat only one basis integral numerically, while the procedure of Ref. [30] in general involves more numerical integrals.

### 5.4. Reduction for small Gram determinant

Let us now derive a reduction scheme that can be used if the Gram determinant $\Delta^{(3)}$ becomes small, but without changing the set of basis integrals, which are thus still the standard scalar integrals $A_{0}, B_{0}, C_{0}, D_{0}$. Multiplying (5.8) with indices $n i_{1} \ldots i_{P}$ by $\tilde{Z}_{j n}^{(3)}$ and summing over $n$ yields

$$
\begin{equation*}
\tilde{X}_{0 j}^{(3)} D_{i_{1} \ldots i_{P}}=-\sum_{n=1}^{N-1} \tilde{Z}_{j n}^{(3)}\left(\hat{S}_{n i_{1} \ldots i_{P}}^{P+1}-2 \sum_{r=1}^{P} \delta_{n i_{r}} D_{00 i_{1} \ldots \hat{i}_{r} \ldots i_{P}}\right)+\Delta^{(3)} D_{j i_{1} \ldots i_{P}} \tag{5.38}
\end{equation*}
$$

[^3]for arbitrary $j=1, \ldots, N-1$ and $i_{r}=0, \ldots, N-1$. In order to arrive at this form, (2.17) and (2.28) have been used. As long as at least one of the quantities $\tilde{X}_{0 j}^{(3)}$, defined in (2.28), is large compared to $\Delta^{(3)}$, (5.38) can be used to determine $D_{i_{1} \ldots i_{P}}$ from $D_{00 i_{1} \ldots \hat{i}_{r} \ldots i_{P}}$ up to terms that are suppressed by the factor $\Delta^{(3)}$.

In order to obtain $D_{00 i_{1} \ldots \hat{i}_{r} \ldots i_{P}}$, we consider for arbitrary $k, l \neq 0$

$$
\begin{align*}
\Delta^{(3)} D_{k l_{3} \ldots i_{P}} & =\sum_{i, j=1}^{N-1} \Delta^{(3)} \delta_{k i} \delta_{l j} D_{i j i_{3} \ldots i_{P}} \\
& =\sum_{i, j=1}^{N-1}\left(\tilde{Z}_{k l}^{(3)} Z_{i j}^{(3)}+\sum_{n, m=1}^{N-1} \tilde{Z}_{(k n)(l m)}^{(3)} Z_{n j}^{(3)} Z_{i m}^{(3)}\right) D_{i j i_{3} \ldots i_{P}} \tag{5.39}
\end{align*}
$$

where (2.21) has been used. The first term on the r.h.s. can be reduced with (5.9), the second term on the r.h.s. upon using (5.8) twice. Collecting terms containing $D_{00 i_{1} \ldots i_{P}}$ and making use of (2.19) and (2.21), we obtain

$$
\begin{align*}
2(6 & \left.+P-N+\sum_{r=1}^{P} \bar{\delta}_{i_{r} 0}\right) \tilde{Z}_{k l}^{(3)} D_{00 i_{1} \ldots i_{P}} \\
= & -2(D-4) \tilde{Z}_{k l}^{(3)} D_{00 i_{1} \ldots i_{P}}-\Delta^{(3)} D_{k l i_{1} \ldots i_{P}} \\
& +\tilde{Z}_{k l}^{(3)} S_{00 i_{1} \ldots i_{P}}^{P+2}+\sum_{n=1}^{N-1}\left(\tilde{Z}_{n l}^{(3)} \hat{S}_{n k i_{1} \ldots i_{P}}^{P+2}-\tilde{Z}_{k l}^{(3)} \hat{S}_{n n i_{1} \ldots i_{P}}^{P+2}\right) \\
& -\sum_{n, m=1}^{N-1} \tilde{\tilde{Z}}_{(k n)(l m)}^{(3)}\left[f_{n} \hat{S}_{m i_{1} \ldots i_{P}}^{P+1}+2 \sum_{r=1}^{P} \delta_{n i_{r}} \hat{S}_{m 00 i_{1} \ldots \hat{i}_{r} \ldots i_{P}}^{P+2}-f_{n} f_{m} D_{i_{1} \ldots i_{P}}\right. \\
& \left.-2 \sum_{r=1}^{P}\left(f_{n} \delta_{m i_{r}}+f_{m} \delta_{n i_{r}}\right) D_{00 i_{1} \ldots i_{r} \ldots i_{P}}-4 \sum_{\substack{r, s=1 \\
r \neq s}}^{P} \delta_{n i_{r}} \delta_{m i_{s}} D_{0000 i_{1} \ldots \hat{i}_{r} \ldots \hat{i}_{s} \ldots i_{P}}\right], \tag{5.40}
\end{align*}
$$

which holds for arbitrary $k, l=1, \ldots, N-1$ and $i_{1}, \ldots, i_{P}=0, \ldots, N-1$. Together with (5.38) this equation allows to iteratively determine the tensor coefficients of 4-point functions in terms of 3-point functions for small Gram determinant $\Delta^{(3)}$. If the 3-point functions are known up to rank $P$, all 4-point tensor coefficients up to this rank can be determined recursively up to terms of order $\Delta^{(3)}$ from these equations by putting all terms involving $\Delta^{(3)}$ to zero. Inserting these results back into the r.h.s. of (5.38) and (5.40) for the terms proportional to $\Delta^{(3)}$, all 4-point tensor coefficients up to rank $(P-1)$ can be determined up to terms of order $\left(\Delta^{(3)}\right)^{2}$, and so on. Finally, the scalar 4-point function is iteratively determined up to terms of order $\left(\Delta^{(3)}\right)^{P+1}$. In order to improve numerical stability, we can choose $j$ in (5.38) such that $X_{0 j}^{(3)}$ is maximal, and $k$ and $l$ in (5.40) such that $\tilde{Z}_{k l}^{(3)}$ is maximal. For $\Delta^{(3)}=0$ this reduction scheme essentially corresponds to the one proposed in Ref. [9].

For the lowest tensor coefficients the explicit results read

$$
\begin{equation*}
\tilde{X}_{0 j}^{(3)} D_{0}=-\sum_{n=1}^{N-1} \tilde{Z}_{j n}^{(3)} \hat{S}_{n}^{1}+\Delta^{(3)} D_{j}, \tag{5.41}
\end{equation*}
$$

$$
\begin{align*}
& 2(6-N) \tilde{Z}_{k l}^{(3)} D_{00}=-2(D-4) \tilde{Z}_{k l}^{(3)} D_{00}-\Delta^{(3)} D_{k l}+\tilde{Z}_{k l}^{(3)} S_{00}^{2} \\
& +\sum_{n=1}^{N-1}\left(\tilde{Z}_{n l}^{(3)} \hat{S}_{n k}^{2}-\tilde{Z}_{k l}^{(3)} \hat{S}_{n n}^{2}\right)-\sum_{n, m=1}^{N-1} \tilde{Z}_{(k n)(l m)}^{(3)}\left[f_{n} \hat{S}_{m}^{1}-f_{n} f_{m} D_{0}\right],  \tag{5.42}\\
& \tilde{X}_{0 j}^{(3)} D_{i_{1}}=-\sum_{n=1}^{N-1} \tilde{Z}_{j n}^{(3)}\left(\hat{S}_{n i_{1}}^{2}-2 \delta_{n i_{1}} D_{00}\right)+\Delta^{(3)} D_{j i_{1}},  \tag{5.43}\\
& 2(8-N) \tilde{Z}_{k l}^{(3)} D_{00 i_{1}} \\
& =-2(D-4) \tilde{Z}_{k l}^{(3)} D_{00 i_{1}}-\Delta^{(3)} D_{k l i_{1}}+\tilde{Z}_{k l}^{(3)} S_{00 i_{1}}^{3} \\
& +\sum_{n=1}^{N-1}\left(\tilde{Z}_{n l}^{(3)} \hat{S}_{n k i_{1}}^{3}-\tilde{Z}_{k l}^{(3)} \hat{S}_{n n i_{1}}^{3}\right)-\sum_{n, m=1}^{N-1} \tilde{\tilde{Z}}_{(k n)(l m)}^{(3)}\left[f_{n} \hat{S}_{m i_{1}}^{2}+2 \delta_{n i_{1}} \hat{S}_{m 00}^{3}\right. \\
& \left.-f_{n} f_{m} D_{i_{1}}-2\left(f_{n} \delta_{m i_{1}}+f_{m} \delta_{n i_{1}}\right) D_{00}\right] \text {, }  \tag{5.44}\\
& \tilde{X}_{0 j}^{(3)} D_{i_{1} i_{2}}=-\sum_{n=1}^{N-1} \tilde{Z}_{j n}^{(3)}\left[\hat{S}_{n i_{1} i_{2}}^{3}-2\left(\delta_{n i_{1}} D_{00 i_{2}}+\delta_{n i_{2}} D_{00 i_{1}}\right)\right]+\Delta^{(3)} D_{j i_{1} i_{2}},  \tag{5.45}\\
& 2(8-N) \tilde{Z}_{k l}^{(3)} D_{0000} \\
& =-2(D-4) \tilde{Z}_{k l}^{(3)} D_{0000}-\Delta^{(3)} D_{00 k l}+\tilde{Z}_{k l}^{(3)} S_{0000}^{4} \\
& +\sum_{n=1}^{N-1}\left(\tilde{Z}_{n l}^{(3)} \hat{S}_{n 00 k}^{4}-\tilde{Z}_{k l}^{(3)} \hat{S}_{n 00 n}^{4}\right)-\sum_{n, m=1}^{N-1} \tilde{\tilde{Z}}_{(k n)(l m)}^{(3)}\left[f_{n} \hat{S}_{m 00}^{3}-f_{n} f_{m} D_{00}\right],  \tag{5.46}\\
& 2(10-N) \tilde{Z}_{k l}^{(3)} D_{00 i_{1} i_{2}} \\
& =-2(D-4) \tilde{Z}_{k l}^{(3)} D_{00 i_{1} i_{2}}-\Delta^{(3)} D_{k l i_{1} i_{2}}+\tilde{Z}_{k l}^{(3)} S_{00 i_{1} i_{2}}^{4}+\sum_{n=1}^{N-1}\left(\tilde{Z}_{n l}^{(3)} \hat{S}_{n k i_{1} i_{2}}^{4}-\tilde{Z}_{k l}^{(3)} \hat{S}_{n n i_{1} i_{2}}^{4}\right) \\
& -\sum_{n, m=1}^{N-1} \tilde{\tilde{Z}}_{(k n)(l m)}^{(3)}\left[f_{n} \hat{S}_{m i_{1} i_{2}}^{3}+2\left(\delta_{n i_{1}} \hat{S}_{m 00 i_{2}}^{4}+\delta_{n i_{2}} \hat{S}_{m 00 i_{1}}^{4}\right)-f_{n} f_{m} D_{i_{1} i_{2}}\right. \\
& -2\left(f_{n} \delta_{m i_{1}}+f_{m} \delta_{n i_{1}}\right) D_{00 i_{2}}-2\left(f_{n} \delta_{m i_{2}}+f_{m} \delta_{n i_{2}}\right) D_{00 i_{1}} \\
& \left.-4\left(\delta_{n i_{1}} \delta_{m i_{2}}+\delta_{n i_{2}} \delta_{m i_{1}}\right) D_{0000}\right], \quad i_{1}, i_{2} \neq 0,  \tag{5.47}\\
& \tilde{X}_{0 j}^{(3)} D_{i_{1} i_{2} i_{3}}=-\sum_{n=1}^{N-1} \tilde{Z}_{j n}^{(3)}\left[\hat{S}_{n i_{1} i_{2} i_{3}}^{4}-2\left(\delta_{n i_{1}} D_{00 i_{2} i_{3}}+\delta_{n i_{2}} D_{00 i_{1} i_{3}}+\delta_{n i_{3}} D_{00 i_{1} i_{2}}\right)\right] \\
& +\Delta^{(3)} D_{j i_{1} i_{2} i_{3}} . \tag{5.48}
\end{align*}
$$

Fig. 4 illustrates a systematic algorithm for this iteration scheme for 4-point integrals in the plane of tensor coefficients similar to the previous sections. Thin arrows indicate that the relation involves a suppression factor $\Delta^{(3)}$. At the beginning of the iteration all 4-point tensor coefficients as well as the scalar integral $D_{0}$ are set to zero, i.e., no 4-point basis integral is needed. The $n$th iteration consists of the $(n+1)$ steps $n \rightarrow(n-1) \rightarrow \cdots \rightarrow 1 \rightarrow 0$ and requires all 3-point coefficients of rank $n$. Step $n$ with $n>0$ starts with the coefficient of rank $(n+1)$ with the highest number of index pairs " 00 ", i.e. with the right-most coefficient in the $(n+1)$ th row in

Step 0

$D_{i_{1} i_{2}}$
$D_{i_{1} i_{2} i_{3}}$

Step 1


Step 2


Step 3
$D_{0}$

$$
D_{i_{1}}
$$



Fig. 4. Schematic illustration of the iteration for small Gram determinants, where thin arrows indicate that the relation involves a suppression factor $\Delta^{(3)}$. In each step the boxed coefficients are calculated in the order indicated by the labels " $a$ ", " $b$ ", etc. The $n$th iteration consists of the following $(n+1)$ steps: $n \rightarrow(n-1) \rightarrow \cdots \rightarrow 1 \rightarrow 0$.
the diagrams in Fig. 4. Within a step, coefficients for rank $(n+1)$ are deduced from the right to the left in the diagram; only for the last coefficient (which has no index pair " 00 ") one has to go one step upwards to rank $n$ in addition. After the $n$th iteration the tensor coefficients $D_{i_{1} i_{2} i_{3} \ldots}$ of rank $n$ without index pairs " 00 " and all coefficients $D_{00 i_{1} i_{2} \ldots}$ of one rank higher with at least one index pair " 00 " are obtained up to terms that are suppressed by a factor $\Delta^{(3)}$. Coefficients of a rank that is lower by a number $m$ are known up to terms suppressed by $\left[\Delta^{(3)}\right]^{m+1}$. Indicating coefficients that are known up to terms of $\mathcal{O}\left(\left[\Delta^{(3)}\right]^{m+1}\right)$ with a superscript " $(m)$ ", the iteration proceeds as follows:

- Iteration 0: $D_{0}^{(0)}$ is calculated; all other coefficients are still zero.
- Iteration 1: Step 1 yields $D_{00}^{(0)}$ and $D_{i_{1}}^{(0)}$; step 0 yields $D_{0}^{(1)}$.
- Iteration 2: Steps 2 to 0 deliver $D_{00 i_{1}}^{(0)}, D_{i_{1} i_{2}}^{(0)}, D_{00}^{(1)}, D_{i_{1}}^{(1)}$, and $D_{0}^{(2)}$.
- Iteration 3: Steps 3 to 0 deliver $D_{0000}^{(0)}, D_{00 i_{1} i_{2}}^{(0)}, D_{i_{1} i_{2} i_{3}}^{(0)}, D_{00 i_{1}}^{(1)}, D_{i_{1} i_{2}}^{(1)}, D_{00}^{(2)}, D_{i_{1}}^{(2)}$, and $D_{0}^{(3)}$.
- Etc.

The reduction method described in this section breaks down if none of the $\tilde{X}_{0 j}^{(3)}$ is large compared to $\Delta^{(3)}$ or if all $\tilde{Z}_{k l}^{(3)}$ become small, since in these cases the iteration does not converge. A reduction for small $\tilde{X}_{0 j}^{(3)}$ is described in Section 5.5. A reduction for small $Z_{k l}^{(3)}$ is given in Section 5.6. For $N=2$ the case of small $Z_{k l}^{(2)}$ is equivalent to small $\tilde{Z}_{k l}^{(2)}$; for $N=3$ the case of small $Z_{k l}^{(3)}$ covers the case of small $\tilde{Z}_{k l}^{(3)}$ apart from exceptional configurations. ${ }^{4}$

### 5.5. Reduction for small Gram determinant and small modified Cayley determinant

If in addition to the Gram determinant $\Delta^{(3)}$ also all quantities $\tilde{X}_{0 j}^{(3)}, j=1, \ldots, N-1$, become small, the reduction scheme of Section 5.4 breaks down. As can be seen from (2.30), in this case the determinant $\operatorname{det}\left(\tilde{X}^{(3)}\right)=\operatorname{det}(Y)$ of (2.25) becomes small, which is a necessary condition for the appearance of leading Landau singularities. In this situation, we can determine the tensor coefficients as follows.

For $i_{r} \neq 0$, Eq. (5.38) can be rewritten as

$$
\begin{equation*}
2 \sum_{r=1}^{P} \tilde{Z}_{k i_{r}}^{(3)} D_{00 i_{1} \ldots \hat{i}_{r} \ldots i_{P}}=\sum_{n=1}^{N-1} \tilde{Z}_{k n}^{(3)} \hat{S}_{n i_{1} \ldots i_{P}}^{P+1}+\tilde{X}_{k 0}^{(3)} D_{i_{1} \ldots i_{P}}-\Delta^{(3)} D_{k i_{1} \ldots i_{P}} . \tag{5.49}
\end{equation*}
$$

This allows to determine $D_{00 i_{1} \ldots \hat{i}_{r} \ldots i_{P}}$ for $i_{1}, \ldots, i_{P} \neq 0$ in terms of 3-point functions as:

$$
\begin{align*}
& 2 P \tilde{Z}_{k l}^{(3)} D_{00} \underbrace{l \ldots l}_{P-1}=\sum_{n=1}^{N-1} \tilde{Z}_{k n}^{(3)} \hat{S}_{n}^{P+1} \underbrace{P+\ldots l}_{P}+\tilde{X}_{k 0}^{(3)} D_{\underbrace{l \ldots l}_{P}}^{l \ldots l}-\Delta^{(3)} D_{k} \underbrace{l \ldots l l}_{P}, \\
& 2(P-1) \tilde{Z}_{k l}^{(3)} D_{00} \underbrace{l \ldots l}_{P-2} i_{1}=-2 \tilde{Z}_{k i_{1}}^{(3)} D_{00} \underbrace{l \ldots l}_{P-1}+\sum_{n=1}^{N-1} \tilde{Z}_{k n}^{(3)} \hat{S}_{n}^{P+1 \ldots l} \underbrace{l \ldots l}_{P-1} i_{1} \\
& +\tilde{X}_{k 0}^{(3)} D_{P-1}^{l \ldots l i_{1}}-\Delta^{(3)} D_{k} \underbrace{l \ldots l}_{P-1} i_{1}, \quad i_{1} \neq 0, l, \\
& 2(P-2) \tilde{Z}_{k l}^{(3)} D_{00} \underbrace{l \ldots l}_{P-3} i_{1} i_{2}=-2 \tilde{Z}_{k i_{1}}^{(3)} D_{00} \underbrace{l \ldots l}_{P-2} i_{2}-2 \tilde{Z}_{k i_{2}}^{(3)} D_{00} \underbrace{l \ldots l}_{P-2} i_{1}+\sum_{n=1}^{N-1} \tilde{Z}_{k n}^{(3)} \hat{S}_{n}^{P+1} \underbrace{l \ldots l}_{P-2} i_{1} i_{2} \\
& +\tilde{X}_{k 0}^{(3)} D_{P-2}^{l \ldots l} i_{1} i_{2}-\Delta^{(3)} D_{k} \underbrace{l \ldots l}_{P-2} i_{1} i_{2}, \quad i_{1}, i_{2} \neq 0, l, \tag{5.50}
\end{align*}
$$

and so on, provided that at least one of the $\tilde{Z}_{k l}^{(3)}$ is not small. Again $k \neq 0$ and $l \neq 0$ can be chosen such that $\tilde{Z}_{k l}^{(3)}$ is maximal in order to improve the numerical stability. The tensor coefficients with more index pairs " 00 " can be determined by equations that are obtained from (5.50) by adding additional index pairs " 00 " to all quantities $S$ and $D$ in (5.50).

