# Electromagnetic Properties of the $\boldsymbol{Z}$ Boson 

II. Ward Identities for $Z \gamma \gamma$ and $Z Z \gamma$ Green's Functions

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#### Abstract

Using the invariance under BRS transformations we derive Ward identities for the $Z \gamma \gamma$ and $Z Z \gamma$ Green's functions. Anomalous contributions to these identities are also obtained using the path integral formalism. The results agree with our previous calculation in perturbation theory.


## 1. Introduction

In a recent paper [1] we have considered the $Z$ boson couplings to photons via the fermion triangle diagram. In particular, it was shown [1] that a real $Z$ has a single, parity violating and CP conserving, electromagnetic moment, called, the anapole moment.

Using the group structure of the standard $S U(2)$ $\times U(1)$ model it is straightforward to see that, at one loop, only the fermions give a contribution to the $Z \gamma \gamma, Z Z \gamma$ and $Z Z Z$ irreducible Green's functions. In [1] the evaluation of the $Z \gamma \gamma$ and $Z Z \gamma$ triangle diagrams was done without the use of any regularization scheme. On the contrary, the calculation relied on symmetry arguments and the application of electromagnetic current conservation.

In the near future one loop radiative corrections should provide a sensitive test of the electroweak model. Then, the three point functions we have been discussing will be of interest for the physics of $e^{+} e^{-}$ colliders. Hence, it is important to extend our previous work and prove that the contraints used can be derived from the Ward identities of the theory. This is the first aim of this paper. Our second objective concerns the well known Adler-Bell-Jackiw anomaly [2] which affects the fermion triangle diagrams

[^0]with axial vector couplings. Recently, a great deal of work has been done in this area (see e.g. [3] for a review). In here we apply Fujikawa's path integral method [4-6] to derive the Wess-Zumino consistent anomaly [7] associated with the axial-vector current that couples to the $Z$ boson.

## 2. Ward Identities

The simplest way of deriving Ward-Slavnov-Taylor identities [8] is to use the fact that Green's functions are invariant under the Becchi-Rouet-Stora (BRS) transformation [9]. We denote by $C_{A}, C_{Z}$ and $C^{ \pm}$ the Faddeev-Popov ghosts associated with the fields $A_{\mu}, Z_{\mu}$ and $W_{\mu}^{ \pm}$respectively. Under a BRS transformation we have

$$
\begin{align*}
& \delta A_{\mu}=\left[\partial_{\mu} C_{A}+i e\left(C^{-} W_{\mu}^{+}-C^{+} W_{\mu}^{-}\right)\right] \theta \\
& \delta Z_{\mu}=\left[\partial_{\mu} C_{Z}-i g \cos \theta_{W}\left(C^{-} W_{\mu}^{+}-C^{+} W_{\mu}^{-}\right)\right] \theta \\
& \delta \bar{C}_{A}=\partial^{\mu} A_{A} \theta  \tag{1}\\
& \delta \bar{C}_{Z}=\left[\partial^{\mu} Z_{\mu}-M_{Z} \Phi_{Z}\right] \theta
\end{align*}
$$

where $\theta$ is a space-time independent Grassmann variable and $\Phi_{Z}$ is the would be Goldstone boson associated with the $Z$. Notice that we are working in the 't Hooft-Feynman gauge with a linear gauge fixing condition. For later use it is also convenient to recall the $Z$ and photon $\left(A_{\mu}\right)$ couplings to fermions, which are:

$$
\begin{align*}
\mathscr{L}= & \frac{g}{\cos \theta_{W}} \bar{\psi} \gamma^{\mu}\left[g_{V f}-g_{A f} \gamma_{5}\right] \psi Z_{\mu} \\
& -e Q_{f} \bar{\psi} \gamma^{\mu} \psi A_{\mu} \tag{2}
\end{align*}
$$

with

$$
\begin{align*}
& g_{V f}=\frac{1}{2} T_{3 f}-Q_{f} \sin ^{2} \theta_{W}  \tag{3}\\
& g_{A f}=\frac{1}{2} T_{3 f} .
\end{align*}
$$

$T_{3 f}$ and $Q_{f}$ are the third component of the weak isospin and the charge (in units $\dot{e}>0$ ) of fermion $f$ respectively.