[^4]In order to derive a relation for the calculation of $D_{i_{1} \ldots i_{P}}$ we rewrite (5.12) as

$$
\begin{align*}
& \left(\begin{array}{ccc}
f_{1} & f_{2} & f_{3} \\
2 p_{1} p_{1} & 2 p_{1} p_{2} & 2 p_{1} p_{3} \\
2 p_{2} p_{1} & 2 p_{2} p_{2} & 2 p_{2} p_{3} \\
2 p_{3} p_{1} & 2 p_{3} p_{2} & 2 p_{3} p_{3}
\end{array}\right)\left(\begin{array}{c}
D_{1 i_{1} \ldots i_{P}} \\
D_{2 i_{1} \ldots i_{P}} \\
D_{3 i_{1} \ldots i_{P}}
\end{array}\right) \\
& \quad=\left(\begin{array}{c}
2(D+1+P-N) D_{00 i_{1} \ldots i_{P}}-C_{i_{1} \ldots i_{P}}(0)-2 m_{0}^{2} D_{i_{1} \ldots i_{P}} \\
\hat{S}_{1 i_{1} \ldots i_{P}}^{P+1}-2 \sum_{r=1}^{P} \delta_{1 i_{r}} D_{00 i_{1} \ldots \hat{i}_{r} \ldots i_{P}}-f_{1} D_{i_{1} \ldots i_{P}} \\
\hat{S}_{2 i_{1} \ldots i_{P}}^{P+1}-2 \sum_{r=1}^{P} \delta_{2 i_{r}} D_{00 i_{1} \ldots \hat{i}_{r} \ldots i_{P}}-f_{2} D_{i_{1} \ldots i_{P}} \\
\hat{S}_{3 i_{1} \ldots i_{P}}^{P+1}-2 \sum_{r=1}^{P} \delta_{3 i_{r}} D_{00 i_{1} \ldots \hat{i}_{r} \ldots i_{P}}-f_{3} D_{i_{1} \ldots i_{P}}
\end{array}\right) . \tag{5.51}
\end{align*}
$$

After discarding the $(j+1)$ th of these equations, where $j=1,2$, or 3 , the remaining three equations have the solution

$$
\begin{align*}
& \sum_{n=1}^{N-1} f_{n} \tilde{Z}_{n j}^{(3)} D_{i i_{1} \ldots i_{P}} \\
& \quad=\tilde{Z}_{i j}^{(3)}\left[2(D+1+P-N) D_{00 i_{1} \ldots i_{P}}-C_{i_{1} \ldots i_{P}}(0)-2 m_{0}^{2} D_{i_{1} \ldots i_{P}}\right] \\
& \quad+\sum_{m, n=1}^{N-1} \tilde{\tilde{Z}}_{(\text {in })(j m)}^{(3)} f_{n}\left[\hat{S}_{m i_{1} \ldots i_{P}}^{P+1}-2 \sum_{r=1}^{P} \delta_{m i_{r}} D_{00 i_{1} \ldots i_{r} \ldots i_{P}}-f_{m} D_{i_{1} \ldots i_{P}}\right] \tag{5.52}
\end{align*}
$$

Using (2.28) this can be written as

$$
\begin{align*}
& \tilde{X}_{i j}^{(3)} D_{i_{1} \ldots i_{P}} \\
& \quad=\tilde{Z}_{i j}^{(3)}\left[2(5+P-N) D_{00 i_{1} \ldots i_{P}}+2(D-4) D_{00 i_{1} \ldots i_{P}}-C_{i_{1} \ldots i_{P}}(0)\right] \\
& \quad+\sum_{m, n=1}^{N-1} \tilde{\tilde{Z}}_{(i n)(j m)}^{(3)} f_{n}\left[\hat{S}_{m i_{1} \ldots i_{P}}^{P+1}-2 \sum_{r=1}^{P} \delta_{m i_{r}} D_{00 i_{1} \ldots \hat{i}_{r} \ldots i_{P}}\right]+\tilde{X}_{0 j}^{(3)} D_{i i_{1} \ldots i_{P}}, \tag{5.53}
\end{align*}
$$

which holds for arbitrary $i, j=1, \ldots, N-1$ and $i_{1}, \ldots, i_{P}=0, \ldots, N-1$. Together with (5.49) this equation allows to iteratively determine the tensor coefficients of 4-point functions in terms of 3-point functions for small Gram determinant $\Delta^{(3)}$ and small $\tilde{X}_{k 0}^{(3)}$ and $\tilde{X}_{0 j}^{(3)}$ as long as at least one of the $\tilde{X}_{i j}^{(3)}$ is not small. Again $i$ and $j$ can be chosen suitably in order to improve the numerical accuracy, e.g. by choosing the maximal $\tilde{X}_{i j}^{(3)}$. If the 3-point functions are known up to rank $P$, all 4-point tensor coefficients up to rank $(P-1)$ can be determined up to terms of order $\Delta^{(3)}, \tilde{X}_{k 0}^{(3)}$, and $\tilde{X}_{0 j}^{(3)}$ from (5.49) and (5.53) by putting all terms proportional to these quantities to zero. Inserting these results back into the r.h.s. of these equations, all 4-point tensor coefficients up to rank $(P-3)$ can be determined up to terms of order $\left[\max \left(\left|\Delta^{(3)}\right|,\left|\tilde{X}_{k 0}^{(3)}\right|,\left|\tilde{X}_{0 j}^{(3)}\right|\right)\right]^{2}$, and so on. Finally, the scalar 4-point function is determined up to terms of order $\left[\max \left(\left|\Delta^{(3)}\right|,\left|\tilde{X}_{k 0}^{(3)}\right|,\left|\tilde{X}_{0 j}^{(3)}\right|\right)\right]^{[(P+1) / 2]}$.

For the tensor coefficients up to rank 3 the reduction formulas explicitly read

$$
\begin{equation*}
2 \tilde{Z}_{k l}^{(3)} D_{00}=\sum_{n=1}^{N-1} \tilde{Z}_{k n}^{(3)} \hat{S}_{n l}^{2}+\tilde{X}_{k 0}^{(3)} D_{l}-\Delta^{(3)} D_{k l} \tag{5.54}
\end{equation*}
$$

$$
\begin{align*}
& \tilde{X}_{i j}^{(3)} D_{0}=\tilde{Z}_{i j}^{(3)}\left[2(5-N) D_{00}+2(D-4) D_{00}-C_{0}(0)\right] \\
& +\sum_{m, n=1}^{N-1} \tilde{\tilde{Z}}_{(i n)(j m)}^{(3)} f_{n} \hat{S}_{m}^{1}+\tilde{X}_{0 j}^{(3)} D_{i},  \tag{5.55}\\
& 4 \tilde{Z}_{k l}^{(3)} D_{00 l}=\sum_{n=1}^{N-1} \tilde{Z}_{k n}^{(3)} \hat{S}_{n l l}^{3}+\tilde{X}_{k 0}^{(3)} D_{l l}-\Delta^{(3)} D_{k l l}, \\
& 2 \tilde{Z}_{k l}^{(3)} D_{00 i_{1}}=-2 \tilde{Z}_{k i_{1}}^{(3)} D_{00 l}+\sum_{n=1}^{N-1} \tilde{Z}_{k n}^{(3)} \hat{S}_{n l i_{1}}^{3}+\tilde{X}_{k 0}^{(3)} D_{l i_{1}}-\Delta^{(3)} D_{k l i_{1}}, \quad i_{1} \neq 0, l,  \tag{5.56}\\
& \tilde{X}_{i j}^{(3)} D_{i_{1}}=\tilde{Z}_{i j}^{(3)}\left[2(6-N) D_{00 i_{1}}+2(D-4) D_{00 i_{1}}-C_{i_{1}}(0)\right] \\
& +\sum_{m, n=1}^{N-1} \tilde{Z}_{(i n)(j m)}^{(3)} f_{n}\left[\hat{S}_{m i_{1}}^{2}-2 \delta_{m i_{1}} D_{00}\right]+\tilde{X}_{0 j}^{(3)} D_{i i_{1}},  \tag{5.57}\\
& 6 \tilde{Z}_{k l}^{(3)} D_{00 l l}=\sum_{n=1}^{N-1} \tilde{Z}_{k n}^{(3)} \hat{S}_{n l l l}^{4}+\tilde{X}_{k 0}^{(3)} D_{l l l}-\Delta^{(3)} D_{k l l l}, \\
& 4 \tilde{Z}_{k l}^{(3)} D_{000 i_{1}}=-2 \tilde{Z}_{k i_{1}}^{(3)} D_{00 l l}+\sum_{n=1}^{N-1} \tilde{Z}_{k n}^{(3)} \hat{S}_{n l l i_{1}}^{4}+\tilde{X}_{k 0}^{(3)} D_{l l i_{1}}-\Delta^{(3)} D_{k l l i_{1}}, \quad i_{1} \neq 0, l, \\
& 2 \tilde{Z}_{k l}^{(3)} D_{00 i_{1} i_{2}}=-2 \tilde{Z}_{k i_{1}}^{(3)} D_{00 i_{2}}-2 \tilde{Z}_{k i_{2}}^{(3)} D_{00 l i_{1}} \\
& +\sum_{n=1}^{N-1} \tilde{Z}_{k n}^{(3)} \hat{S}_{n l i_{1} i_{2}}^{4}+\tilde{X}_{k 0}^{(3)} D_{l i_{1} i_{2}}-\Delta^{(3)} D_{k l i_{1} i_{2}}, \quad i_{1}, i_{2} \neq 0, l,  \tag{5.58}\\
& \tilde{X}_{i j}^{(3)} D_{i_{1} i_{2}}=\tilde{Z}_{i j}^{(3)}\left[2(7-N) D_{00 i_{1} i_{2}}+2(D-4) D_{00 i_{1} i_{2}}-C_{i_{1} i_{2}}(0)\right] \\
& +\sum_{m, n=1}^{N-1} \tilde{\tilde{Z}}_{(i n)(j m)}^{(3)} f_{n}\left[\hat{S}_{m i_{1} i_{2}}^{3}-2 \delta_{m i_{1}} D_{00 i_{2}}-2 \delta_{m i_{2}} D_{00 i_{1}}\right]+\tilde{X}_{0 j}^{(3)} D_{i i_{1} i_{2}},  \tag{5.59}\\
& 8 \tilde{Z}_{k l}^{(3)} D_{00 l l l}=\sum_{n=1}^{N-1} \tilde{Z}_{k n}^{(3)} \hat{S}_{n l l l l}^{5}+\tilde{X}_{k 0}^{(3)} D_{l l l l}-\Delta^{(3)} D_{k l l l l}, \\
& 6 \tilde{Z}_{k l}^{(3)} D_{00 l l i_{1}}=-2 \tilde{Z}_{k i_{1}}^{(3)} D_{00 l l l}+\sum_{n=1}^{N-1} \tilde{Z}_{k n}^{(3)} \hat{S}_{n l l i_{1}}^{5}+\tilde{X}_{k 0}^{(3)} D_{l l l i_{1}}-\Delta^{(3)} D_{k l l i_{1}}, \quad i_{1} \neq 0, l, \\
& 4 \tilde{Z}_{k l}^{(3)} D_{00 l i_{1} i_{2}}=-2 \tilde{Z}_{k i_{1}}^{(3)} D_{00 l l i_{2}}-2 \tilde{Z}_{k i_{2}}^{(3)} D_{00 l l i_{1}} \\
& +\sum_{n=1}^{N-1} \tilde{Z}_{k n}^{(3)} \hat{S}_{n l l_{1} i_{2}}^{5}+\tilde{X}_{k 0}^{(3)} D_{l l i_{1} i_{2}}-\Delta^{(3)} D_{k l l_{1} i_{2}}, \quad i_{1}, i_{2} \neq 0, l, \\
& 2 \tilde{Z}_{k l}^{(3)} D_{00 i_{1} i_{2} i_{3}}=-2 \tilde{Z}_{k i_{1}}^{(3)} D_{000 i_{2} i_{3}}-2 \tilde{Z}_{k i_{2}}^{(3)} D_{00 i_{1} i_{3}}-2 \tilde{Z}_{k i_{3}}^{(3)} D_{00 i_{1} i_{2}}
\end{align*}
$$

$$
\begin{align*}
& +\sum_{n=1}^{N-1} \tilde{Z}_{k n}^{(3)} \hat{S}_{n l i_{1} i_{2} i_{3}}^{5}+\tilde{X}_{k 0}^{(3)} D_{l i_{1} i_{2} i_{3}}-\Delta^{(3)} D_{k l i_{1} i_{2} i_{3}}, \quad i_{1}, i_{2}, i_{3} \neq 0, l  \tag{5.60}\\
\tilde{X}_{i j}^{(3)} D_{i_{1} i_{2} i_{3}}= & \tilde{Z}_{i j}^{(3)}\left[2(8-N) D_{00 i_{1} i_{2} i_{3}}+2(D-4) D_{00 i_{1} i_{2} i_{3}}-C_{i_{1} i_{2} i_{3}}(0)\right] \\
& +\sum_{m, n=1}^{N-1} \tilde{\tilde{Z}}_{(i n)(j m)}^{(3)} f_{n}\left[\hat{S}_{m i_{1} i_{2} i_{3}}^{4}-2 \delta_{m i_{1}} D_{00 i_{2} i_{3}}-2 \delta_{m i_{2}} D_{00 i_{1} i_{3}}-2 \delta_{m i_{3}} D_{00 i_{1} i_{2}}\right] \\
& +\tilde{X}_{0 j}^{(3)} D_{i i_{1} i_{2} i_{3}} . \tag{5.61}
\end{align*}
$$

Fig. 5 illustrates a systematic algorithm for the iteration scheme for 4-point integrals in the plane of tensor coefficients similar to the previous sections. Thin arrows indicate that the relation involves a suppression factor $\Delta^{(3)}$, $\tilde{X}_{k 0}^{(3)}$, or $\tilde{X}_{0 j}^{(3)}$. At the beginning of the iteration all 4-point tensor coefficients as well as the scalar integral $D_{0}$ are set to zero, i.e., no 4-point basis integral is needed. The $n$th iteration consists of the $(n+1)$ steps $n \rightarrow(n-1) \rightarrow \cdots \rightarrow 1 \rightarrow 0$ and requires all 3-point coefficient functions up to rank $2(n+1)$. Step $n$ starts with the two coefficients of ranks $(2 n+2)$ and $(2 n+3)$ that have exactly one index pair " 00 ", i.e. which belong to the second column in the respective rows in the diagrams in Fig. 5. Within a step, first the two coefficients are calculated that are reached upon omitting the index pair " 00 " from the starting coefficients; they are located in the first column two rows above the starting rows in the diagram. Then all coefficients that lie to the right of the starting coefficients are calculated column by column. After the $n$th iteration the tensor coefficients $D_{i_{1} i_{2} i_{3} \ldots}$ of ranks $2 n$ and $(2 n+1)$ without index pairs " 00 " and all coefficients $D_{00 i_{1} i_{2} \ldots}$ of two ranks higher with at least one index pair " 00 " are obtained up to terms that are suppressed by a factor $\Delta^{(3)}, \tilde{X}_{k 0}^{(3)}$, or $\tilde{X}_{0 j}^{(3)}$. Coefficients of a rank that

Step 0

$D_{i_{1} i_{2} i_{3} i_{4}} \quad D_{00 i_{1} i_{2}}$
$\begin{array}{cc}D_{i_{1} i_{2} i_{3} i_{4} i_{5}} & D_{00 i_{1} i_{2} i_{3}} \\ \vdots & \vdots\end{array}$


Step 1

$$
D_{0}
$$

$$
D_{i_{1}}
$$



Fig. 5. Schematic illustration of the iteration for small Gram and modified Cayley determinants, where thin arrows indicate that the relation involves a suppression factor $\Delta^{(3)}, \tilde{X}_{k 0}^{(3)}$, or $\tilde{X}_{0 j}^{(3)}$. In each step the boxed coefficients are calculated in the order indicated by the labels " $a$ ", " $b$ ", etc. The $n$th iteration consists of the following $(n+1)$ steps: $n \rightarrow(n-1) \rightarrow \cdots \rightarrow 1 \rightarrow 0$.
is lower by a number $2 m$ are known up to terms suppressed by $\left[\max \left(\left|\Delta^{(3)}\right|,\left|\tilde{X}_{k 0}^{(3)}\right|,\left|\tilde{X}_{0 j}^{(3)}\right|\right)\right]^{m+1}$. The iteration proceeds as follows:

- Iteration 0: $D_{00}^{(0)}, D_{00 i_{1}}^{(0)}, D_{0}^{(0)}$, and $D_{i_{1}}^{(0)}$ are calculated; all other coefficients are still zero.
- Iteration 1: Step 1 yields $D_{00 i_{1} i_{2}}^{(0)}, D_{00 i_{1} i_{2} i_{3}}^{(0)}, D_{i_{1} i_{2}}^{(0)}, D_{i_{1} i_{2} i_{3}}^{(0)}, D_{0000}^{(0)}$, and $D_{0000 i_{1}}^{(0)}$, step 0 yields $D_{00}^{(1)}, D_{00 i_{1}}^{(1)}, D_{0}^{(1)}$, and $D_{i_{1}}^{(1)}$.
- Etc.

The reduction described in this section breaks down if none of $\tilde{X}_{i j}^{(3)}$ is large compared to $\Delta^{(3)}$ and $\tilde{X}_{0 j}^{(3)}$, or if all $\tilde{Z}_{k l}^{(3)}$ become small, since in these cases the iteration does not converge. A reduction for small $Z_{k l}^{(3)}$, and thus for small $\tilde{Z}_{k l}^{(3)}$ in non-exceptional configurations, is described in Section 5.6. If both $\Delta^{(3)}$ and all $\tilde{X}_{k 0}^{(3)}$ and $\tilde{X}_{i j}^{(3)}$ become small, in some cases the alternative Passarino-Veltman reduction of Section 5.2 works. In other cases, none of the discussed reduction methods is really good. However, this happens only in exceptional cases, and one of the discussed methods yields at least crude results. ${ }^{5}$

### 5.6. Reduction for small momenta

Finally, we provide a reduction scheme for the case where all $Z_{k l}^{(3)}$ and thus all momenta become small. Note that in this case also all of the quantities $\Delta^{(3)}, \tilde{Z}_{k l}^{(3)}, \tilde{X}_{0 k}^{(3)}$, and $\tilde{X}_{k l}^{(3)}$ become small. If the $f_{k}$ are not small as well, we can proceed as follows. We rewrite (5.8) as

$$
\begin{align*}
& f_{k} D_{i_{1} \ldots i_{P}}=\hat{S}_{k i_{1} \ldots i_{P}}^{P++1}-2 \sum_{r=1}^{P} \delta_{k i_{r}} D_{00 i_{1} \ldots \hat{i}_{r} \ldots i_{P}}-\sum_{m=1}^{N-1} Z_{k m}^{(3)} D_{m i_{1} \ldots i_{P}}, \\
& k=1, \ldots, N-1, \quad i_{1}, \ldots, i_{P}=0, \ldots, N-1, \tag{5.62}
\end{align*}
$$

and (5.9) as

$$
\begin{align*}
& 2\left(4+P+\sum_{r=1}^{P} \bar{\delta}_{i_{r} 0}\right) D_{00 i_{1} \ldots i_{P}} \\
& \quad=-2(D-4) D_{00 i_{1} \ldots i_{P}}+2 C_{i_{1} \ldots i_{P}}(0)+2 m_{0}^{2} D_{i_{1} \ldots i_{P}} \\
& \quad-\sum_{n, m=1}^{N-1} Z_{n m}^{(3)} D_{n m i_{1} \ldots i_{P}}, \quad i_{1}, \ldots, i_{P}=0, \ldots, N-1 . \tag{5.63}
\end{align*}
$$

By using these equations iteratively, we can determine $D_{i_{1} \ldots i_{P}}$ and $D_{0 i_{1} \ldots i_{P}}$ for given 3-point functions for small $Z_{k l}^{(3)}$. If the 3-point functions are known up to rank $P$, we can determine the coefficients of the 4-point functions with rank $P$ up to terms of order $Z_{k l}^{(3)}$, those of rank $(P-1)$

[^5]up to terms of order $\left[Z_{k l}^{(3)}\right]^{2}, \ldots$, and those of order 0 up to terms of order $\left[Z_{k l}^{(3)}\right]^{P+1}$. In order to improve numerical stability, we can choose $k$ such that $f_{k}$ is maximal. Note that the structure of (5.62) and (5.63) is similar to the one of (5.38) and (5.40). In fact, a systematic algorithm for this iteration scheme for 4-point integrals is given by Fig. 4, if the arrows that point vertically downwards or horizontally to the left are omitted.

Up to tensor rank 3 the explicit formulas read:

$$
\begin{align*}
& f_{k} D_{0}=\hat{S}_{k}^{1}-\sum_{m=1}^{N-1} Z_{k m}^{(3)} D_{m},  \tag{5.64}\\
& 8 D_{00}=-2(D-4) D_{00}+2 C_{0}(0)+2 m_{0}^{2} D_{0}-\sum_{n, m=1}^{N-1} Z_{n m}^{(3)} D_{n m},  \tag{5.65}\\
& f_{k} D_{i_{1}}=\hat{S}_{k i_{1}}^{2}-2 \delta_{k i_{1}} D_{00}-\sum_{m=1}^{N-1} Z_{k m}^{(3)} D_{m i_{1}},  \tag{5.66}\\
& 12 D_{00 i_{1}}=-2(D-4) D_{00 i_{1}}+2 C_{i_{1}}(0)+2 m_{0}^{2} D_{i_{1}}-\sum_{n, m=1}^{N-1} Z_{n m}^{(3)} D_{n m i_{1}},  \tag{5.67}\\
& f_{k} D_{i_{1} i_{2}}=\hat{S}_{k i_{1} i_{2}}^{3}-2 \delta_{k i_{1}} D_{00 i_{2}}-2 \delta_{k i_{2}} D_{00 i_{1}}-\sum_{m=1}^{N-1} Z_{k m}^{(3)} D_{m i_{1} i_{2}},  \tag{5.68}\\
& 16 D_{00 i_{1} i_{2}}=-2(D-4) D_{00 i_{1} i_{2}}+2 C_{i_{1} i_{2}}(0)+2 m_{0}^{2} D_{i_{1} i_{2}}-\sum_{n, m=1}^{N-1} Z_{n m}^{(3)} D_{n m i_{1} i_{2}},  \tag{5.69}\\
& f_{k} D_{i_{1} i_{2} i_{3}}=\hat{S}_{k i_{1} i_{2} i_{3}}^{4}-2 \delta_{k i_{1}} D_{00 i_{2} i_{3}}-2 \delta_{k i_{2}} D_{00 i_{1} i_{3}}-2 \delta_{k i_{3}} D_{00 i_{1} i_{2}}-\sum_{m=1}^{N-1} Z_{k m}^{(3)} D_{m i_{1} i_{2} i_{3}} . \tag{5.70}
\end{align*}
$$

If also all the $f_{k}$ become small we can rewrite (5.62) and (5.63) as

$$
\begin{align*}
& 2 \sum_{r=1}^{P} \delta_{k i_{r}} D_{00 i_{1} \ldots \hat{i}_{r} \ldots i_{P}}=\hat{S}_{k i_{1} \ldots i_{P}}^{P+1}-f_{k} D_{i_{1} \ldots i_{P}}-\sum_{m=1}^{N-1} Z_{k m}^{(3)} D_{m i_{1} \ldots i_{P}} \\
& \quad k=1, \ldots, N-1, \quad i_{1}, \ldots, i_{P}=0, \ldots, N-1 \tag{5.71}
\end{align*}
$$

and

$$
\begin{align*}
2 m_{0}^{2} D_{i_{1} \ldots i_{P}}= & 2\left(4+P+\sum_{r=1}^{P} \bar{\delta}_{i_{r} 0}\right) D_{00 i_{1} \ldots i_{P}}+2(D-4) D_{00 i_{1} \ldots i_{P}} \\
& -2 C_{i_{1} \ldots i_{P}}(0)+\sum_{n, m=1}^{N-1} Z_{n m}^{(3)} D_{n m i_{1} \ldots i_{P}}, \quad i_{1}, \ldots, i_{P}=0, \ldots, N-1 . \tag{5.72}
\end{align*}
$$

By using these equations iteratively, we can determine $D_{i_{1} \ldots i_{P}}$ and $D_{00 i_{1} \ldots i_{P}}$ for given 3-point functions for small $Z_{k l}^{(3)}$ and small $f_{k}$. The structure of (5.71) and (5.72) is similar to the one of (5.49) and (5.53). If the 3-point functions are known up to rank $P$, we can determine the coefficients of the 4-point functions with rank $(P-1)$ up to terms of order $\max \left(\left|Z_{k l}^{(3)}\right|,\left|f_{n}\right|\right)$,

Table 1
Summary of features of the reduction schemes for 3- and 4-point tensor integrals. The type of the method is either "reduction (red.)" or "expansion (exp.)"; Gram and Cayley determinants are generically indicated by $|Z|$ and $|X|$, respectively

| Section | 5.1 | 5.2 | 5.3 | 5.4 | 5.5 | 5.6 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Method | PV | $\mathrm{PV}^{\prime}$ | Cayley | Gram | Gram/Cayley | Momenta |
| Type | red. | red. | red. | exp. | exp. | exp. |
|  |  |  |  | $\|Z\| \rightarrow 0$ | $\|Z\| \rightarrow 0$ |  |
| Applicability | $\|Z\| \neq 0$ | $\|Z\| \neq 0$ | $\|X\| \neq 0$ | $\tilde{X}_{0 j} \neq 0$ | $\tilde{X}_{0 k}, \tilde{X}_{0 j} \rightarrow 0$ | $Z \rightarrow 0$ |
|  |  |  |  | $\tilde{X}_{i j} \neq 0$ | $f_{k} \neq 0$ |  |
| Stable for $\|Z\| \rightarrow 0 ?$ | no | no | yes | yes | yes | yes |
| Stable for $\|X\| \rightarrow 0 ?$ | yes | no | no |  | yes | yes |
| Stable for $Z \rightarrow 0 ?$ | no | no | no | no | no | yes |
| Fast? | yes | yes | no | yes | yes | yes |

those of rank $(P-3)$ up to terms of order $\left[\max \left(\left|Z_{k l}^{(3)}\right|,\left|f_{k}\right|\right)\right]^{2}$, and so on. Finally, the scalar 4-point function is determined up to terms of order $\left[\max \left(\left|Z_{k l}^{(3)}\right|,\left|f_{k}\right|\right)\right]^{[(P+1) / 2]}$.

### 5.7. Summary of reduction schemes and application to $\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow 4 f$ at one loop

Table 1 briefly summarizes some of the features of the described reduction schemes for 3and 4-point tensor integrals. The type of the method, "reduction (red.)" or "expansion (exp.)" is indicated in the third row. In the fourth row we summarize the conditions for the applicability of the schemes. Conditions that depend on indices $i, j, k, l$ have to be fulfilled for at least one choice of these indices. The "yes" and "no" in the last rows indicate whether a method is stable or unstable in the corresponding limits or if the method is fast in terms of CPU time. A blank entry means that the method can be stable or unstable in the considered limit.

The reduction schemes described above have been successfully applied in the calculation of the complete one-loop corrections to the charged-current processes $\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow 4 f$ as presented in Ref. [35]. As described there, actually two independent calculations of the corrections have been carried out employing two different procedures (called "rescue systems" there) for the evaluation of the one-loop tensor integrals in the numerically delicate kinematical configurations. Both procedures make use of the conventional Passarino-Veltman reduction (see Section 5.1) as long as internal consistency checks prove this method to be reliable. If this is not the case, the procedures differ:
(i) Procedure 1: reduction with modified Cayley determinants and further exception handling.

If conventional Passarino-Veltman reduction seems not to be trustworthy, since consistency relations among the tensor coefficients are valid only to very few digits or even violated, the method with modified Cayley determinants is used as described in Section 5.3. Because of the vanishing modified Cayley determinant this is not possible for the IR-singular (i.e. soft or collinear divergent) 3-point functions. Therefore, these cases are evaluated as described in Appendix B, yielding perfectly stable results.

The described procedure fails if both the Gram and the modified Cayley determinants are very small. In practice, this happens only at a small fraction of events that hardly contribute to
the $\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow 4 f$ cross section. However, if this limitation of the procedure becomes serious in other cases, the double limit of small Gram and modified Cayley determinants can be covered using the method of Section 5.5, etc., as it is done in Procedure 2.
(ii) Procedure 2: expansions for small Gram determinants, etc.

The second procedure is based on the two versions of Passarino-Veltman reduction of Sections 5.1 and 5.2 and on the expansion methods described in Sections 5.4-5.6. If the PassarinoVeltman reduction fails, at least the Gram determinant of the corresponding integral (or an integral related to a subgraph) is small. The question which of the different expansions is most appropriate is decided by estimating the number of valid digits in each of the expansion variants; the variant promising the highest precision is taken.

For the application to the processes $\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow 4 f$, it turned out to be sufficient to implement the expansions for small Gram determinant (Section 5.4), for small Gram and modified Cayley determinants (Section 5.5), and for small momenta (Section 5.6) up to tensor rank 4 for 4 -point functions and the corresponding formulas for 3-point functions up to tensor rank 5. The implementation of the modified procedure for small $f_{k}$ was not required. We also did not yet implement the schemes mentioned in footnotes 4 and 5.

Note that in the one-loop diagrams for $\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow 4 f 3$ - and 4-point functions appear only up to rank 3, i.e. the implemented reductions go beyond taking pure limits of vanishing determinants. For these processes, the exceptional cases where none of the expansions is good appeared only for a very small fraction of events and did not yield sizeable contributions to integrated physical quantities.

### 5.8. UV and IR divergences in dimensional regularization and terms of order $(D-4)$

In the preceding equations we have kept all terms of order $(D-4)$ that multiply one-loop coefficient integrals. These terms give rise to finite terms in dimensional regularization if these integrals are divergent in four dimensions. It is convenient to discuss UV divergences, which formally result from loop momenta $q$ tending to infinity, and IR divergences, which arise from finite loop momenta but specific kinematical configurations, separately:

## (i) UV divergences.

UV divergences are universal in the sense that the divergent terms in an integral are regular functions of the external momenta $p_{k}$ and internal masses $m_{k}$, but these terms do not change if these kinematical quantities approach exceptional configurations (zero limits, on-shell configurations, etc.). At one loop, UV divergences generally have the form $1 /(D-4)$ times a polynomial in $p_{k}$ and $m_{k}$. Therefore, the terms $(D-4) T_{i_{1} \ldots i_{P}}^{N}$ contained in the above formulas are finite polynomials in $p_{k}$ and $m_{k}$. We have listed these $(D-4) T_{i_{1} \ldots i_{P}}^{N}$ terms for 1-point functions of arbitrary rank, 2-point functions up to rank 5, 3-point functions up to rank 7, 4-point functions up to rank 7, and five-point functions up to rank 6 in Appendix A.

## (ii) IR divergences.

IR divergences at one loop originate either from soft or collinear configurations of a loop momentum [41]. These types of divergences have the property that they do not show up in tensor
coefficients with at least one index pair " 00 ", i.e., all tensor structures containing at least one factor of the metric tensor are IR finite.

These fact can be seen by inspecting the Feynman-parameter integrals of the tensor coefficients or by the following arguments. A soft singularity results from the limit of zero-momentum transfer of a massless particle $(q \rightarrow 0)$ between two on-shell particles. Assuming that the massless particle correspond to the propagator denominator $N_{0}$, power counting in $q$ shows that soft divergences can appear only in the scalar integral, but not in tensor integrals, because loop momenta in the numerator render the limit $q \rightarrow 0$ in the integral non-singular. Thus, in the general case, where the massless particle corresponds to any propagator denominator $N_{k}$, soft divergent parts of tensor integrals are always proportional to powers of the momenta $p_{k}$, as can be seen by performing a shift $q \rightarrow q-p_{k}$, which maps $N_{k}$ to $N_{0}$. A collinear singularity results from the range where the loop momentum $q$ is parallel to the momentum $p_{k}$ of a light external on-shell particle that splits into two light particles. If a tensor $q_{\mu_{1}} \ldots q_{\mu_{P}}$ is present in the loop integral, the divergence can only show up in covariants that are built up in the singular region. Thus, collinear divergences of tensor integrals appear in covariants containing only the momentum $p_{k}$.

In the reduction formulas given above the factor $(D-4)$ appears only in front of tensor coefficients $T_{00 i_{3} \ldots i_{P}}^{N}$ containing at least one index pair " 00 ", which have been shown to be IR finite. Therefore, all the reduction formulas are valid without modification if IR singularities are regularized dimensionally. All terms $(D-4) T_{00 i_{3} \ldots i_{P}}^{N}$ can be taken from Appendix A; if more of these terms are needed, they can be easily derived from the reduction formulas themselves.

## 6. Reduction of 5-point integrals

In four space-time dimensions, 5-point integrals can be reduced to 4-point integrals. In Ref. [26] we have given relations that express 5-point tensor integrals of rank $P$ by 4-point tensor integrals of rank $P$ (see also Appendix C). This method follows the strategy proposed in Ref. [22] for the reduction of scalar integrals and was actually used in the calculation of oneloop corrections to $\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow 4 f$ [35]. Here we derive formulas that directly reduce 5-point tensor integrals of rank $P$ to 4-point tensor integrals of rank $(P-1)$. While similar results have been presented in Ref. [19], our derivation is more transparent.

We start by considering the determinant

$$
\begin{aligned}
\mathcal{E} & =\left|\begin{array}{ccccc}
q^{\mu} & -2 q^{2} & 2 q p_{1} & \cdots & 2 q p_{4} \\
0 & 2 m_{0}^{2} & f_{1} & \cdots & f_{4} \\
p_{1}^{\mu} & -2 p_{1} q & 2 p_{1} p_{1} & \cdots & 2 p_{1} p_{4} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{4}^{\mu} & -2 p_{4} q & 2 p_{4} p_{1} & \cdots & 2 p_{4} p_{4}
\end{array}\right| \\
& =\left|\begin{array}{ccccc}
q^{\mu} & -N_{0}-2 m_{0}^{2} & 2 q p_{1} & \cdots & 2 q p_{4} \\
0 & 2 m_{0}^{2} & f_{1} & \cdots & f_{4} \\
p_{1}^{\mu} & f_{1} & 2 p_{1} p_{1} & \cdots & 2 p_{1} p_{4} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{4}^{\mu} & f_{4} & 2 p_{4} p_{1} & \cdots & 2 p_{4} p_{4}
\end{array}\right|+\left|\begin{array}{ccccc}
q^{\mu} & -N_{0} & 2 q p_{1} & \cdots & 2 q p_{4} \\
0 & 0 & f_{1} & \cdots & f_{4} \\
p_{1}^{\mu} & N_{0}-N_{1} & 2 p_{1} p_{1} & \cdots & 2 p_{1} p_{4} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{4}^{\mu} & N_{0}-N_{4} & 2 p_{4} p_{1} & \cdots & 2 p_{4} p_{4}
\end{array}\right|
\end{aligned}
$$

$$
\begin{align*}
= & \left|\begin{array}{ccccc}
q^{\mu} & -N_{0} & N_{1}-N_{0} & \cdots & N_{4}-N_{0} \\
0 & 2 m_{0}^{2} & f_{1} & \cdots & f_{4} \\
p_{1}^{\mu} & f_{1} & 2 p_{1} p_{1} & \cdots & 2 p_{1} p_{4} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{4}^{\mu} & f_{4} & 2 p_{4} p_{1} & \cdots & 2 p_{4} p_{4}
\end{array}\right| \\
& +\left|\begin{array}{ccccc}
g_{\alpha}^{\mu} & -N_{0} & 2 p_{1, \alpha} & \cdots & 2 q p_{4, \alpha} \\
0 & 0 & f_{1} & \cdots & f_{4} \\
p_{1}^{\mu} & q^{\alpha}\left(N_{0}-N_{1}\right) & 2 p_{1} p_{1} & \cdots & 2 p_{1} p_{4} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{4}^{\mu} & q^{\alpha}\left(N_{0}-N_{4}\right) & 2 p_{4} p_{1} & \cdots & 2 p_{4} p_{4}
\end{array}\right| . \tag{6.1}
\end{align*}
$$

In the first manipulation, we have split the determinant in the second column, and in the second we have added the second row of the first determinant to its first row and we have moved $q^{\alpha}$ from the first row to the second column in the second determinant. Moreover, we have used the definitions (2.2) and (2.24).

In four dimensions, the determinant $\mathcal{E}$ vanishes, as can be seen from its defining form, because $q$ is linearly dependent on the four momenta $p_{i}, i=1, \ldots, 4$. Since we want to derive a relation that also holds in dimensional regularization we do not use this fact, but translate the integral over $\mathcal{E}$ into a form that has a factor of $\mathcal{O}(D-4)$ rendering the whole contribution zero for finite integrals. Inserting the first form of $\mathcal{E}$ in (6.1) into the integrand of the tensor integral $E^{\mu_{1} \ldots \mu_{P}}$ results in

$$
\begin{align*}
\int \mathcal{E} \equiv & \equiv \frac{(2 \pi \mu)^{4-D}}{\mathrm{i} \pi^{2}} \int \mathrm{~d}^{D} q \frac{q^{\mu_{1}} \cdots q^{\mu_{P}}}{N_{0} N_{1} \cdots N_{4}} \mathcal{E} \\
& =2 m_{0}^{2} E^{\alpha \mu_{1} \ldots \mu_{P}}\left|\begin{array}{cccc}
g_{\alpha}^{\mu} & 2 p_{1, \alpha} & \cdots & 2 p_{4, \alpha} \\
p_{1}^{\mu} & 2 p_{1} p_{1} & \cdots & 2 p_{1} p_{4} \\
\vdots & \vdots & \ddots & \vdots \\
p_{4}^{\mu} & 2 p_{4} p_{1} & \cdots & 2 p_{4} p_{4}
\end{array}\right| \\
& +E^{\alpha \beta \mu_{1} \ldots \mu_{P}}\left|\begin{array}{ccccc}
g_{\alpha}^{\mu} & -2 g_{\alpha \beta} & 2 p_{1, \alpha} & \cdots & 2 p_{4, \alpha} \\
0 & 0 & f_{1} & \cdots & f_{4} \\
p_{1}^{\mu} & -2 p_{1, \beta} & 2 p_{1} p_{1} & \cdots & 2 p_{1} p_{4} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{4}^{\mu} & -2 p_{4, \beta} & 2 p_{4} p_{1} & \cdots & 2 p_{4} p_{4}
\end{array}\right| . \tag{6.2}
\end{align*}
$$

This form can be written more compactly by introducing the four-dimensional metric tensor

$$
g_{(4)}^{\mu \nu}=\sum_{j, k=1}^{4} 2 p_{j}^{\mu} p_{k}^{\nu}\left(Z^{(4)}\right)_{k j}^{-1}=-\frac{1}{\Delta^{(4)}}\left|\begin{array}{cccc}
0 & 2 p_{1}^{v} & \cdots & 2 p_{4}^{v}  \tag{6.3}\\
p_{1}^{\mu} & 2 p_{1} p_{1} & \cdots & 2 p_{1} p_{4} \\
\vdots & \vdots & \ddots & \vdots \\
p_{4}^{\mu} & 2 p_{4} p_{1} & \cdots & 2 p_{4} p_{4}
\end{array}\right|,
$$

leading to the result

$$
\begin{align*}
\int \mathcal{E}= & 2 m_{0}^{2} \Delta^{(4)}\left(g_{\alpha}^{\mu}-g_{\left.(4)_{\alpha}^{\mu}\right)}^{\mu}\right) E^{\alpha \mu_{1} \ldots \mu_{P}} \\
& +2 \sum_{n=1}^{4} \tilde{X}_{n 0}^{(4)}\left[p_{n}^{\mu}\left(g_{\alpha \beta}-g_{\left.(4)_{\alpha \beta}\right)}-p_{n, \beta}\left(g_{\alpha}^{\mu}-g_{(4)}^{\alpha}{ }_{\alpha}^{\mu}\right)\right] E^{\alpha \beta \mu_{1} \ldots \mu_{P}} .\right. \tag{6.4}
\end{align*}
$$

The second term is obtained by expanding the second determinant in (6.2) along the first two rows and the first two columns according to (2.17) and using (2.20).