Let us consider the $Z \gamma \gamma$ irreducible Green's function $G^{\rho \mu \nu}\left(q, k_{1}, k_{2}\right)$, where $q^{\rho}$ is the 4-momentum of the incoming $Z$ and $k_{1}^{\mu}$ and $k_{2}^{v}$ are the 4 -momentum of the outgoing photons. From $\delta\langle 0| T Z_{\rho} \bar{C}_{A} A_{v}|0\rangle=0$ and using (1) we obtain

$$
\begin{align*}
& \langle 0| T Z_{\rho} \partial^{\mu} A_{\mu} A_{v}|0\rangle=\langle 0| T \partial_{\rho} C_{Z} \bar{C}_{A} A_{v}|0\rangle \\
& +\langle 0| T Z_{\rho} \partial_{v} C_{A} \bar{C}_{A}|0\rangle \\
& -i g \cos \theta_{W}\langle 0| T\left(C^{-} W_{\rho}^{+}-C^{+} W_{\rho}^{-}\right) \bar{C}_{A} A_{v}|0\rangle \\
& +i e\langle 0| T Z_{\rho}\left(C^{-} W_{v}^{+}-C^{+} W_{v}^{-}\right) \bar{C}_{A}|0\rangle \tag{4}
\end{align*}
$$

In momentum space, the first term of the right hand side (RHS) of (4) is $q_{\rho} T_{v}$ where $T_{v}$ is shown in Fig. 1. Evaluating the loop diagrams one obtains $T_{v}=0$ since there is a cancellation between the $C^{-}$and $C^{+}$ ghosts, i.e., the diagrams change sign under the substitution $C^{-} \rightarrow C^{+}$. This argument shows that the second contribution to the RHS of (4), $k_{2 v} T_{\rho}^{\prime}$, is also zero, i.e., $T_{\rho}^{\prime}=0$ at one loop.

The Fourier transform of the third term of the RHS of (4), $T_{\rho v}$, is represented in Fig. 2. It is again easy to show that, at one loop, the diagrams are symmetric under the substitution $C^{-} \rightarrow C^{+}$. This in turn implies $T_{\rho v}=0$. A similar argument applied to the last term of (4) shows that it also vanishes. Hence, we obtain

$$
\begin{equation*}
-i k_{1}^{\mu} G_{\mu \mu^{\prime}}\left(k_{1}\right) G_{v v^{\prime}}\left(k_{2}\right) G_{\rho \rho^{\prime}}(q) G^{\rho^{\prime} v^{\prime} \mu^{\prime}}\left(q, k_{1}, k_{2}\right)=0, \tag{5}
\end{equation*}
$$

where $G_{\rho \rho^{\prime}}, G_{\mu \mu}$, and $G_{v v^{\prime}}$, are the external particle propagators. In a previous paper [10] we showed that the photon and $Z$ propagators obey the identities
$k_{1}^{\mu} G_{\mu \mu^{\prime}}\left(k_{1}\right)=-k_{1 \mu^{\prime}} \Delta_{C_{A}}+F_{\mu^{\prime}} \Delta_{C_{A}}$
and
$q^{\rho} G_{\rho \rho^{\prime}}(q)=-q_{\rho^{\prime}} \Delta_{C_{Z}}+E_{\rho^{\prime}} \Delta_{C_{Z}}$,
respectively. $\Delta_{C_{A}}$ and $\Delta_{C_{Z}}$ are the ghosts propagators and the quantities $F_{\mu}$ and $E_{\rho}$ are at least of second order in the coupling constant and so do not play any role in the following. Using (6a) in (5) and multiplying by the inverse propagators we finally obtain that, in lowest order,
$k_{1 \mu} G^{\rho \mu \nu}\left(q, k_{1}, k_{2}\right)=0$,
and similarly
$k_{2 v} G^{\rho \mu v}\left(q, k_{1}, k_{2}\right)=0$.
These are the relations used in [1] to calculate the $Z \gamma \gamma$ fermion triangle. Recalling that, at one loop




Fig. 1. One loop contributions to $T_{v}$ (4)


Fig. 2. One loop contributions to $T_{\rho v}$ (4)
level, $G^{\rho \mu v}$ is entirely due to this diagram (7) justifies the previous procedure.