Alternatively integrating over the last form of $\mathcal{E}$ in (6.1), we obtain

$$
\begin{align*}
\int \mathcal{E}= & \left|\begin{array}{ccccc}
E^{\mu \mu_{1} \ldots \mu_{P}} & -D^{\mu_{1} \ldots \mu_{P}}(0) & D^{\mu_{1} \ldots \mu_{P}}(1)-D^{\mu_{1} \ldots \mu_{P}}(0) & \cdots & D^{\mu_{1} \ldots \mu_{P}}(4)-D^{\mu_{1} \ldots \mu_{P}}(0) \\
0 & 2 m_{0}^{2} & f_{1} & \cdots & f_{4} \\
p_{1}^{\mu} & f_{1} & 2 p_{1} p_{1} & \cdots & 2 p_{1} p_{4} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{4}^{\mu} & f_{4} & 2 p_{4} p_{1} & \cdots & 2 p_{4} p_{4}
\end{array}\right| \\
& +\left|\begin{array}{ccccc}
g_{\alpha}^{\mu} & -D^{\mu_{1} \ldots \mu_{P}}(0) & 2 p_{1, \alpha} & \cdots & 2 p_{4, \alpha} \\
0 & 0 & f_{1} & \cdots & f_{4} \\
p_{1}^{\mu} & D^{\alpha \mu_{1} \ldots \mu_{P}}(0)-D^{\alpha \mu_{1} \ldots \mu_{P}}(1) & 2 p_{1} p_{1} & \cdots & 2 p_{1} p_{4} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{4}^{\mu} & D^{\alpha \mu_{1} \ldots \mu_{P}}(0)-D^{\alpha \mu_{1} \ldots \mu_{P}}(4) & 2 p_{4} p_{1} & \cdots & 2 p_{4} p_{4}
\end{array}\right| . \tag{6.5}
\end{align*}
$$

The last determinant can be written as

$$
\begin{align*}
& \left|\begin{array}{ccccc}
g_{\alpha}^{\mu} & -D^{\mu_{1} \ldots \mu_{P}}(0) & 2 p_{1, \alpha} & \cdots & 2 p_{4, \alpha} \\
0 & 0 & f_{1} & \cdots & f_{4} \\
p_{1}^{\mu} & D^{\alpha \mu_{1} \ldots \mu_{P}}(0)-D^{\alpha \mu_{1} \ldots \mu_{P}}(1) & 2 p_{1} p_{1} & \cdots & 2 p_{1} p_{4} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{4}^{\mu} & D^{\alpha \mu_{1} \ldots \mu_{P}}(0)-D^{\alpha \mu_{1} \ldots \mu_{P}}(4) & 2 p_{4} p_{1} & \cdots & 2 p_{4} p_{4}
\end{array}\right| \\
& =\left|\begin{array}{ccccc}
g_{\alpha}^{\mu} & 0 & 2 p_{1, \alpha} & \cdots & 2 p_{4, \alpha} \\
0 & 0 & f_{1} & \cdots & f_{4} \\
p_{1}^{\mu} & D^{\alpha \mu_{1} \ldots \mu_{P}}(0)+p_{1}^{\alpha} D^{\mu_{1} \ldots \mu_{P}}(0)-D^{\alpha \mu_{1} \ldots \mu_{P}}(1) & 2 p_{1} p_{1} & \cdots & 2 p_{1} p_{4} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{4}^{\mu} & D^{\alpha \mu_{1} \ldots \mu_{P}}(0)+p_{1}^{\alpha} D^{\mu_{1} \ldots \mu_{P}}(0)-D^{\alpha \mu_{1} \ldots \mu_{P}}(4) & 2 p_{4} p_{1} & \cdots & 2 p_{4} p_{4}
\end{array}\right| \\
& =\left|\begin{array}{ccccc}
g_{\alpha}^{\mu} & 0 & 2 p_{1, \alpha} & \cdots & 2 p_{4, \alpha} \\
0 & 0 & f_{1} & \cdots & f_{4} \\
p_{1}^{\mu} & -\mathcal{D}^{\alpha \mu_{1} \ldots \mu_{P}}(1) & 2 p_{1} p_{1} & \cdots & 2 p_{1} p_{4} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{4}^{\mu} & -\mathcal{D}^{\alpha \mu_{1} \ldots \mu_{P}}(4) & 2 p_{4} p_{1} & \cdots & 2 p_{4} p_{4}
\end{array}\right| . \tag{6.6}
\end{align*}
$$

The first equality in (6.6) can be easily checked by expanding along the second column. In order to explain the second equality, we introduce the Lorentz-covariant decompositions

$$
\begin{align*}
& D^{\alpha \mu_{1} \ldots \mu_{P}}(i)=\left[D^{\alpha \mu_{1} \ldots \mu_{P}}(i)\right]^{(p)}+\left[D^{\alpha \mu_{1} \ldots \mu_{P}}(i)\right]^{(g)}, \quad i=0, \ldots, 4, \\
& {\left[D^{\alpha \mu_{1} \ldots \mu_{P}}(i)\right]^{(p)}=\sum_{\substack{n=1 \\
n \neq i}}^{4} p_{n}^{\alpha} x_{n}^{\mu_{1} \ldots \mu_{P}}(i), \quad\left[D^{\alpha \mu_{1} \ldots \mu_{P}}(i)\right]^{(g)}=\sum_{r=1}^{P} g^{\alpha \mu_{r}} y_{r}^{\mu_{1} \ldots \hat{\mu}_{r} \ldots \mu_{P}}(i),} \\
& {\left[D^{\alpha \mu_{1} \ldots \mu_{P}}(0)+p_{1}^{\alpha} D^{\mu_{1} \ldots \mu_{P}}(0)\right]^{(p)}=\sum_{n=2}^{4}\left(p_{n}-p_{1}\right)^{\alpha} z_{n}^{\mu_{1} \ldots \mu_{P}} .} \tag{6.7}
\end{align*}
$$

The operation " $(g)$ " isolates all tensor structures in which the first Lorentz index appears at a metric tensor; the remaining part of the tensor furnishes the " $(p)$ " contribution. The last
decomposition in (6.7) becomes obvious after performing a shift $q \rightarrow q-p_{1}$ in the integral. From (6.7) it follows immediately that the terms in the second line of (6.6) that involve $\left[D_{\alpha \mu_{1} \ldots \mu_{P}}(i)\right]^{(p)}, i=1, \ldots, 4$, drop out when expanding the determinant along the second column, because the resulting determinants vanish. Similarly, the contribution proportional to $\left[D_{\alpha \mu_{1} \ldots \mu_{P}}(0)+p_{1 \alpha} D_{\mu_{1} \ldots \mu_{P}}(0)\right]^{(p)}$ vanishes after summation over all contributions. The remaining terms involving $[D]^{(g)}$ are collected in the quantity

$$
\begin{equation*}
\mathcal{D}^{\alpha \mu_{1} \ldots \mu_{P}}(i)=\left[D^{\alpha \mu_{1} \ldots \mu_{P}}(i)-D^{\alpha \mu_{1} \ldots \mu_{P}}(0)\right]^{(g)}, \quad i=1, \ldots, 4 \tag{6.8}
\end{equation*}
$$

Inserting (6.6) into (6.5) and expanding the determinants we find

$$
\begin{align*}
\int \mathcal{E}= & \operatorname{det}\left(X^{(4)}\right) E^{\mu \mu_{1} \ldots \mu_{P}}-\sum_{n, m=1}^{4} \tilde{X}_{m n}^{(4)} p_{m}^{\mu}\left[D^{\mu_{1} \ldots \mu_{P}}(n)-D^{\mu_{1} \ldots \mu_{P}}(0)\right] \\
& -\sum_{n=1}^{4} \tilde{X}_{n 0}^{(4)}\left[-p_{n}^{\mu} D^{\mu_{1} \ldots \mu_{P}}(0)+\mathcal{D}^{\mu \mu_{1} \ldots \mu_{P}}(n)\right] \\
& +\sum_{n=1}^{4} \mathcal{D}^{\alpha \mu_{1} \ldots \mu_{P}}(n) \sum_{m, l=1}^{4} 2 p_{m, \alpha} p_{l}^{\mu} \tilde{\tilde{X}}_{(l n)(0 m)}^{(4)} \tag{6.9}
\end{align*}
$$

where $\tilde{\tilde{X}}_{(l n)(0 m)}^{(4)}$ is given in (2.29). Setting this equal to (6.4), we obtain

$$
\begin{align*}
& \operatorname{det}\left(X^{(4)}\right) E^{\mu \mu_{1} \ldots \mu_{P}} \\
& \quad=\sum_{n, m=1}^{4} \tilde{X}_{m n}^{(4)} p_{m}^{\mu}\left[D^{\mu_{1} \ldots \mu_{P}}(n)-D^{\mu_{1} \ldots \mu_{P}}(0)\right] \\
& \quad+\sum_{n=1}^{4} \tilde{X}_{n 0}^{(4)}\left[-p_{n}^{\mu} D^{\mu_{1} \ldots \mu_{P}}(0)+\mathcal{D}^{\mu \mu_{1} \ldots \mu_{P}}(n)\right] \\
& \quad-\sum_{n=1}^{4} \mathcal{D}^{\alpha \mu_{1} \ldots \mu_{P}}(n) \sum_{m, l=1}^{4} 2 p_{m, \alpha} p_{l}^{\mu} \tilde{\tilde{X}}_{(l n)(0 m)}^{(4)}+2 m_{0}^{2} \Delta^{(4)}\left(g_{\alpha}^{\mu}-g_{(4)}^{\alpha}\right) E^{\alpha \mu_{1} \ldots \mu_{P}} \\
& \quad+2 \sum_{n=1}^{4} \tilde{X}_{n 0}^{(4)}\left[p_{n}^{\mu}\left(g_{\alpha \beta}-g_{(4){ }_{\alpha \beta}}\right)-p_{n, \beta}\left(g_{\alpha}^{\mu}-g_{(4)}^{\mu}\right)\right] E^{\alpha \beta \mu_{1} \ldots \mu_{P}} \tag{6.10}
\end{align*}
$$

In this result, all inverse Gram determinants have been absorbed in the four-dimensional metric tensor, which appears only in the difference $\left(g-g_{(4)}\right)$. In four dimensions, all these terms vanish identically. In dimensional regularization they contribute only if $E^{\alpha \beta \mu_{1} \ldots \mu_{P}}$ involves singularities, i.e., only the singular terms in $E^{\alpha \beta \mu_{1} \ldots \mu_{P}}$ are relevant. As explained in Section 5.8, IR singularities of $E^{\alpha \mu_{1} \ldots \mu_{P}}$ appear only in contributions that are proportional to a momentum $p_{k}^{\alpha}$. These contributions vanish exactly in (6.10) as long as the external momenta have only nonvanishing components in the four-dimensional subspace. UV singularities appear only if $P \geqslant 4$. Therefore, we can omit the last two terms in (6.10) for $P<4 .{ }^{6}$ For $P \geqslant 4$ the inverse Gram de-

[^6]terminant that is implicitly contained in $g_{(4)}$ can always be cancelled by a prefactor $\Delta^{(4)}$. In the last-but-one term of (6.10) this prefactor is already explicit; for the last contribution it is straightforward to check ${ }^{7}$ that this factor always arises after symmetrizing the r.h.s. of (6.10) w.r.t. the indices $\mu, \mu_{1}, \ldots, \mu_{P}$.

The next step consists in the insertion of the decompositions of tensor integrals into Lorentz covariants. Here and in the following we omit the terms involving $\left(g-g_{(4)}\right)$ if $P<4$. The general tensor decompositions up to rank 5 explicitly read

$$
\begin{align*}
E^{\mu}= & \sum_{i_{1}=1}^{4} p_{i_{1}}^{\mu} E_{i_{1}}, \quad E^{\mu \nu}=\sum_{i_{1}, i_{2}=1}^{4} p_{i_{1}}^{\mu} p_{i_{2}}^{\nu} E_{i_{1} i_{2}}+g^{\mu \nu} E_{00}, \\
E^{\mu \nu \rho}= & \sum_{i_{1}, i_{2}, i_{3}=1}^{4} p_{i_{1}}^{\mu} p_{i_{2}}^{\nu} p_{i_{3}}^{\rho} E_{i_{1} i_{2} i_{3}}+\sum_{i_{1}=1}^{4}\{g p\}_{i_{1}}^{\mu \nu \rho} E_{00 i_{1}}, \\
E^{\mu \nu \rho \sigma}= & \sum_{i_{1}, i_{2}, i_{3}, i_{4}=1}^{4} p_{i_{1}}^{\mu} p_{i_{2}}^{\nu} p_{i_{3}}^{\rho} p_{i_{4}}^{\sigma} E_{i_{1} i_{2} i_{3} i_{4}}+\sum_{i_{1}, i_{2}=1}^{4}\{g p p\}_{i_{1} i_{2}}^{\mu \nu \sigma} E_{00 i_{1} i_{2}}+\{g g\}^{\mu \nu \rho \sigma} E_{0000}, \\
E^{\mu \nu \rho \sigma \tau}= & \sum_{i_{1}, i_{2}, i_{3}, i_{4}, i_{5}=1}^{4} p_{i_{1}}^{\mu} p_{i_{2}}^{\nu} p_{i_{3}}^{\rho} p_{i_{4}}^{\sigma} p_{i_{5}}^{\tau} E_{i_{1} i_{2} i_{3} i_{4} i_{5}}+\sum_{i_{1}, i_{2}, i_{3}=1}^{4}\{g p p p\}_{i_{1} i_{2} i_{3}}^{\mu \nu \rho \tau} E_{00 i_{1} i_{2} i_{3}} \\
& +\sum_{i_{1}=1}^{4}\{g g p\}_{i_{1}}^{\mu \nu \rho \sigma \tau} E_{0000 i_{1} .} . \tag{6.11}
\end{align*}
$$

In four dimensions, the covariants involving metric tensors are redundant in these decompositions, since the metric tensor could be replaced by (6.3). By keeping these coefficients we can avoid the appearance of explicit inverse Gram determinants in the reduction formulas.

Inserting the Lorentz decompositions of the tensor integrals into (6.10), we find the following reduction equations for the tensor coefficients upon comparing coefficients of covariants,

$$
\begin{align*}
& \operatorname{det}\left(X^{(4)}\right) \bar{E}_{k i_{1} \ldots i_{P}} \\
& \quad=\sum_{n=1}^{4} \tilde{X}_{k n}^{(4)}\left[D_{\left(i_{1}\right)_{n} \ldots\left(i_{P}\right)_{n}}(n) \bar{\delta}_{i_{1} n} \ldots \bar{\delta}_{i_{P} n}-D_{i_{1} \ldots i_{P}}(0)\right]-\tilde{X}_{k 0}^{(4)} D_{i_{1} \ldots i_{P}}(0) \\
& \quad-2 \sum_{n=1}^{4} \sum_{r=1}^{P} \tilde{\tilde{X}}_{(k n)\left(0 i_{r}\right)}^{(4)}\left[D_{00\left(i_{1}\right)_{n} \ldots\left(\widehat{\left.i_{r}\right)_{n} \ldots\left(i_{P}\right)_{n}}\right.}(n) \bar{\delta}_{i_{1} n} \ldots \bar{\delta}_{i_{r-1} n} \bar{\delta}_{i_{r+1} n} \ldots \bar{\delta}_{i_{P} n}\right. \\
& \left.\quad-D_{00 i_{1} \ldots \hat{i}_{r} \ldots i_{P}}(0)\right], \quad k=1, \ldots, 4, \quad P<4,  \tag{6.12}\\
& \operatorname{det}\left(X^{(4)}\right) \bar{E}_{00 i_{2} \ldots i_{P}}=\sum_{n=1}^{4} \tilde{X}_{n 0}^{(4)}\left[D_{00\left(i_{2}\right)_{n} \ldots\left(i_{P}\right)_{n}}(n) \bar{\delta}_{i_{2} n} \ldots \bar{\delta}_{i_{P} n}-D_{00 i_{2} \ldots i_{P}}(0)\right], \quad P<4 . \tag{6.13}
\end{align*}
$$

[^7]Since we have distinguished the index $k$ in the derivation of (6.12), the resulting tensor coefficients $\bar{E}_{\ldots}$ are not symmetric under the exchange of $k$ with one of the indices $i_{r}, r=1, \ldots, P$. In order to distinguish them from the symmetric tensor coefficients $E_{\ldots}$, we marked them with a bar. Symmetric tensor coefficients can be easily obtained by adding all $P$ results with $k$ exchanged with one of the $i_{r}$ and dividing the sum by $P$, e.g.,

$$
\begin{align*}
& E_{i_{1} i_{2} i_{3}}=\frac{1}{3}\left(\bar{E}_{i_{1} i_{2} i_{3}}+\bar{E}_{i_{2} i_{1} i_{3}}+\bar{E}_{i_{3} i_{2} i_{1}}\right),  \tag{6.14}\\
& E_{00 i_{1}}=\frac{1}{3}\left(\bar{E}_{00 i_{1}}+\bar{E}_{0 i_{1} 0}+\bar{E}_{i_{1} 00}\right) . \tag{6.15}
\end{align*}
$$

In (6.15), $\bar{E}_{00 i_{1}}$ and $\bar{E}_{0 i_{1} 0}$ are determined from (6.13), while $\bar{E}_{i_{1} 00}$ is determined from (6.12).
For $P \geqslant 4$ extra terms of order $(D-4) E_{00 \ldots}$.. have to be added to Eqs. (6.12) and (6.13). For $P=4$ the last-but-one contribution in (6.10) is of $\mathcal{O}(D-4)$, but the last term yields a finite contribution for $D \rightarrow 4$, because the coefficient $E_{000000}$ is UV divergent. We calculate this contribution upon inserting $\left.E^{\alpha \beta \mu_{1} \ldots \mu_{4}}\right|_{\text {div }}=\left.\{g g g\}^{\alpha \beta \mu_{1} \ldots \mu_{4}} E_{000000}\right|_{\text {div }}$ into (6.4) and using $(D-4) E_{000000}$ from (A.5). After symmetrizing in the Lorentz indices, we get

$$
\begin{equation*}
\int \mathcal{E}=-\frac{1}{240} \sum_{n=1}^{4} \tilde{X}_{n 0}^{(4)}\{g g p\}_{n}^{\mu \mu_{1} \ldots \mu_{4}} \tag{6.16}
\end{equation*}
$$

This contribution to the coefficients $E_{i 0000}$ can be included by replacing - $\tilde{X}_{i 0}^{(4)} D_{0000}(0)$ in (6.12) by $-\tilde{X}_{i 0}^{(4)}\left[D_{0000}(0)+\frac{1}{48}\right]$. The cases $P>4$ can be treated analogously, but usually do not appear in renormalizable quantum field theories.

After the symmetrization, we thus find for the tensor coefficients up to rank 5:

$$
\begin{align*}
\operatorname{det}\left(X^{(4)}\right) E_{i_{1}}= & \sum_{n=1}^{4} \tilde{X}_{i_{1} n}^{(4)}\left[D_{0}(n)-D_{0}(0)\right]-\tilde{X}_{i_{1} 0}^{(4)} D_{0}(0),  \tag{6.17}\\
\operatorname{det}\left(X^{(4)}\right) E_{00}= & \sum_{n=1}^{4} \tilde{X}_{n 0}^{(4)}\left[D_{00}(n)-D_{00}(0)\right], \\
2 \operatorname{det}\left(X^{(4)}\right) E_{i_{1} i_{2}}= & \left\{\sum_{n=1}^{4} \tilde{X}_{i_{1} n}^{(4)}\left[D_{\left(i_{2}\right)_{n}}(n) \bar{\delta}_{i_{2} n}-D_{i_{2}}(0)\right]-\tilde{X}_{i_{1} 0}^{(4)} D_{i_{2}}(0)\right. \\
& \left.-2 \sum_{n=1}^{4} \tilde{\tilde{X}}_{\left(i_{1} n\right)\left(0 i_{2}\right)}^{(4)}\left[D_{00}(n)-D_{00}(0)\right]\right\}+\left(i_{1} \leftrightarrow i_{2}\right),  \tag{6.18}\\
3 \operatorname{det}\left(X^{(4)}\right) E_{00 i_{1}=}= & 2 \sum_{n=1}^{4} \tilde{X}_{n 0}^{(4)}\left[D_{00\left(i_{1}\right)_{n}}(n) \bar{\delta}_{i_{1} n}-D_{00 i_{1}}(0)\right] \\
& +\sum_{n=1}^{4} \tilde{X}_{i_{1} n}^{(4)}\left[D_{00}(n)-D_{00}(0)\right]-\tilde{X}_{i_{1} 0}^{(4)} D_{00}(0), \\
3 \operatorname{det}\left(X^{(4)}\right) E_{i_{1} i_{2} i_{3}}= & \left\{\sum_{n=1}^{4} \tilde{X}_{i_{1} n}^{(4)}\left[D_{\left(i_{2}\right)_{n}\left(i_{3}\right)_{n}}(n) \bar{\delta}_{i_{2} n} \bar{\delta}_{i_{3} n}-D_{i_{2} i_{3}}(0)\right]-\tilde{X}_{i_{1} 0}^{(4)} D_{i_{2} i_{3}}(0)\right.
\end{align*}
$$

$$
\begin{align*}
& -2 \sum_{n=1}^{4} \tilde{\tilde{X}}_{\left(i_{1} n\right)\left(0 i_{2}\right)}^{(4)}\left[D_{00\left(i_{3}\right)_{n}}(n) \bar{\delta}_{i_{3} n}-D_{00 i_{3}}(0)\right] \\
& \left.-2 \sum_{n=1}^{4} \tilde{\tilde{X}}_{\left(i_{1} n\right)\left(0 i_{3}\right)}^{(4)}\left[D_{00\left(i_{2}\right)_{n}}(n) \bar{\delta}_{i_{2} n}-D_{00 i_{2}}(0)\right]\right\} \\
& +\left(i_{1} \leftrightarrow i_{2}\right)+\left(i_{1} \leftrightarrow i_{3}\right),  \tag{6.19}\\
& \operatorname{det}\left(X^{(4)}\right) E_{0000}=\sum_{n=1}^{4} \tilde{X}_{n 0}^{(4)}\left[D_{0000}(n)-D_{0000}(0)\right], \\
& 4 \operatorname{det}\left(X^{(4)}\right) E_{00 i_{1} i_{2}}=2 \sum_{n=1}^{4} \tilde{X}_{n 0}^{(4)}\left[D_{00\left(i_{1}\right)_{n}\left(i_{2}\right)_{n}}(n) \bar{\delta}_{i_{1} n} \bar{\delta}_{i_{2} n}-D_{00 i_{1} i_{2}}(0)\right] \\
& +\left\{\sum_{n=1}^{4} \tilde{X}_{i_{1} n}^{(4)}\left[D_{00\left(i_{2}\right)_{n}}(n) \bar{\delta}_{i_{2} n}-D_{00 i_{2}}(0)\right]-\tilde{X}_{i_{1} 0}^{(4)} D_{00 i_{2}}(0)\right. \\
& \left.-2 \sum_{n=1}^{4} \tilde{\tilde{X}}_{\left(i_{11} n\right)\left(0 i_{2}\right)}^{(4)}\left[D_{0000}(n)-D_{0000}(0)\right]+\left(i_{1} \leftrightarrow i_{2}\right)\right\}, \\
& 4 \operatorname{det}\left(X^{(4)}\right) E_{i_{1} i_{2} i_{3} i_{4}}=\left\{\sum_{n=1}^{4} \tilde{X}_{i_{1} n}^{(4)}\left[D_{\left(i_{2}\right)_{n}\left(i_{3}\right)_{n}\left(i_{4}\right)_{n}}(n) \bar{\delta}_{i_{2} n} \bar{\delta}_{i_{3} n} \bar{\delta}_{i_{4} n}-D_{i_{2} i_{3} i_{4}}(0)\right]\right. \\
& -\tilde{X}_{i_{1} 0}^{(4)} D_{i_{2} i_{3} i_{4}}(0) \\
& -2 \sum_{n=1}^{4} \tilde{\tilde{X}}_{\left(i_{1} n\right)\left(0 i_{2}\right)}^{(4)}\left[D_{00\left(i_{3}\right)_{n}\left(i_{4}\right)_{n}}(n) \bar{\delta}_{i_{3} n} \bar{\delta}_{i_{4} n}-D_{00 i_{3} i_{4}}(0)\right] \\
& -2 \sum_{n=1}^{4} \tilde{\tilde{X}}_{\left(i_{1} n\right)\left(0 i_{3}\right)}^{(4)}\left[D_{00\left(i_{2}\right)_{n}\left(i_{4}\right)_{n}}(n) \bar{\delta}_{i_{2} n} \bar{\delta}_{i_{4} n}-D_{00 i_{2} i_{4}}(0)\right] \\
& \left.-2 \sum_{n=1}^{4} \tilde{\tilde{X}}_{\left(i_{1} n\right)\left(0 i_{4}\right)}^{(4)}\left[D_{00\left(i_{2}\right)_{n}\left(i_{3}\right)_{n}}(n) \bar{\delta}_{i_{2} n} \bar{\delta}_{i_{3} n}-D_{00 i_{2} i_{3}}(0)\right]\right\} \\
& +\left(i_{1} \leftrightarrow i_{2}\right)+\left(i_{1} \leftrightarrow i_{3}\right)+\left(i_{1} \leftrightarrow i_{4}\right),  \tag{6.20}\\
& 5 \operatorname{det}\left(X^{(4)}\right) E_{0000 i_{1}}=4 \sum_{n=1}^{4} \tilde{X}_{n 0}^{(4)}\left[D_{0000\left(i_{1}\right)_{n}}(n) \bar{\delta}_{i_{1} n}-D_{0000 i_{1}}(0)\right] \\
& +\sum_{n=1}^{4} \tilde{X}_{i_{1} n}^{(4)}\left[D_{0000}(n)-D_{0000}(0)\right]-\tilde{X}_{i_{1} 0}^{(4)}\left[D_{0000}(0)+\frac{1}{48}\right] \text {, } \\
& 5 \operatorname{det}\left(X^{(4)}\right) E_{00 i_{1} i_{2} i_{3}}=2 \sum_{n=1}^{4} \tilde{X}_{n 0}^{(4)}\left[D_{00\left(i_{1}\right)_{n}\left(i_{2}\right)_{n}\left(i_{3}\right)_{n}}(n) \bar{\delta}_{i_{1} n} \bar{\delta}_{i_{2} n} \bar{\delta}_{i_{3} n}-D_{00 i_{1} i_{2} i_{3}}(0)\right] \\
& +\left\{\sum_{n=1}^{4} \tilde{X}_{i_{1} n}^{(4)}\left[D_{00\left(i_{2}\right)_{n}\left(i_{3}\right)_{n}}(n) \bar{\delta}_{i_{2} n} \bar{\delta}_{i_{3} n}-D_{00 i_{2} i_{3}}(0)\right]-\tilde{X}_{i_{1} 0}^{(4)} D_{00 i_{2} i_{3}}(0)\right.
\end{align*}
$$

$$
\begin{align*}
& -2 \sum_{n=1}^{4} \tilde{\tilde{X}}_{\left(i_{1} n\right)\left(0 i_{2}\right)}^{(4)}\left[D_{0000\left(i_{3}\right)_{n}}(n) \bar{\delta}_{i_{3} n}-D_{0000 i_{3}}(0)\right] \\
& -2 \sum_{n=1}^{4} \tilde{\tilde{X}}_{\left(i_{1} n\right)\left(0 i_{3}\right)}^{(4)}\left[D_{0000\left(i_{2}\right)_{n}}(n) \bar{\delta}_{i_{2} n}-D_{0000 i_{2}}(0)\right] \\
& \left.+\left(i_{1} \leftrightarrow i_{2}\right)+\left(i_{1} \leftrightarrow i_{3}\right)\right\}, \\
& 5 \operatorname{det}\left(X^{(4)}\right) E_{i_{1} i_{2} i_{3} i_{4} i_{5}}=\left\{\sum_{n=1}^{4} \tilde{X}_{i_{1} n}^{(4)}\left[D_{\left(i_{2}\right)_{n}\left(i_{3}\right)_{n}\left(i_{4}\right)_{n}\left(i_{5}\right)_{n}}(n) \bar{\delta}_{i_{2} n} \bar{\delta}_{i_{3} n} \bar{\delta}_{i_{4} n} \bar{\delta}_{i_{5} n}-D_{i_{2} i_{3} i_{4} i_{5}}(0)\right]\right. \\
& -\tilde{X}_{i_{1} 0}^{(4)} D_{i_{2} i_{3} i_{4} i_{5}}(0) \\
& -2 \sum_{n=1}^{4} \tilde{\tilde{X}}_{\left(i_{1} n\right)\left(0 i_{2}\right)}^{(4)}\left[D_{00\left(i_{3}\right)_{n}\left(i_{4}\right)_{n}\left(i_{5}\right)_{n}}(n) \bar{\delta}_{i_{3} n} \bar{\delta}_{i_{4} n} \bar{\delta}_{i_{5} n}-D_{00 i_{3} i_{4} i_{5}}(0)\right] \\
& -2 \sum_{n=1}^{4} \tilde{\tilde{X}}_{\left(i_{1} n\right)\left(0 i_{3}\right)}^{(4)}\left[D_{00\left(i_{2}\right)_{n}\left(i_{4}\right)_{n}\left(i_{5}\right)_{n}}(n) \bar{\delta}_{i_{2} n} \bar{\delta}_{i_{4} n} \bar{\delta}_{i_{5} n}-D_{00 i_{2} i_{4} i_{5}}(0)\right] \\
& -2 \sum_{n=1}^{4} \tilde{\tilde{X}}_{\left(i_{1} n\right)\left(0 i_{4}\right)}^{(4)}\left[D_{00\left(i_{2}\right)_{n}\left(i_{3}\right)_{n}\left(i_{5}\right)_{n}}(n) \bar{\delta}_{i_{2} n} \bar{\delta}_{i_{3} n} \bar{\delta}_{i_{5} n}-D_{00 i_{2} i_{3} i_{5}}(0)\right] \\
& \left.-2 \sum_{n=1}^{4} \tilde{\tilde{X}}_{\left(i_{1} n\right)\left(0 i_{5}\right)}^{(4)}\left[D_{00\left(i_{2}\right)_{n}\left(i_{3}\right)_{n}\left(i_{4}\right)_{n}}(n) \bar{\delta}_{i_{2} n} \bar{\delta}_{i_{3} n} \bar{\delta}_{i_{4} n}-D_{00 i_{2} i_{3} i_{4}}(0)\right]\right\} \\
& +\left(i_{1} \leftrightarrow i_{2}\right)+\left(i_{1} \leftrightarrow i_{3}\right)+\left(i_{1} \leftrightarrow i_{4}\right)+\left(i_{1} \leftrightarrow i_{5}\right) . \tag{6.21}
\end{align*}
$$

For the 4 -point tensor coefficients that result from omitting $N_{0}$ in the 5 -point integrals, we have introduced the auxiliary quantities

$$
\begin{align*}
& D_{i_{1}}(0)=\tilde{D}_{i_{1}-1}(0), \quad i_{1}=2,3,4, \\
& D_{1}(0)=-\sum_{n=2}^{4} D_{n}(0)-D_{0}(0),  \tag{6.22}\\
& D_{i_{1} i_{2}}(0)=\tilde{D}_{i_{1}-1, i_{2}-1}(0), \quad i_{1}, i_{2}=2,3,4, \\
& D_{1 i_{1}}(0)=-\sum_{n=2}^{4} D_{n i_{1}}(0)-D_{i_{1}}(0), \quad i_{1}=1, \ldots, 4,  \tag{6.23}\\
& D_{i_{1} i_{2} i_{3}}(0)=\tilde{D}_{i_{1}-1, i_{2}-1, i_{3}-1}(0), \quad i_{1}, i_{2}, i_{3}=2,3,4, \\
& D_{1 i_{1} i_{2}}(0)=-\sum_{n=2}^{4} D_{n i_{1} i_{2}}(0)-D_{i_{1} i_{2}}(0), \quad i_{1}, i_{2}=1, \ldots, 4,  \tag{6.24}\\
& D_{i_{1} i_{2} i_{3} i_{4}}(0)=\tilde{D}_{i_{1}-1, i_{2}-1, i_{3}-1, i_{4}-1}(0), \quad i_{1}, i_{2}, i_{3}, i_{4}=2,3,4, \\
& D_{1 i_{1} i_{2} i_{3}}(0)=-\sum_{n=2}^{4} D_{n i_{1} i_{2} i_{3}}(0)-D_{i_{1} i_{2} i_{3}}(0), \quad i_{1}, i_{2}, i_{3}=1, \ldots, 4, \tag{6.25}
\end{align*}
$$

and similar quantities resulting from these relations with index pairs " 00 " added to the $D_{\text {.... (0) }}$ functions on both sides.

## 7. Reduction of 6-point integrals

Following the guideline of the reduction of the scalar 6-point integral to six scalar 5-point integrals [22], the 6-point tensor integrals of rank $P$ can be reduced to six 5-point tensor integrals of rank $P$ as described in Ref. [24]. This method, which was used in the calculation of one-loop corrections to $\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow 4 f$ [35], is more explicitly worked out in Appendix D.

In the following we describe a method that reduces 6-point tensor integrals of rank $P$ to 5-point tensor integrals of rank $(P-1)$. The scalar 6-point integral should be treated following Refs. [22,35] as explicitly described in Appendix D. The tensor reduction can be derived by considering the determinant

$$
\mathcal{F}=\left|\begin{array}{cccc}
q^{\mu} & 2 q p_{1} & \cdots & 2 q p_{5}  \tag{7.1}\\
p_{1}^{\mu} & 2 p_{1} p_{1} & \cdots & 2 p_{1} p_{5} \\
\vdots & \vdots & \ddots & \vdots \\
p_{k-1}^{\mu} & 2 p_{k-1} p_{1} & \cdots & 2 p_{k-1} p_{5} \\
0 & f_{1} & \cdots & f_{5} \\
p_{k+1}^{\mu} & 2 p_{k+1} p_{1} & \cdots & 2 p_{k+1} p_{5} \\
\vdots & \vdots & \ddots & \vdots \\
p_{5}^{\mu} & 2 p_{5} p_{1} & \cdots & 2 p_{5} p_{5}
\end{array}\right|=\left|\begin{array}{cccc}
q^{\mu} & N_{1}-N_{0} & \cdots & N_{5}-N_{0} \\
p_{1}^{\mu} & 2 p_{1} p_{1} & \cdots & 2 p_{1} p_{5} \\
\vdots & \vdots & \ddots & \vdots \\
p_{k-1}^{\mu} & 2 p_{k-1} p_{1} & \cdots & 2 p_{k-1} p_{5} \\
0 & f_{1} & \cdots & f_{5} \\
p_{k+1}^{\mu} & 2 p_{k+1} p_{1} & \cdots & 2 p_{k+1} p_{5} \\
\vdots & \vdots & \ddots & \vdots \\
p_{5}^{\mu} & 2 p_{5} p_{1} & \cdots & 2 p_{5} p_{5}
\end{array}\right| .
$$

The r.h.s. is obtained by adding the $(k+1)$ th row to the first row and using (5.4).
In four dimensions, this determinant vanishes, as can be seen from the first form in (7.1), because $q$ is linearly dependent on the four (non-exceptional) momenta $p_{i}, i=1, \ldots, 5, i \neq k$. We again do not use this fact, but translate the integral over $\mathcal{F}$ into a form that has a factor of $\mathcal{O}(D-4)$ rendering the whole contribution zero for finite integrals. Inserting (7.1) into the integrand of the tensor integral $F^{\mu_{1} \ldots \mu_{P}}$ results in

$$
\begin{align*}
\int \mathcal{F} & \equiv \frac{(2 \pi \mu)^{4-D}}{\mathrm{i} \pi^{2}} \int \mathrm{~d}^{D} q \frac{q^{\mu_{1}} \cdots q^{\mu_{P}}}{N_{0} N_{1} \cdots N_{5}} \mathcal{F} \\
& =F^{\alpha \mu_{1} \ldots \mu_{P}}\left|\begin{array}{cccc}
g_{\alpha}^{\mu} & 2 p_{1 \alpha} & \cdots & 2 p_{5 \alpha} \\
p_{1}^{\mu} & 2 p_{1} p_{1} & \cdots & 2 p_{1} p_{5} \\
\vdots & \vdots & \ddots & \vdots \\
p_{k-1}^{\mu} & 2 p_{k-1} p_{1} & \cdots & 2 p_{k-1} p_{5} \\
0 & f_{1} & \cdots & f_{5} \\
p_{k+1}^{\mu} & 2 p_{k+1} p_{1} & \cdots & 2 p_{k+1} p_{5} \\
\vdots & \vdots & \ddots & \vdots \\
p_{5}^{\mu} & 2 p_{5} p_{1} & \cdots & 2 p_{5} p_{5}
\end{array}\right| . \tag{7.2}
\end{align*}
$$