Now let us think about the $Z Z_{\gamma}$ three point function $T^{\rho v \mu}\left(q_{1}, q_{2}, k\right)$, where $k^{\mu}$ is the outgoing photon momentum and $q_{1}^{p}, q_{2}^{v}$ are the incoming and outgoing $Z$ momenta respectively. Starting with $\delta\langle 0| T Z_{\rho} Z_{v} \bar{C}_{A}|0\rangle=0$ and following a path similar to the previous one it is immediate to obtain
$k_{\mu} T^{\rho v \mu}\left(q_{1}, q_{2}, k\right)=0$.

$$
\mathrm{T}^{\nu \mu} \equiv
$$


$=$


Fig. 3. One loop contributions to $T^{v \mu}(9)$

A second Ward identity for $T^{\rho v \mu}$ can be obtained from
$\delta\langle 0| T \bar{C}_{Z} Z_{v} A_{\mu}|0\rangle=0$,
which gives

$$
\begin{align*}
& \langle 0| T \partial^{\rho} Z_{\rho} Z_{v} A_{\mu}|0\rangle=\langle 0| T Z_{v} \partial_{\mu} C_{A} \bar{C}_{Z}|0\rangle \\
& +\langle 0| T \partial_{v} C_{Z} \bar{C}_{Z} A_{\mu}|0\rangle \\
& -i g \cos \theta_{W}\langle 0| T\left(C^{-} W_{v}^{+}-C^{+} W_{v}^{-}\right) A_{\mu} \bar{C}_{Z}|0\rangle \\
& +i e\langle 0|\left(C^{-} W_{\mu}^{+}-C^{+} W_{\mu}^{-}\right) Z_{v} \bar{C}_{Z}|0\rangle . \\
& +M_{Z}\langle 0| T \Phi_{Z} Z_{v} A_{\mu}|0\rangle . \tag{9}
\end{align*}
$$

Again, using the same arguments as before, it is straightforward to see that, in lowest order, only the last term on the RHS of (9) gives a non zero contribution. Therefore we obtain
$q_{1_{\rho}} T^{\rho \nu \mu}\left(q_{1}, q_{2}, k\right)=i M_{Z} T^{\nu \mu}$,
where $T^{\nu \mu}$ is represented in Fig. 3. In the same figure we also show the one loop contributions to $T^{\nu \mu}$ which are entirely due to the fermions. Evaluating these diagrams we have
$q_{1 \rho} T^{\rho \nu \mu}=\frac{e}{\pi^{2}}\left(\frac{g}{\cos \theta_{W}}\right)^{2} \sum_{f} g_{A f} g_{V f} Q_{f^{\rho}} \varepsilon^{\rho \nu \mu \alpha} q_{1 \rho} k_{\alpha}$
$\cdot \int_{0}^{1} d x_{1} \int_{0}^{1-x_{1}} d x_{2} m_{f}^{2} / \Delta$,
with

$$
\Delta=m_{f}^{2}+k^{2} x_{1}\left(x_{1}-1\right)+q_{1}^{2} x_{2}\left(x_{2}-1\right)+2 q_{1} \cdot k x_{1} x_{2} .
$$

This result coincides with the non-anomalous contribution calculated in [1]. From our derivation it should be clear that a similar result can be derived for $q_{2 v} T^{\rho v \mu}$. To obtain the fermion mass, $m_{f}$, independent contribution, the so called anomaly, we need to consider the general problem of anomalous Ward identities [3-7].

## 3. $\operatorname{SU}(\mathbf{2}) \times U(1)$ Anomalies

The existence of anomalies in currents that are coupled to gauge bosons breaks the gauge invariance of the theory. This is by now a well known fact and since gauge invariance is a necessary condition for renormalizability it is sometimes used to place constraints on the fermion contents of the theory*. Hence, in the $S U(2) \times U(1)$ electroweak model the quark and lepton multiplets are such that the anomalies are cancelled.