We expand the determinant along the $(k+1)$ th row and use the fact that the four-dimensional metric tensor can be written as

$$
g_{(4)}^{\mu \nu}\left|\begin{array}{ccc}
2 k_{1} p_{1} & \cdots & 2 k_{1} p_{4}  \tag{7.3}\\
\vdots & \ddots & \vdots \\
2 k_{4} p_{1} & \cdots & 2 k_{4} p_{4}
\end{array}\right|=-\left|\begin{array}{cccc}
0 & 2 p_{1}^{v} & \cdots & 2 p_{4}^{v} \\
k_{1}^{\mu} & 2 k_{1} p_{1} & \cdots & 2 k_{1} p_{4} \\
\vdots & \vdots & \ddots & \vdots \\
k_{4}^{\mu} & 2 k_{4} p_{1} & \cdots & 2 k_{4} p_{4}
\end{array}\right|
$$

for two arbitrary sets of linear independent momenta $p_{1}, p_{2}, p_{3}, p_{4}$ and $k_{1}, k_{2}, k_{3}, k_{4}$. This yields

$$
\begin{equation*}
\int \mathcal{F}=-\tilde{X}_{k 0}^{(5)} F^{\alpha \mu_{1} \ldots \mu_{P}}\left(g_{\alpha}^{\mu}-g_{(4)}^{\mu}\right) \tag{7.4}
\end{equation*}
$$

Inserting the r.h.s. of (7.1) into the integrand of the tensor integral $F^{\mu_{1} \ldots \mu_{P}}$ results in

$$
\int \mathcal{F}=\left|\begin{array}{cccc}
F^{\mu \mu_{1} \ldots \mu_{P}} & E^{\mu_{1} \ldots \mu_{P}}(1)-E^{\mu_{1} \ldots \mu_{P}}(0) & \cdots & E^{\mu_{1} \ldots \mu_{P}}(5)-E^{\mu_{1} \ldots \mu_{P}}(0)  \tag{7.5}\\
p_{1}^{\mu} & 2 p_{1} p_{1} & \cdots & 2 p_{1} p_{5} \\
\vdots & \vdots & \ddots & \vdots \\
p_{k-1}^{\mu} & 2 p_{k-1} p_{1} & \cdots & 2 p_{k-1} p_{5} \\
0 & f_{1} & \cdots & f_{5} \\
p_{k+1}^{\mu} & 2 p_{k+1} p_{1} & \cdots & 2 p_{k+1} p_{5} \\
\vdots & \vdots & \ddots & \vdots \\
p_{5}^{\mu} & 2 p_{5} p_{1} & \cdots & 2 p_{5} p_{5}
\end{array}\right| .
$$

Expanding the determinant along the first row and the first column according to the analogue of (2.22), yields

$$
\begin{equation*}
\int \mathcal{F}=-\tilde{X}_{k 0}^{(5)} F^{\mu \mu_{1} \ldots \mu_{P}}-\sum_{n, m=1}^{5} \tilde{\tilde{X}}_{(k m)(0 n)}^{(5)} p_{m}^{\mu}\left[E^{\mu_{1} \ldots \mu_{P}}(n)-E^{\mu_{1} \ldots \mu_{P}}(0)\right] \tag{7.6}
\end{equation*}
$$

where $\tilde{\tilde{X}}_{(k m)(0 n)}^{(5)}$ is given in (2.29).
From (7.4) and (7.6) we obtain

$$
\begin{align*}
\tilde{X}_{k 0}^{(5)} F^{\mu \mu_{1} \ldots \mu_{P}}= & -\sum_{n, m=1}^{5} \tilde{\tilde{X}}_{(k m)(0 n)}^{(5)} p_{m}^{\mu}\left[E^{\mu_{1} \ldots \mu_{P}}(n)-E^{\mu_{1} \ldots \mu_{P}}(0)\right] \\
& +\tilde{X}_{k 0}^{(5)} F^{\alpha \mu_{1} \ldots \mu_{P}}\left(g_{\alpha}^{\mu}-g_{(4)}{ }_{\alpha}^{\mu}\right) \tag{7.7}
\end{align*}
$$

The last term in (7.7) only contributes in dimensional regularization if $F^{\alpha \mu_{1} \ldots \mu_{P}}$ is singular. For UV singularities this is the case if $P \geqslant 7$, which is usually not needed in renormalizable theories. As explained in Section 5.8, IR (soft and collinear) singularities of $F^{\alpha \mu_{1} \ldots \mu_{P}}$ only appear in contributions that are proportional to a momentum $p_{i}^{\alpha}$. These contributions vanish exactly in (7.7). Therefore, the terms involving $\left(g-g_{(4)}\right)$ in (7.7) can be omitted for $P<7 .{ }^{8}$ For $P \geqslant 7$ the inverse determinant that is implicitly contained in $g_{(4)}$ can always be cancelled. ${ }^{9}$

[^8]Introducing the matrix

$$
M_{(k)}=\left(\begin{array}{ccc}
2 p_{1} p_{1} & \cdots & 2 p_{1} p_{5}  \tag{7.8}\\
\vdots & \ddots & \vdots \\
2 p_{k-1} p_{1} & \cdots & 2 p_{k-1} p_{5} \\
f_{1} & \cdots & f_{5} \\
2 p_{k+1} p_{1} & \cdots & 2 p_{k+1} p_{5} \\
\vdots & \ddots & \vdots \\
2 p_{5} p_{1} & \cdots & 2 p_{5} p_{5}
\end{array}\right)
$$

(7.7) can be written as

$$
\begin{align*}
F^{\mu \mu_{1} \ldots \mu_{P}}= & \sum_{n=1}^{5} \sum_{\substack{m=1 \\
m \neq k}}^{5}\left(M_{(k)}^{-1}\right)_{n m} p_{m}^{\mu}\left[E^{\mu_{1} \ldots \mu_{P}}(n)-E^{\mu_{1} \ldots \mu_{P}}(0)\right] \\
& +F^{\alpha \mu_{1} \ldots \mu_{P}}\left(g_{\alpha}^{\mu}-g_{(4)}^{\mu}\right) \tag{7.9}
\end{align*}
$$

which expresses the 6-point tensor integral of rank $P$ in terms of six 5-point tensor integrals of rank $(P-1)$. The inverse of $M_{(k)}$ is given by

$$
\begin{equation*}
\left(M_{(k)}^{-1}\right)_{i j}=-\tilde{\tilde{X}}_{(k j)(0 i)}^{(5)} / \tilde{X}_{k 0}^{(5)}, \quad i, j, k=1, \ldots, N \tag{7.10}
\end{equation*}
$$

In the form (7.9) our result can easily be extended to the reduction of $N$-point functions with $N>6$ by simply forming a matrix similar to $M_{(k)}$ by selecting five momenta for the columns and four momenta for the rows out of the $(N-1)$ available momenta of the $N$-point function.

Eq. (7.7) can also be used to derive an alternative reduction of tensor 6-point integrals. Multiplying it with $X_{k 0}$, summing over $k=1, \ldots, N$, and using (2.30), for $\tilde{\tilde{X}}_{(k m)(0 n)}^{(5)}$ yields

$$
\begin{align*}
\operatorname{det}\left(X^{(5)}\right) F^{\mu \mu_{1} \ldots \mu_{P}}= & \sum_{n, m=1}^{5} \tilde{X}_{n m}^{(5)} p_{m}^{\mu}\left[E^{\mu_{1} \ldots \mu_{P}}(n)-E^{\mu_{1} \ldots \mu_{P}}(0)\right] \\
& +\operatorname{det}\left(X^{(5)}\right) F^{\alpha \mu_{1} \ldots \mu_{P}}\left(g_{\alpha}^{\mu}-g_{(4)}^{\alpha}\right) \tag{7.11}
\end{align*}
$$

Here, as in (6.10), all inverse Gram determinants have been absorbed in the four-dimensional metric tensor, which appears only in the difference $\left(g-g_{(4)}\right)$. The result (7.11) is equivalent to Eq. (64) of Ref. [19].

Finally, we insert the decompositions of tensor 6-point integrals into Lorentz covariants in order to derive explicit reduction formulas for the tensor coefficients. Since we consider only tensors up to rank 3, we can omit the terms involving $\left(g-g_{(4)}\right)$. The tensor decompositions explicitly read

$$
\begin{align*}
& F^{\mu}=\sum_{i_{1}=1}^{5} p_{i_{1}}^{\mu} F_{i_{1}}, \quad F^{\mu \nu}=\sum_{i_{1}, i_{2}=1}^{5} p_{i_{1}}^{\mu} p_{i_{2}}^{\nu} F_{i_{1} i_{2}}+g^{\mu \nu} F_{00}, \\
& F^{\mu \nu \rho}=\sum_{i_{1}, i_{2}, i_{3}=1}^{5} p_{i_{1}}^{\mu} p_{i_{2}}^{\nu} p_{i_{3}}^{\rho} F_{i_{1} i_{2} i_{3}}+\sum_{i_{1}=1}^{5}\{g p\}_{i_{1}}^{\mu \nu \rho} F_{00 i_{1}} . \tag{7.12}
\end{align*}
$$

In four dimensions, some covariants in these decompositions are redundant in the sense that they can be expressed by the others. For instance, in the decomposition of $F^{\mu}$ one of the five covariants $p_{i_{1}}^{\mu} F_{i_{1}}$ is redundant, because one of the momenta $p_{i_{1}}$ can be expressed by the other four linearly independent vectors. Similarly, all covariants involving metric tensors are redundant. However, by keeping these coefficients we can avoid the appearance of explicit inverse Gram determinants in the reduction formulas.

Inserting the Lorentz decompositions of the tensor integrals in the reduction formulas given above, we can read off the reduction formulas for the tensor coefficients upon comparing coefficients of covariants on both sides. Generically we find

$$
\begin{equation*}
\bar{F}_{j i_{1} \ldots i_{P}}=\sum_{n=1}^{5} c_{j n}\left[E_{\left(i_{1}\right)_{n} \ldots\left(i_{P}\right)_{n}}(n) \bar{\delta}_{i_{1} n} \ldots \bar{\delta}_{i_{P} n}-E_{i_{1} \ldots i_{P}}(0)\right], \quad P<7, \tag{7.13}
\end{equation*}
$$

with

$$
\begin{equation*}
c_{0 n}=c_{k n}=0, \quad c_{j n}=\left(M_{(k)}^{-1}\right)_{n j}, \quad j, n=1, \ldots, 5, \quad j \neq k \tag{7.14}
\end{equation*}
$$

for the reduction given in (7.9) and with

$$
\begin{equation*}
c_{0 n}=0, \quad c_{j n}=\tilde{X}_{n j}^{(5)} / \operatorname{det}\left(X^{(5)}\right)=\left(X^{(5)}\right)_{j n}^{-1}, \quad j, n=1, \ldots, 5 \tag{7.15}
\end{equation*}
$$

for the reduction given in (7.11). In the numerical reduction we can select the equation that is numerically most stable. For example, in (7.14) we can choose $k$ such that the modulus of $\tilde{X}_{k 0}^{(5)}=-\operatorname{det} M_{(k)}$ is maximal.

Since we have distinguished one momentum in the derivation of (7.9) the resulting tensor coefficients $\bar{F}$... are not symmetric under the exchange of $j$ with one of the indices $i_{r}$. This can be easily cured as in the case of 5-point functions [see (6.14) and (6.15)] by adding all $P$ results with $j$ exchanged with one of the $i_{r}$, and dividing the sum by $P$.

Thus, we find from (7.13) for the tensor coefficients up to rank 3

$$
\begin{align*}
& F_{i_{1}}=\sum_{n=1}^{5} c_{i_{1} n}\left[E_{0}(n)-E_{0}(0)\right], \quad i_{1}=1, \ldots, 5,  \tag{7.16}\\
& F_{00}=0, \\
& F_{i_{1} i_{2}}=\frac{1}{2} \sum_{n=1}^{5}\left\{c_{i_{1} n}\left[E_{\left(i_{2}\right)_{n}}(n) \bar{\delta}_{i_{2} n}-E_{i_{2}}(0)\right]+\left(i_{1} \leftrightarrow i_{2}\right)\right\}, \quad i_{1}, i_{2}=1, \ldots, 5,  \tag{7.17}\\
& F_{00 i_{1}}=\frac{1}{3} \sum_{n=1}^{5} c_{i_{1} n}\left[E_{00}(n)-E_{00}(0)\right], \\
& F_{i_{1} i_{2} i_{3}}=\frac{1}{3} \sum_{n=1}^{5}\left\{c_{i_{1} n}\left[E_{\left(i_{2}\right)_{n}\left(i_{3}\right)_{n}}(n) \bar{\delta}_{i_{2} n} \bar{\delta}_{i_{3} n}-E_{i_{2} i_{3}}(0)\right]+\left(i_{1} \leftrightarrow i_{2}\right)+\left(i_{1} \leftrightarrow i_{3}\right)\right\}, \\
& i_{1}, i_{2}, i_{3}=1, \ldots, 5 . \tag{7.18}
\end{align*}
$$

For the 5-point tensor coefficients that result from omitting $N_{0}$ in the 6-point integrals, we have again used the auxiliary quantities

$$
E_{i_{1}}(0)=\tilde{E}_{i_{1}-1}(0), \quad i_{1}=2, \ldots, 5
$$

$$
\begin{align*}
& E_{1}(0)=-\sum_{n=2}^{5} E_{n}(0)-E_{0}(0),  \tag{7.19}\\
& E_{i_{1} i_{2}}(0)=\tilde{E}_{i_{1}-1, i_{2}-1}(0), \quad i_{1}, i_{2}=2, \ldots, 5, \\
& E_{1 i_{1}}(0)=-\sum_{n=2}^{5} E_{n i_{1}}(0)-E_{i_{1}}(0), \quad i_{1}=1, \ldots, 5,  \tag{7.20}\\
& E_{i_{1} i_{2} i_{3}}(0)=\tilde{E}_{i_{1}-1, i_{2}-1, i_{3}-1}(0), \quad i_{1}, i_{2}, i_{3}=2, \ldots, 5, \\
& E_{1 i_{1} i_{2}}(0)=-\sum_{n=2}^{5} E_{n i_{1} i_{2}}(0)-E_{i_{1} i_{2}}(0), \quad i_{1}, i_{2}=1, \ldots, 5 . \tag{7.21}
\end{align*}
$$

## 8. Summary

Methods for a systematic evaluation of one-loop tensor integrals have been described for graphs with up to six external legs. The results are presented in a form that can be directly translated into a computer code; only the scalar 3- and 4-point integrals have to be taken from elsewhere.

While UV divergences are treated in dimensional regularization, possible IR (soft or collinear) divergences can be regularized either dimensionally or with small mass parameters; the described results are valid in either IR regularization scheme. Moreover, the results hold if internal masses are complex parameters, which naturally appear for unstable internal particles. The generalization of the proposed methods to functions with more than six external lines is straightforward.

Particular attention is paid to the issue of numerical stability. For 1- and 2-point integrals of arbitrary tensor rank, general numerically stable results are presented. For 3- and 4-point tensor integrals, serious numerical instabilities are known to arise in the frequently used PassarinoVeltman reduction if Gram determinants built of external momenta become small. For these cases we have developed dedicated reduction techniques. One of the techniques replaces the standard scalar integral by a specific tensor coefficient that can be safely evaluated numerically and reduces the remaining tensor coefficients as well as the standard scalar integral to the new set of basis integrals. In this scheme no dangerous inverse Gram determinants occur, but inverse modified Cayley determinants instead. In a second class of techniques we keep the basis set of standard scalar integrals and iteratively deduce the tensor coefficients up to terms that are systematically suppressed by small Gram determinants or by other kinematical determinants in specific kinematical configurations. The convergence of the iteration can be systematically improved upon including higher tensor ranks. For 5- and 6-point tensor integrals, we describe reductions to 5-and 4-point integrals, respectively, that do not involve inverse Gram determinants either. Compared to some other existing methods, the described methods are distinguished by the fact that the reduction from 6-(5-) to 5- (4-)point integrals decreases the tensor rank at the same time.

We finally emphasize that the presented methods have already been successfully applied in the calculation of a complete one-loop correction to a $2 \rightarrow 4$ scattering reaction, viz. the electroweak corrections to the charged-current processes $\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow 4 f$. The described methods, thus, have proven their reliability in practice and will certainly be used in future loop calculations for interesting many-particle production processes at the LHC and ILC.

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## Appendix A. UV-divergent parts of tensor integrals

In the reduction formulas given above, products of $(D-4)$ with tensor integrals appear. These give rise to finite terms originating from UV singularities in the loop integrals. As mentioned above, no IR-singular integrals multiplied with $(D-4)$ appear in the reduction formulas. The UV-singular parts of the loop integrals can be derived easily from the Feynman-parameter representation or by using (5.10) for these parts only. In the following, we list results for ( $D-4$ ) times one-loop integrals omitting terms of order $\mathcal{O}(D-4)$. For the 1-point functions $A_{\ldots}$ ( $m_{0}$ ) we get

$$
\begin{equation*}
(D-4) A_{0}=-2 m_{0}^{2}, \quad(D-4) A_{2 n}^{0 \ldots 0}=-\frac{m_{0}^{2 n+2}}{2^{n-1}(n+1)!}, \quad n=1,2, \ldots \tag{A.1}
\end{equation*}
$$

For the IR-finite 2-point functions $B_{\ldots}\left(p_{1}, m_{0}, m_{1}\right)$, i.e. excluding the case $p_{1}^{2}=m_{0}^{2}=m_{1}^{2}=0$, we obtain

$$
\begin{align*}
& (D-4) B_{0}=-2, \quad(D-4) B_{1}=1, \\
& (D-4) B_{00}=\frac{1}{6}\left(p_{1}^{2}-3 m_{0}^{2}-3 m_{1}^{2}\right), \quad(D-4) B_{11}=-\frac{2}{3}, \\
& (D-4) B_{001}=-\frac{1}{12}\left(p_{1}^{2}-2 m_{0}^{2}-4 m_{1}^{2}\right), \quad(D-4) B_{111}=\frac{1}{2}, \\
& (D-4) B_{0000}=-\frac{1}{120}\left[p_{1}^{4}-5 p_{1}^{2}\left(m_{0}^{2}+m_{1}^{2}\right)+10\left(m_{0}^{4}+m_{0}^{2} m_{1}^{2}+m_{1}^{4}\right)\right] \\
& (D-4) B_{0011}=\frac{1}{60}\left(3 p_{1}^{2}-5 m_{0}^{2}-15 m_{1}^{2}\right), \quad(D-4) B_{1111}=-\frac{2}{5} \\
& (D-4) B_{00001}=\frac{1}{240}\left[p_{1}^{4}-4 p_{1}^{2} m_{0}^{2}-6 p_{1}^{2} m_{1}^{2}+5 m_{0}^{4}+10 m_{0}^{2} m_{1}^{2}+15 m_{1}^{4}\right] \\
& (D-4) B_{00111}=-\frac{1}{60}\left(2 p_{1}^{2}-3 m_{0}^{2}-12 m_{1}^{2}\right), \quad(D-4) B_{11111}=\frac{1}{3} \tag{A.2}
\end{align*}
$$