Following the leading work of Fujikawa [4] and after some controversy [5, 6] it became clear [11] how to derive, in the framework of the path integral formalism, the anomaly of gauge currents. Furthermore, it was shown by Wess and Zumino [7] that these anomalies must satisfy certain consistency conditions, which in turn imply that the non-singlet non-Abelian anomaly cannot have a covariant expression. In here we want to apply all these results to construct explicitely the anomaly per fermion associated with the axial-vector current that couples to the $Z$ boson. Summed over fermions our results are obviously zero. Nevertheless, we think that the expressions derived below could provide useful checks for calculations done in perturbation theory.

Let $t^{a}$ denote the generators of $S U(2)$ and $Y$ the weak hypercharge. Then for each family doublet of quarks or leptons we can define left and right covariant derivatives which are
$D_{\mu L}=\partial_{\mu}-i g t^{a} A_{\mu}^{a}-i g^{\prime} Y B_{\mu}$
$D_{\mu R}=\hat{o}_{\mu}+i g^{\prime} Q B_{\mu}$,
with $Q=t^{3}-Y$. Rewriting (12) in the form
$D_{\mu}=\partial_{\mu}-V_{\mu}-C_{\mu} \gamma_{5}$,
with
$V_{\mu}=\sum_{a=1,2} i \frac{g}{2} t^{a} A_{\mu}^{a}+i / 2\left(g A_{\mu}^{3}-g^{\prime} B_{\mu}\right) t^{3}+i g^{\prime} Y B_{\mu}$
and

$$
\begin{equation*}
C_{\mu}=-\sum_{a=1,2} i \frac{g}{2} t^{a} A_{\mu}^{a}-i / 2\left(g A_{\mu}^{3}+g^{\prime} B_{\mu}\right) t^{3} \tag{13b}
\end{equation*}
$$

and introducing the notation

$$
\begin{align*}
& G_{\mu \nu}^{V}=\partial_{\mu} V_{v}-\partial_{v} V_{\mu}-\left[V_{\mu}, V_{v}\right]-\left[C_{\mu}, C_{v}\right]  \tag{14a}\\
& G_{\mu \nu}^{A}=\partial_{\mu} C_{v}-\partial_{v} C_{\mu}-\left[V_{\mu}, C_{v}\right]-\left[C_{\mu}, V_{v}\right] \tag{14b}
\end{align*}
$$

[^1]one can show [6] that the divergency of the isovector axial current, $J_{5 \mu}^{a}$, is:
\[

$$
\begin{align*}
& \partial^{\mu} J_{5 \mu}^{a}=-\frac{1}{8 \pi^{2}} \varepsilon^{\mu \nu \alpha \beta} \operatorname{Tr}\left\{t ^ { a } \left[\frac{1}{2} G_{\mu \nu}^{V} G_{\alpha \beta}^{V}\right.\right. \\
& \left.\left.+\frac{1}{6} G_{\mu \nu}^{A} G_{\alpha \beta}^{A}+4 / 3\left(C_{\mu} C_{\nu} G_{\alpha \beta}^{V}+G_{\mu \nu}^{V} C_{\alpha} C_{\beta}+C_{\mu} G_{v \alpha}^{V} C_{\beta}\right)\right]\right\} \tag{15}
\end{align*}
$$
\]

From this equation and after a lengthy but straightforward calculation one obtains