For the 3-point functions $C \ldots\left(p_{1}, p_{2}, m_{0}, m_{1}, m_{2}\right)$ we obtain, denoting $\left(p_{1}-p_{2}\right)^{2}=s_{12}$,

$$
\begin{aligned}
& (D-4) C_{00}=-\frac{1}{2}, \quad(D-4) C_{00 i}=\frac{1}{6}, \\
& (D-4) C_{0000}=\frac{1}{48}\left[s_{12}+p_{1}^{2}+p_{2}^{2}\right]-\frac{1}{12}\left(m_{0}^{2}+m_{1}^{2}+m_{2}^{2}\right), \\
& (D-4) C_{00 i i}=-\frac{1}{12}, \quad(D-4) C_{00 i j}=-\frac{1}{24}, \\
& (D-4) C_{0000 i}=-\frac{1}{240}\left[2 s_{12}-5 m_{0}^{2}+\sum_{n=1}^{2}\left(p_{n}^{2}-5 m_{n}^{2}\right)\left(1+\delta_{i n}\right)\right],
\end{aligned}
$$

$$
\begin{align*}
& (D-4) C_{00 i i i}=\frac{1}{20}, \quad(D-4) C_{00 i i j}=\frac{1}{60}, \\
& \text { ( } D-4 \text { ) } C_{000000} \\
& =-\frac{1}{2880}\left[2 s_{12}^{2}-6 s_{12} m_{0}^{2}+30 m_{0}^{4}+2 s_{12} \sum_{n=1}^{2}\left(p_{n}^{2}-6 m_{n}^{2}\right)-6 m_{0}^{2} \sum_{n=1}^{2}\left(2 p_{n}^{2}-5 m_{n}^{2}\right)\right. \\
& \left.+\sum_{m, n=1}^{2}\left(p_{m}^{2} p_{n}^{2}-6 p_{m}^{2} m_{n}^{2}+15 m_{m}^{2} m_{n}^{2}\right)\left(1+\delta_{m n}\right)\right] \text {, } \\
& (D-4) C_{0000 i i}=\frac{1}{720}\left[3 s_{12}-6 m_{0}^{2}+\sum_{n=1}^{2}\left(p_{n}^{2}-6 m_{n}^{2}\right)\left(1+2 \delta_{i n}\right)\right] \text {, } \\
& (D-4) C_{0000 i j}=\frac{1}{720}\left[2 s_{12}-3 m_{0}^{2}+\sum_{n=1}^{2}\left(p_{n}^{2}-6 m_{n}^{2}\right)\right] \text {, } \\
& (D-4) C_{00 i i i i}=-\frac{1}{30}, \quad(D-4) C_{00 i i i j}=-\frac{1}{120}, \quad(D-4) C_{00 i i j j}=-\frac{1}{180}, \\
& (D-4) C_{000000 i}=\frac{1}{10080}\left[3 s_{12}^{2}-7 s_{12} m_{0}^{2}+21 m_{0}^{4}+s_{12} \sum_{n=1}^{2}\left(p_{n}^{2}-7 m_{n}^{2}\right)\left(2+\delta_{i n}\right)\right. \\
& -7 m_{0}^{2} \sum_{n=1}^{2}\left(p_{n}^{2}-3 m_{n}^{2}\right)\left(1+\delta_{i n}\right) \\
& \left.+\sum_{m, n=1}^{2}\left(p_{m}^{2} p_{n}^{2}-7 p_{m}^{2} m_{n}^{2}+21 m_{m}^{2} m_{n}^{2}\right)\left(1+2 \delta_{i m} \delta_{i n}\right)\right] \text {, } \\
& (D-4) C_{0000 i i i}=-\frac{1}{1680}\left[4 s_{12}-7 m_{0}^{2}+\sum_{n=1}^{2}\left(p_{n}^{2}-7 m_{n}^{2}\right)\left(1+3 \delta_{i n}\right)\right] \text {, } \\
& (D-4) C_{0000 i i j}=-\frac{1}{5040}\left[6 s_{12}-7 m_{0}^{2}+\sum_{n=1}^{2}\left(p_{n}^{2}-7 m_{n}^{2}\right)\left(2+\delta_{i n}\right)\right] \text {, } \\
& (D-4) C_{00 i i i i i}=\frac{1}{42}, \quad(D-4) C_{00 i i i i j}=\frac{1}{210}, \quad(D-4) C_{00 i i i j j}=\frac{1}{420}, \tag{A.3}
\end{align*}
$$

where $i, j=1,2$ but $i \neq j$. All other 3-point tensor coefficients up to rank 7 are UV finite, so that for them $(D-4) C_{\ldots}=0$ if they are IR finite.

For the 4-point functions $D_{\ldots}\left(p_{1}, p_{2}, p_{3}, m_{0}, m_{1}, m_{2}, m_{3}\right)$ we find, denoting $\left(p_{1}-p_{2}\right)^{2}=s_{12}$, $\left(p_{1}-p_{3}\right)^{2}=s_{13}$, and $\left(p_{2}-p_{3}\right)^{2}=s_{23}$ :

$$
\begin{aligned}
& (D-4) D_{0000}=-\frac{1}{12}, \quad(D-4) D_{0000 i}=\frac{1}{48}, \\
& (D-4) D_{000000}=\frac{1}{480}\left[s_{12}+s_{13}+s_{23}+p_{1}^{2}+p_{2}^{2}+p_{3}^{2}\right]-\frac{1}{96}\left(m_{0}^{2}+m_{1}^{2}+m_{2}^{2}+m_{3}^{2}\right), \\
& (D-4) D_{0000 i i}=-\frac{1}{120}, \quad(D-4) D_{0000 i j}=-\frac{1}{240},
\end{aligned}
$$

$$
\begin{align*}
& (D-4) D_{000000 i}=-\frac{1}{2880}\left[\sum_{n=1}^{3} p_{n}^{2}\left(1+\delta_{i n}\right)+\sum_{\substack{m, n=1 \\
m>n}}^{3} s_{m n}\left(1+\delta_{i n}+\delta_{i m}\right)\right]+\frac{1}{480} \sum_{n=0}^{3} m_{n}^{2}, \\
& (D-4) D_{0000 i i i}=\frac{1}{240}, \quad(D-4) D_{0000 i i j}=\frac{1}{720}, \\
& (D-4) D_{0000 i j k}=\frac{1}{1440}, \tag{A.4}
\end{align*}
$$

where $i, j, k=1,2,3$ but are pairwise different. All other 4-point tensor coefficients up to rank 7 are UV finite.

For the 5-point functions $E_{\ldots}\left(p_{1}, p_{2}, p_{3}, p_{4}, m_{0}, m_{1}, m_{2}, m_{3}, m_{4}\right)$, there is only one UVsingular tensor coefficient up to rank 6,

$$
\begin{equation*}
(D-4) E_{000000}=-\frac{1}{96} . \tag{A.5}
\end{equation*}
$$

## Appendix B. Tensor coefficients of singular 3-point functions

The vanishing of the modified Cayley determinant $\operatorname{det}\left(X^{(N)}\right)$, as defined via (2.25), is a necessary condition for the existence of a leading Landau singularity in a one-loop $N$-point integral. For 3-point integrals this means that $\operatorname{det} X^{(3)}=0$ for IR-singular (either soft or collinear) integrals, so that the reduction methods of Sections 5.2 and 5.3 are not applicable in this case. If in addition the Gram determinant is small, for IR-singular 3-point integrals also the $\tilde{X}_{0 j}$ are small, and the reduction method of Section 5.4 cannot be used either. One could still, however, use the method of Section 5.5. In the following we describe a way of evaluating these specific 3-point functions that does not make use of an iteration technique, but is based on analytical simplifications that are admitted by the simple structure of the special cases.

The simplifications are achieved by directly using the analytical results for the standard scalar integrals and for the tensor coefficients, as obtained with the Passarino-Veltman reduction, and by rewriting them in such a way that the limit of vanishing Gram determinant does not involve numerical cancellations. To this end, the scalar integrals are split into two parts: one contains the asymptotic behaviour of the integral in the limit of vanishing Gram determinant $\Delta^{(3)}$ up to a specific order $n$ and a corresponding remainder which is of $\mathcal{O}\left(\left[\Delta^{(3)}\right]^{n+1}\right)$. We symbolize this splitting by introducing the asymptotic operators $\mathcal{T}_{x \rightarrow x_{0}}^{(n)}$ and $\mathcal{R}_{x \rightarrow x_{0}}^{(n)}$, which define the asymptotic behaviour of a function $f(x)$ for $x \rightarrow x_{0}$ by

$$
\begin{align*}
& f(x)=\mathcal{T}_{x \rightarrow x_{0}}^{(n)}[f(x)]+\mathcal{R}_{x \rightarrow x_{0}}^{(n)}[f(x)], \\
& \mathcal{R}_{x \rightarrow x_{0}}^{(n)}[f(x)]=\mathcal{O}\left(\left(x-x_{0}\right)^{n+1}\right), \quad n=0,1, \ldots . \tag{B.1}
\end{align*}
$$

If the function $f(x)$ is analytical at $x=x_{0}, \mathcal{T}_{x \rightarrow x_{0}}^{(n)}$ is the usual operator for a Taylor expansion up to order $n$.

Making use of these definitions, we now describe the treatment of the IR-singular 3-point tensor integrals that were needed in the calculation of the one-loop corrections to $\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow 4 f$ [35]. It is convenient to switch from the original definition (2.1) of arguments on tensor coefficients to the new notation

$$
\begin{align*}
& B_{\ldots}\left(p_{1}^{2}, m_{0}, m_{1}\right) \equiv B_{\ldots}\left(p_{1}, m_{0}, m_{1}\right) \\
& C_{\ldots}\left(p_{1}^{2},\left(p_{2}-p_{1}\right)^{2}, p_{2}^{2}, m_{0}, m_{1}, m_{2}\right) \equiv C_{\ldots}\left(p_{1}, p_{2}, m_{0}, m_{1}, m_{2}\right) \tag{B.2}
\end{align*}
$$

(i) Collinear-singular case with two off-shell legs: $C \ldots\left(m^{2}, s, s^{\prime}, 0, m, M\right)$.

Here $m$ denotes a small real mass, which will be neglected whenever possible. In this limit the relevant scalar integrals read

$$
\begin{align*}
& B_{0}(0)=B_{0}(s, m, M)=\Delta+\ln \left(\frac{\mu^{2}}{M^{2}}\right)+2+\left(\frac{M^{2}}{s}-1\right) \ln \left(\frac{M^{2}-s}{M^{2}}\right), \\
& B_{0}(1)=B_{0}\left(s^{\prime}, 0, M\right)=\Delta+\ln \left(\frac{\mu^{2}}{M^{2}}\right)+2+\left(\frac{M^{2}}{s^{\prime}}-1\right) \ln \left(\frac{M^{2}-s^{\prime}}{M^{2}}\right), \\
& B_{0}(2)=B_{0}\left(m^{2}, 0, m\right)=\Delta+\ln \left(\frac{\mu^{2}}{m^{2}}\right)+2, \\
& C_{0}= \\
& \frac{1}{s-s^{\prime}}\left\{\ln \left(\frac{M^{2}-s}{m^{2}}\right) \ln \left(\frac{M^{2}-s}{M^{2}}\right)-\ln \left(\frac{M^{2}-s^{\prime}}{m^{2}}\right) \ln \left(\frac{M^{2}-s^{\prime}}{M^{2}}\right)\right.  \tag{B.3}\\
& \left.\quad-2 \operatorname{Li}_{2}\left(\frac{s-s^{\prime}}{M^{2}-s^{\prime}}\right)+\operatorname{Li}_{2}\left(\frac{s}{M^{2}}\right)-\operatorname{Li}_{2}\left(\frac{s^{\prime}}{M^{2}}\right)\right\},
\end{align*}
$$

where $M^{2}$ is complex with a finite or infinitesimal negative imaginary part, which is also present for vanishing $M^{2}$. The Gram determinant is given by

$$
\begin{equation*}
\Delta^{(2)}=-\left(s-s^{\prime}\right)^{2} \tag{B.4}
\end{equation*}
$$

so that the delicate limit is $\delta s \equiv s^{\prime}-s \rightarrow 0$. The asymptotic expansions of the scalar integrals in (B.3) for this limit can be worked out easily; the first few terms read

$$
\begin{align*}
B_{0}(1) & =B_{0}(0)-\frac{\delta s}{s}\left[1+\frac{M^{2}}{s} \ln \left(\frac{M^{2}-s}{M^{2}}\right)\right]+\mathcal{R}_{\delta s \rightarrow 0}^{(1)}\left[B_{0}(1)\right], \\
C_{0}= & \frac{1}{s-M^{2}}\left\{\left[1+\frac{\delta s}{2\left(M^{2}-s\right)}\right] \ln \left(\frac{M^{2}-s}{m^{2}}\right)+\frac{M^{2}}{s}\left[1-\frac{\delta s\left(M^{2}-2 s\right)}{2 s\left(M^{2}-s\right)}\right] \ln \left(\frac{M^{2}-s}{M^{2}}\right)\right. \\
& \left.+2-\frac{\delta s\left(M^{2}-2 s\right)}{2 s\left(M^{2}-s\right)}\right\}+\mathcal{R}_{\delta s \rightarrow 0}^{(1)}\left[C_{0}\right], \tag{B.5}
\end{align*}
$$

where we have kept $s$ fixed. Inserting these or forms with more explicit terms of the asymptotic expansion for the scalar integrals into the explicit formulas for the tensor coefficients, one obtains expressions like

$$
\begin{aligned}
C_{1}= & \frac{M^{2}\left[\left(M^{2}-s\right) s-\delta s\left(4 M^{2}-5 s\right)\right]}{2 s^{2}\left(M^{2}-s\right)^{2}}+\frac{M^{2}-s+\delta s}{2\left(M^{2}-s\right)^{2}} \ln \left(\frac{M^{2}-s}{m^{2}}\right) \\
& +\frac{M^{2}\left[M^{2} s\left(M^{2}-s\right)-\delta s\left(4 M^{4}-7 M^{2} s+2 s^{2}\right)\right]}{2\left(M^{2}-s\right)^{2} s^{3}} \ln \left(\frac{M^{2}-s}{M^{2}}\right) \\
& -\frac{2(s+\delta s)}{(\delta s)^{2}} \mathcal{R}_{\delta s \rightarrow 0}^{(2)}\left[B_{0}(1)\right]+\frac{M^{2}-s-\delta s}{\delta s} \mathcal{R}_{\delta s \rightarrow 0}^{(1)}\left[C_{0}\right], \\
C_{2}= & -\frac{M^{2}}{s^{2}} \ln \left(\frac{M^{2}-s}{M^{2}}\right)-\frac{1}{s}+\frac{1}{\delta s} \mathcal{R}_{\delta s \rightarrow 0}^{(1)}\left[B_{0}(1)\right], \\
C_{00}= & \frac{1}{4} \Delta+\frac{1}{4} \ln \left(\frac{\mu^{2}}{M^{2}-s}\right)+\frac{M^{2}\left(M^{2}-\delta s\right)}{4 s^{2}} \ln \left(\frac{M^{2}-s}{M^{2}}\right)+\frac{M^{2}+2 s-\delta s}{4 s}
\end{aligned}
$$

$$
\begin{equation*}
-\frac{M^{2}-s-\delta s}{4 \delta s} \mathcal{R}_{\delta s \rightarrow 0}^{(1)}\left[B_{0}(1)\right] \tag{B.6}
\end{equation*}
$$

Here the orders $n$ in the $\mathcal{R}^{(n)}$ operators are chosen in such a way that all terms involving $\mathcal{R}^{(n)}$ contribute only in $\mathcal{O}(\delta s)$ in spite of the enhancement factors $1 /(\delta s)^{m}$. Note that no delicate cancellations for $\delta s \rightarrow 0$ appear in the other terms, although the original Passarino-Veltman results contain plenty of terms involving $1 /(\delta s)^{m}$ in front of linear combinations of scalar integrals. Thus, the above forms are numerically stable as long as the remainder terms $\mathcal{R}^{(n)}$ can be evaluated in a stable way. This task is, however, easily achieved upon expanding the scalar integrals as in (B.5) to a high order, e.g., with computer-algebraic methods, and dropping the first $n$ orders. The resulting series are easy to evaluate, and an arbitrarily high precision can be achieved by including sufficiently high orders in the expansions. On the other hand, if $\delta s$ is not small, the $\mathcal{R}^{(n)}$ terms can safely be evaluated upon numerically subtracting the $\mathcal{T}^{(n)}$ terms from the scalar integrals. In this way an arbitrarily high precision can be achieved as long as $s, s^{\prime} \neq 0$ and $s \neq M^{2}$. The case $s=M^{2}$ does not occur in our application, the cases $s=0$ and $s^{\prime}=0$ are treated below.
(ii) Collinear-singular case with one off-shell leg: $C \ldots\left(m^{2}, 0, s^{\prime}, 0, m, M\right)$.

Specializing the previous case to $s=0$, the scalar integrals read

$$
\begin{align*}
& B_{0}(0)=B_{0}(0, m, M)=\Delta+\ln \left(\frac{\mu^{2}}{M^{2}}\right)+1 \\
& C_{0}=\frac{1}{s^{\prime}}\left\{\ln \left(\frac{M^{2}}{m^{2}}\right) \ln \left(\frac{M^{2}-s^{\prime}}{M^{2}}\right)-\operatorname{Li}_{2}\left(\frac{s^{\prime}}{M^{2}}\right)\right\} \tag{B.7}
\end{align*}
$$

with $B_{0}(1)$ and $B_{0}(2)$ still as given in (B.3). The limit of vanishing Gram determinant is now reached for $s^{\prime} \rightarrow 0$, where the scalar integrals can be expanded according to

$$
\begin{align*}
& B_{0}(1)=B_{0}(0)+\frac{s^{\prime}}{2 M^{2}}+\mathcal{R}_{s^{\prime} \rightarrow 0}^{(1)}\left[B_{0}(1)\right], \\
& C_{0}=-\frac{1}{M^{2}}\left[\left(1+\frac{s^{\prime}}{2 M^{2}}\right) \ln \left(\frac{M^{2}}{m^{2}}\right)+1+\frac{s^{\prime}}{4 M^{2}}\right]+\mathcal{R}_{s^{\prime} \rightarrow 0}^{(1)}\left[C_{0}\right], \tag{B.8}
\end{align*}
$$

or to higher orders if needed. Making use of these expansions, the first few tensor coefficients can be written as

$$
\begin{align*}
& C_{1}=\frac{M^{2}+s^{\prime}}{2 M^{4}} \ln \left(\frac{M^{2}}{m^{2}}\right)-\frac{M^{2}-s^{\prime}}{4 M^{4}}-\frac{2}{s^{\prime}} \mathcal{R}_{s^{\prime} \rightarrow 0}^{(1)}\left[B_{0}(1)\right]+\frac{M^{2}-s^{\prime}}{s^{\prime}} \mathcal{R}_{s^{\prime} \rightarrow 0}^{(1)}\left[C_{0}\right], \\
& C_{2}=\frac{1}{2 M^{2}}+\frac{1}{s^{\prime}} \mathcal{R}_{s^{\prime} \rightarrow 0}^{(1)}\left[B_{0}(1)\right], \\
& C_{00}=\frac{1}{4} \Delta+\frac{1}{4} \ln \left(\frac{\mu^{2}}{M^{2}}\right)+\frac{3 M^{2}+s^{\prime}}{8 M^{2}}-\frac{M^{2}-s^{\prime}}{4 s^{\prime}} \mathcal{R}_{s^{\prime} \rightarrow 0}^{(1)}\left[B_{0}(1)\right] . \tag{B.9}
\end{align*}
$$

The $\mathcal{R}^{(n)}$ terms, which are suppressed by a factor $\left(s^{\prime}\right)^{n+1}$, can be evaluated to arbitrary precision for all values of $s^{\prime}$ as described above.
(iii) Collinear-singular case with one off-shell leg: $C \ldots\left(m^{2}, s, 0,0, m, M\right)$.