$$
\begin{align*}
& \partial^{\mu} J_{5 \mu}^{Z}=-\sum_{f} \frac{g}{16 \pi^{2} \cos \theta_{W}} g_{A f} \varepsilon^{\epsilon^{\epsilon^{v} \alpha}} \partial_{\mu} \\
& \cdot\left\{-\frac{1}{3} g^{2}\left(W_{v}^{+} \partial_{\alpha} W_{\beta}^{-}+W_{v}^{-} \partial_{\alpha} W_{\beta}^{+}\right)\right. \\
& -4 e^{2} Q_{f}^{2} A_{v} \partial_{\alpha} A_{\beta}-4\left(\frac{g}{\cos \theta_{W}}\right)^{2}\left(g_{V f}^{2}+\frac{1}{3} g_{A f}^{2}\right) \\
& \cdot Z_{v} \partial_{\alpha} Z_{\beta}+4 \frac{e g}{\cos \theta_{W}} g_{V f} Q_{f}\left(A_{v} \partial_{\alpha} Z_{\beta}+Z_{v} \partial_{\alpha} A_{\beta}\right) \\
& -i / 3\left[2 e g^{2} W_{v}^{+} W_{\alpha}^{-} A_{\beta}\right. \\
& \left.\left.-\frac{g^{3}}{\cos \theta_{W}}\left(3 / 2-2 \sin ^{2} \theta_{W}\right) W_{v}^{+} W_{\alpha}^{-} Z_{\beta}\right]\right\}, \tag{16}
\end{align*}
$$

where $J_{5 \mu}^{Z}$ is the axial-vector part of the current that couples to the $Z$ boson (2), and the sum is over all fermions including a factor of three due to the colour quantum number. Applying this equation to the $Z \gamma \gamma$ and $Z Z_{\gamma}$ Green's functions it is immediate to derive the following relations:

$$
\begin{align*}
& q_{\rho} G^{\rho \mu \nu}\left(q, k_{1}, k_{2}\right) \\
& =\sum_{f} \frac{e^{2} g}{2 \pi^{2} \cos \theta_{W}} g_{A f} Q_{f}^{2} \varepsilon^{\mu \nu \alpha \beta} k_{1 \alpha} k_{2 \beta}+\ldots \tag{17a}
\end{align*}
$$

and
$q_{1 \rho} T^{\rho \nu \mu}\left(q_{1}, q_{2}, k\right)$
$=-\sum_{f} \frac{e g^{2}}{2 \pi^{2} \cos \theta_{W}}-g_{A f} g_{V f} Q_{f} \varepsilon^{\mu \nu \alpha \beta} q_{2 \alpha} k_{\beta}+\ldots$,
where the dots on the right hand side stand for the normal contributions that we have discussed before. Again, (17) are in agreement with the results of [1]. Furthermore, from (16) we see that the anomaly associated with the triangle with three $Z$ bosons is
$q_{1 \rho} T^{\rho v \mu}\left(q_{1}, q_{2}, q_{3}\right)$
$=\sum_{f}\left(\frac{g}{\cos \theta_{W}}\right)^{3} \frac{1}{2 \pi^{2}} g_{A f}\left(g_{V f}^{2}+\frac{1}{3} g_{A f}^{2}\right) \varepsilon^{\nu \mu \alpha \beta} q_{2 \alpha} q_{3 \beta}$.
Recently, this Green's function was calculated by Boudjema [12] in the framework of perturbation theory using a dimensional regularization scheme. Or results, which are essentially non-perturbative, agree with theirs. In particular, notice the factor $1 / 3$ in front of the $g_{A f}$ term which correctly reproduces the perturbation theory factor associated with the triangle with three axial-vector currents [12, 13].

The remaining terms of (16) show that anomalies are also present in the $Z W^{+} W^{-}, Z W^{+} W^{-} \gamma$ and $Z Z W^{+} W^{-}$Green's functions. In all cases the cancellation due to the sum over fermions is clearly seen. However, if the standard $S U(2) \times U(1)$ model is an effective field theory [14], which may be valid up to the one-loop order, anomalies are phenomenologically allowed, if for exampley the top quark should be absent. Some of the one-loop effects such as in (17) and (18) may then give rise to testable results. For instance, the decay $Z \rightarrow e^{+} e^{-} \gamma$ will show a small deviation from the standard model prediction [1].

Summarizing, let us stress that the Ward identities that we derived here fully support the perturbative results of [1] and [12]. Hence, we confirm that, like a Majorana neutrino, the only electromagnetic moment of the $Z$ is the anapole moment.

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[^1]:    * Even more daring the same fact is used to constrain the dimension of manifolds in string theory