Specializing case (i) to $s^{\prime}=0$, the scalar integrals read

$$
\begin{align*}
& B_{0}(1)=B_{0}(0,0, M)=\Delta+\ln \left(\frac{\mu^{2}}{M^{2}}\right)+1 \\
& C_{0}=\frac{1}{s}\left\{\ln \left(\frac{M^{2}-s}{m^{2}}\right) \ln \left(\frac{M^{2}-s}{M^{2}}\right)-\operatorname{Li}_{2}\left(\frac{s}{M^{2}}\right)\right\}, \tag{B.10}
\end{align*}
$$

with $B_{0}(0)$ and $B_{0}(2)$ still as given in (B.3). The limit of vanishing Gram determinant is reached for $s \rightarrow 0$, where the scalar integrals can be expanded according to

$$
\begin{align*}
& B_{0}(0)=B_{0}(1)+\frac{s}{2 M^{2}}+\mathcal{R}_{s \rightarrow 0}^{(1)}\left[B_{0}(0)\right], \\
& C_{0}=-\frac{1}{M^{2}}\left[\left(1+\frac{s}{2 M^{2}}\right) \ln \left(\frac{M^{2}}{m^{2}}\right)+1-\frac{3 s}{4 M^{2}}\right]+\mathcal{R}_{s \rightarrow 0}^{(1)}\left[C_{0}\right], \tag{B.11}
\end{align*}
$$

or to higher orders if needed. Making use of these expansions, the first few tensor coefficients can be written as

$$
\begin{align*}
& C_{1}=\frac{1}{2 M^{2}} \ln \left(\frac{M^{2}}{m^{2}}\right)-\frac{1}{4 M^{2}}+\frac{1}{s} \mathcal{R}_{s \rightarrow 0}^{(1)}\left[B_{0}(0)\right]-\frac{M^{2}}{s} \mathcal{R}_{s \rightarrow 0}^{(1)}\left[C_{0}\right], \\
& C_{2}=\frac{1}{2 M^{2}}+\frac{1}{s} \mathcal{R}_{s \rightarrow 0}^{(1)}\left[B_{0}(0)\right], \\
& C_{00}=\frac{1}{4} \Delta+\frac{1}{4} \ln \left(\frac{\mu^{2}}{M^{2}}\right)+\frac{3 M^{2}+s}{8 M^{2}}-\frac{M^{2}-s}{4 s} \mathcal{R}_{s \rightarrow 0}^{(1)}\left[B_{0}(0)\right] . \tag{B.12}
\end{align*}
$$

The $\mathcal{R}^{(n)}$ terms, which are suppressed by a factor $s^{n+1}$, can be evaluated to arbitrary precision for all values of $s$ as described above.
(iv) Soft-singular case: $C \ldots\left(m_{1}^{2}, s, m_{2}^{2}, \lambda, m_{1}, m_{2}\right)$.

For processes with external fermions in the massless limit ( $m_{i} \rightarrow 0$ ), the Passarino-Veltman reduction of this case turns out to be less delicate than the previous ones. In fact, no special treatment was necessary for $\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow 4 f$ [35], although one could also improve the stability as described in the previous sections. We attribute the robustness of this case to the following reasons. Firstly, because $\lambda$ is an infinitesimal photon mass, $f_{1}=f_{2}=0$ and all 3-point tensor coefficients are directly obtained from 2-point coefficients without further recursions. Thus, instabilities do not accumulate. Secondly, for massless fermions the Gram determinant $\Delta^{(2)}=-s^{2}$ vanishes only for $s \rightarrow 0$, and this case appears for $\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow 4 f$ only in regions of phase space that are suppressed by $\Gamma_{\mathrm{W}} / M_{\mathrm{W}}$.

## Appendix C. Alternative reduction of 5-point integrals

In Ref. [26] we have worked out a reduction of 5-point tensor integrals that follows the strategy proposed in Ref. [22] for scalar integrals in four space-time dimensions. Here we briefly describe the derivation of this method in $D$ dimensions, to make closer contact to the methods used in this paper.

The reduction is based on different ways of evaluating the determinant

$$
\mathcal{E}^{\prime}=\left|\begin{array}{cccc}
2 q^{2} & 2 q p_{1} & \cdots & 2 q p_{4}  \tag{C.1}\\
2 p_{1} q & 2 p_{1} p_{1} & \cdots & 2 p_{1} p_{4} \\
\vdots & \vdots & \ddots & \vdots \\
2 p_{4} q & 2 p_{4} p_{1} & \cdots & 2 p_{4} p_{4}
\end{array}\right|
$$

which vanishes in four dimensions owing to the linear dependence of any five momenta. In $D$ dimensions the integral over $\mathcal{E}^{\prime}$ can be easily evaluated to

$$
\begin{align*}
\frac{(2 \pi \mu)^{4-D}}{\mathrm{i} \pi^{2}} \int \mathrm{~d}^{D} q \frac{q^{\mu_{1}} \cdots q^{\mu_{P}}}{N_{0} N_{1} \cdots N_{4}} \mathcal{E}^{\prime} & =E^{\alpha \beta \mu_{1} \ldots \mu_{P}}\left|\begin{array}{cccc}
2 g_{\alpha \beta} & 2 p_{1, \alpha} & \cdots & 2 p_{4, \alpha} \\
2 p_{1, \beta} & 2 p_{1} p_{1} & \cdots & 2 p_{1} p_{4} \\
\vdots & \vdots & \ddots & \vdots \\
2 p_{4, \beta} & 2 p_{4} p_{1} & \cdots & 2 p_{4} p_{4}
\end{array}\right| \\
& =2 E^{\alpha \beta \mu_{1} \ldots \mu_{P}} \Delta^{(4)}\left(g_{\alpha \beta}-g_{(4), \alpha \beta}\right), \tag{C.2}
\end{align*}
$$

where we have identified the form (6.3) of the metric tensor $g_{(4), \alpha \beta}$ in four dimensions.
On the other hand, the integral over $\mathcal{E}^{\prime}$ can be evaluated in terms of 4-point functions as described in Section 2 of Ref. [26] with the only difference that no additional UV regularization is needed, because we now keep the dimension $D$ general. In detail, this means that the factor $-\Lambda^{2} /\left(q^{2}-\Lambda^{2}\right)$ introduced in (2.5) of Ref. [26] is absent, and the result analogous to (2.19) of Ref. [26] becomes

$$
\begin{align*}
\operatorname{det}(Y) E^{\mu_{1} \ldots \mu_{P}}= & -\sum_{n=0}^{4} \operatorname{det}\left(Y_{n}\right) D^{\mu_{1} \ldots \mu_{P}}(n)+\sum_{n, m=1}^{4} \tilde{Z}_{n m}^{(4)} 2 p_{m, \alpha} \mathcal{D}^{\alpha \mu_{1} \ldots \mu_{P}}(n) \\
& +2 E^{\alpha \beta \mu_{1} \ldots \mu_{P}} \Delta^{(4)}\left(g_{\alpha \beta}-g_{(4), \alpha \beta}\right) \tag{C.3}
\end{align*}
$$

where $Y=\left(Y_{i j}\right), i, j=0, \ldots, 4$, was defined in (2.26), and $Y_{n}$ is obtained from the 5-dimensional modified Cayley matrix $Y$ by replacing all entries in the $n$th column by 1 . The last term of (C.3), which results from (C.2), contributes only if $E^{\alpha \beta \mu_{1} \ldots \mu_{P}}$ involves a divergent coefficient $E_{00} \ldots$ corresponding to a covariant containing a metric tensor. As explained in Section 5.8, such coefficients are free of IR divergences, and power counting shows that UV divergences only occur for $P \geqslant 4$. Therefore, the last term in (C.3) is of $\mathcal{O}(D-4)$, and thus irrelevant, for $P \leqslant 3$. For $P=4$, this term can be explicitly evaluated using (A.5) yielding

$$
\begin{align*}
\operatorname{det}(Y) E^{\mu_{1} \ldots \mu_{4}}= & -\sum_{n=0}^{4} \operatorname{det}\left(Y_{n}\right) D^{\mu_{1} \ldots \mu_{4}}(n)+\sum_{n, m=1}^{4} \tilde{Z}_{n m}^{(4)} 2 p_{m, \alpha} \mathcal{D}^{\alpha \mu_{1} \ldots \mu_{4}}(n) \\
& -\frac{1}{48} \Delta^{(4)}\{g g\}^{\mu_{1} \ldots \mu_{4}}-\frac{1}{24(D-4)} \Delta^{(4)}\left\{\left(g-g_{(4)}\right) g\right\}^{\mu_{1} \ldots \mu_{4}}, \tag{C.4}
\end{align*}
$$

where $\left\{\left(g-g_{(4)}\right) g\right\}^{\mu_{1} \ldots \mu_{4}}$ is a symmetric tensor of rank 4 constructed according to the rules explained in Section 2,

$$
\begin{align*}
& \left\{\left(g-g_{(4)}\right) g\right\}^{\mu_{1} \ldots \mu_{4}} \\
& \quad=\left(g-g_{(4)}\right)^{\mu_{1} \mu_{2}} g^{\mu_{3} \mu_{4}}+\left(g-g_{(4)}\right)^{\mu_{1} \mu_{3}} g^{\mu_{2} \mu_{4}}+\left(g-g_{(4)}\right)^{\mu_{1} \mu_{4}} g^{\mu_{2} \mu_{3}} \\
& \quad+\left(g-g_{(4)}\right)^{\mu_{2} \mu_{3}} g^{\mu_{1} \mu_{4}}+\left(g-g_{(4)}\right)^{\mu_{2} \mu_{4}} g^{\mu_{1} \mu_{3}}+\left(g-g_{(4)}\right)^{\mu_{3} \mu_{4}} g^{\mu_{1} \mu_{2}} \tag{C.5}
\end{align*}
$$

The first term in the last line of (C.4) is just the finite contribution $U^{\mu_{1} \ldots \mu_{4}}$ defined in (2.15) of Ref. [26], and the UV-divergent terms of the second term in the last line exactly cancel the UV divergences of the 4-point integrals in the first line. Thus, the result for $E^{\mu_{1} \ldots \mu_{4}}$ exactly receives the form of (2.19) of Ref. [26],

$$
\begin{align*}
\operatorname{det}(Y) E^{\mu_{1} \ldots \mu_{4}}= & -\sum_{n=0}^{4} \operatorname{det}\left(Y_{n}\right) D^{(\mathrm{fin}) \mu_{1} \ldots \mu_{4}}(n)+\sum_{n, m=1}^{4} \tilde{Z}_{n m}^{(4)} 2 p_{m, \alpha} \mathcal{D}^{(\mathrm{fin}) \alpha \mu_{1} \ldots \mu_{4}}(n) \\
& -\frac{1}{48} \Delta^{(4)}\{g g\}^{\mu_{1} \ldots \mu_{4}}, \tag{C.6}
\end{align*}
$$

where the superscript "(fin)" indicates that the UV parts have to be consistently omitted, as e.g. following the $\overline{\mathrm{MS}}$ prescription.

## Appendix D. Alternative reduction of 6-point integrals

Here we describe the reduction of 6-point tensor integrals of rank $P$ (including the scalar case $P=0$ ) to six 5-point tensor integrals of equal rank that is based on the strategy of Ref. [24]. This reduction is related to the reduction of 5-point functions as given in Ref. [26] and Appendix C and has been used in the calculation of the electroweak corrections to $\mathrm{e}^{+} \mathrm{e}^{+} \rightarrow 4 f$ [35]. Moreover, it is needed to reduce the scalar 6-point function to 5-point functions [22].

It starts from the observation that

$$
\frac{(2 \pi \mu)^{4-D}}{\mathrm{i} \pi^{2}} \int \mathrm{~d}^{D} q \frac{q^{\mu_{1}} \cdots q^{\mu_{P}}}{N_{0} N_{1} \cdots N_{5}}\left|\begin{array}{cccc}
N_{0}+Y_{00} & 2 q p_{1} & \cdots & 2 q p_{5}  \tag{D.1}\\
Y_{10}-Y_{00} & 2 p_{1} p_{1} & \cdots & 2 p_{1} p_{5} \\
\vdots & \vdots & \ddots & \vdots \\
Y_{50}-Y_{00} & 2 p_{5} p_{1} & \cdots & 2 p_{5} p_{5}
\end{array}\right|=0
$$

which is correct in any space-time dimension $D$ as long as the five four-momenta $p_{i}$ ( $i=$ $1, \ldots, 5$ ) are linearly dependent, and thus for four-dimensional $p_{i}$, because then the five last columns of the determinant are linearly dependent for an arbitrary $D$-dimensional momentum $q$. The l.h.s. of this relation is practically the same as in Eq. (2.10) of Ref. [26], where the reduction of 5-point integrals is described. The same manipulations as described there lead to the result

$$
\left|\begin{array}{ccccc}
F^{\mu_{1} \ldots \mu_{P}} & -E^{\mu_{1} \ldots \mu_{P}}(0) & -E^{\mu_{1} \ldots \mu_{P}}(1) & \cdots & -E^{\mu_{1} \ldots \mu_{P}}(5)  \tag{D.2}\\
1 & Y_{00} & Y_{01} & \cdots & Y_{05} \\
1 & Y_{10} & Y_{11} & \cdots & Y_{15} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & Y_{50} & Y_{51} & \cdots & Y_{55}
\end{array}\right|=0 .
$$

Eq. (D.2) expresses $F^{\mu_{1} \ldots \mu_{P}}$ in terms of six 5-point integrals,

$$
\begin{equation*}
F^{\mu_{1} \ldots \mu_{P}}=-\sum_{n=0}^{5} \eta_{n} E^{\mu_{1} \ldots \mu_{P}}(n) \quad \text { with } \quad \eta_{n}=\frac{\operatorname{det}\left(Y_{n}\right)}{\operatorname{det}(Y)} \tag{D.3}
\end{equation*}
$$

where $Y=\left(Y_{i j}\right), i, j=0, \ldots, 5$, and $Y_{n}$ is obtained from the 6-dimensional modified Cayley matrix $Y$ by replacing all entries in the $n$th column by 1 . For the scalar integral $F_{0}$, this result is identical with the one of Ref. [22].

By inserting the Lorentz decompositions as given in (7.12), we can derive explicit formulas for the scalar 6-point function and the coefficients of tensor 6-point integrals from (D.3):

$$
\begin{align*}
& F_{0}=-\sum_{n=0}^{5} \eta_{n} E_{0}(n),  \tag{D.4}\\
& F_{i_{1}}=-\sum_{n=1}^{5} \eta_{n} E_{\left(i_{1}\right)_{n}}(n) \bar{\delta}_{i_{1} n}-\eta_{0} E_{i_{1}}(0), \quad i_{1}=1, \ldots, 5,  \tag{D.5}\\
& F_{00}=-\sum_{n=0}^{5} \eta_{n} E_{00}(n), \\
& F_{i_{1} i_{2}}=-\sum_{n=1}^{5} \eta_{n} E_{\left(i_{1}\right)_{n}\left(i_{2}\right)_{n}(n)} \bar{\delta}_{i_{1} n} \bar{\delta}_{i_{2} n}-\eta_{0} E_{i_{1} i_{2}}(0), \quad i_{1}, i_{2}=1, \ldots, 5,  \tag{D.6}\\
& F_{00 i_{1}}=-\sum_{n=1}^{5} \eta_{n} E_{00\left(i_{1}\right)_{n}}(n) \bar{\delta}_{i_{1} n}-\eta_{0} E_{00 i_{1}}(0), \quad i_{1}=1, \ldots, 5, \\
& F_{i_{1} i_{2} i_{3}}=-\sum_{n=1}^{5} \eta_{n} E_{\left(i_{1}\right)_{n}\left(i_{2}\right)_{n}\left(i_{3}\right)_{n}(n) \bar{\delta}_{i_{1} n} \bar{\delta}_{i_{2} n} \bar{\delta}_{i_{3} n}-\eta_{0} E_{i_{1} i_{2} i_{3}}(0), \quad i_{1}, i_{2}, i_{3}=1, \ldots, 5 .} \tag{D.7}
\end{align*}
$$

The 5-point tensor coefficients that result from omitting $N_{0}$ in the 6-point integrals have been given in (7.19).

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[^0]:    * Corresponding author.

    E-mail address: ansgar.denner@psi.ch (A. Denner).

[^1]:    1 The GRACE-loop Collaboration has recently reported on progress towards one-loop calculations for $2 \rightarrow 4$ particle processes. Using the methods described in Refs. [7,27], first results on $\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow \nu \bar{\nu} \mathrm{HH}$ have been shown at conferences [36], and a status report on $\mathrm{e}^{+} \mathrm{e}^{-} \rightarrow \mu^{-} \bar{v}_{\mu} \mathrm{ud}$ has been given in Ref. [37].

[^2]:    ${ }^{2}$ A similar idea, where tensor coefficients are iteratively determined from higher rank tensors has been described in Ref. [18].

[^3]:    ${ }^{3}$ Note that the Feynman-parameter integral of $D_{00}$ is not logarithmic, so that the calculation of $D_{i_{1} i_{2} i_{3}}$ from $D_{0000}$ and $D_{000000}$ requires the use of the recurrence relations twice.

[^4]:    ${ }^{4}$ In an alternative approach, one could disregard (5.40) and use (5.38) also to determine $D_{00 i_{1} \ldots i_{r} \ldots i_{P}}$. This reduction method would also work if all $\tilde{Z}_{k l}^{(3)}$ are small. However, in this case, tensor integrals of higher rank would be needed. For instance, to calculate $D_{i_{1} i_{2} i_{3}}$ in leading order in $\Delta^{(3)}$ one would have to calculate $D_{000000}$ and $C_{000000}$ first.

[^5]:    5 An alternative reduction could be derived, by considering $\sum_{r=1}^{P+1} \tilde{X}_{i i_{r}}^{(3)} D_{i_{1} \ldots \hat{i}_{r} \ldots i_{P+1}}$, using (5.53) and inserting (5.49) on the r.h.s. of the resulting equation to eliminate $D_{00 i_{1} \ldots \hat{i}_{r} \ldots i_{P+1}}$. From the obtained relation, all tensor coefficients could be calculated. This reduction method would also work if all $\tilde{Z}_{k l}^{(3)}$ are small. However, in this case, tensor integrals of higher rank would be needed.

[^6]:    ${ }^{6}$ This result is in agreement with the observation made in Ref. [29] that in the absence of UV divergences reduction formulas valid in 4 dimensions remain valid in $D$ dimensions up to terms of $\mathcal{O}(D-4)$, independent of the possible occurrence of IR singularities.

[^7]:    ${ }^{7}$ Contributions to $E^{\alpha \beta \mu_{1} \ldots \mu_{P}}$ involving $p_{k}^{\alpha}$ vanish, and those involving $g^{\alpha \mu_{i}} g^{\beta \mu_{j}}$ cancel after symmetrizing w.r.t. the indices $\mu, \mu_{1}, \ldots, \mu_{P}$. In terms involving $g^{\alpha \beta}$ the surviving $\left(g-g_{(4)}\right)$ turns into ( $D-4$ ). Finally, terms involving $p_{k}^{\beta}$ get a factor $\sum_{n=1}^{4} Z_{i n}^{(4)} \tilde{X}_{n 0}^{(4)}=-f_{i} \Delta^{(4)}$ owing to (2.28) and (2.13).

[^8]:    ${ }^{8}$ This is again in agreement with the observation [29] that in the absence of UV divergences reduction formulas valid in 4 dimensions remain valid in $D$ dimensions up to terms of $\mathcal{O}(D-4)$, independent of possible IR singularities.
    ${ }^{9}$ According to (2.28), $\tilde{X}_{k 0}^{(5)}=-\sum_{n=1}^{5} \tilde{Z}_{k n}^{(5)} f_{n}$. For each of these terms, $\tilde{Z}_{k n}^{(5)} g_{(4)}$ can be expressed via (7.3) by a determinant without denominator.

