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$D = 1 + 2$ Phase Diagram for Z_N with a Variational Method

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Abstract. Trial ground state are constructed which for Z_N gauge theories in $D = 1 + 2$ dimensions lead to phase diagrams which are in agreement both with theoretical expectations and Monte Carlo results. The nature of the phase transition is correctly predicted for each N.

1. Introduction

Lattice gauge theories [1] have by now proved to be a good way, perhaps the only one known, to study the nonperturbative aspects of quantum field theory. The ultimate goal of such a formulation is to obtain non-perturbative quantitative results for the continuum limit of the theory. At large couplings all lattice gauge theories are in a confinement phase. Therefore the first thing one has to do is to search for phase transitions when passing from the strong to the weak coupling regime.

This program has been carried out either using numerical methods, as in Monte Carlo simulations or using analytical tools such as strong coupling expansions, mean field techniques or variational methods. Although the Monte Carlo simulations [2] have by now confirmed many theoretically anticipated features of these models, they have the drawback, besides the need for very large machines, of not giving analytical control over the nature of the solutions.

In this paper we want to report on work done in Z_N lattice gauge theories in $d = 1 + 2$ dimensions within the Hamiltonian formulation [3]. Our trial ground states are of a similar form to those proposed in [4] for $U(1)$ and $SU(2)$ gauge theories.

Variational methods for lattice gauge theories have already been used in connection both with the Euclidean [5] and Hamiltonian formulations [6]. However for the trial functions that have been used matrix elements cannot in general be evaluated exactly and numerical computation implies a loss of analytical control over the results. If a trial ground state is to be useful to define a theory, matrix elements should have simple analytical forms and, at least for weak and strong coupling, variational parameters should display an explicit analytical dependence on the coupling constant.

The paper is organized as follows: In Sect. 2 we define the Z_N gauge theory. In Sect. 3 we define the trial ground states and calculate the matrix elements for the Hamiltonian. Section 4 presents the results for $N = 1, 3, 4$, and in Sect. 5 the results for $N > 4$ are discussed. Section 6 contains a brief summary.

2. The Z_N-Gauge Model

Z_N lattice gauge theories are defined in terms of the elements $U_{n,n+\hat{\mu}}$ of the gauge group. These U's live on the links of a hypercubical lattice and are parametrized by an integer (mod N) according to

$$U_{n,n+\hat{\mu}} \exp\left(i\frac{2\pi}{N} p_{\hat{\mu}}(n)\right), \; p_{\hat{\mu}}(n) = 0, 1, \ldots N-1 \qquad (2.1)$$

where the index n labels the sites and $\hat{\mu}$ is the unit vector in the $\hat{\mu}$ direction. The oriented product of group elements around a given plaquette $p \equiv (n, \hat{\mu}, \hat{\nu})$ is a gauge invariant quantity that for simplicity we denote by $B_N(p)$ or $U_N U_N U_N^* U_N^*(p)$ and it is defined by

$$B_N(p) \equiv U_N U_N U_N^* U_N^* \equiv U_{n,\hat{\mu}} U_{n+\hat{\mu},\hat{\nu}} U^*_{n+\hat{\nu},\hat{\mu}} U^*_{n,\hat{\nu}} \qquad (2.2)$$

where U^* is the complex conjugate of U, and the index N remainds us of which Z_N theory we are dealing with.

The Hamiltonian for the Z_N model is then given by [7]:

$$H_N = -\frac{1}{2}\sum_{\text{links}} [P_N(l) + P_N^+(l)] - \frac{\lambda}{2}\sum_{\text{plaq}} (B_N(p) + h.c) \qquad (2.3)$$

where the operators $P_N(l)$ live on the links and are defined through the following relations

$$(P_N(l))^N = 1,\ P_N^*(l)P_N(l) = 1$$

$$P_N^+(l)U_N(l)P_N(l) = e^{i\delta}U_N(l),\ \delta = \frac{2\pi}{N}$$

$$P_N^+(l)U_N^*(l)P_N(l) = e^{-i\delta}U_N^*(l) \tag{2.4}$$

For different links, $l \neq l'$ the $P_N(l)$ commute with the group elements $U_N(l')$.

3. Trial Ground States and Matrix Elements

Z_2

To explain more clearly our "ansätze" for the ground state we start with the Z_2 case. Then the Hamiltonian is

$$H = -\sum_{\text{links}} \sigma_1(l) - \lambda \sum_{\text{plaq}} \sigma_3\sigma_3\sigma_3\sigma_3(p). \tag{3.1}$$

The starting point it is then to notice that in the $\lambda = 0$ limit the ground state is an eigenstate of σ_1, that is $\sigma_1|0\rangle = |0\rangle$ where

$$|0\rangle \equiv \prod_{\text{links}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{2}} \end{pmatrix};\ \langle 0|0\rangle = 1. \tag{3.2}$$

As it has been shown [8] that the ground state of a lattice gauge theory must be gauge invariant we take for our trial ground state one that has already this property built-in. More specifically we choose

$$|\Omega\rangle = \prod_{\text{plaq}} (\alpha + \gamma\sigma_3\sigma_3\sigma_3\sigma_3)|0\rangle. \tag{3.3}$$

It we take our state to be normalized, $\langle\Omega|\Omega\rangle = 1$, we obtain (we are in $d = 2$ spatial dimensions)

$$|\alpha|^2 + |\gamma|^2 = 1 \tag{3.4}$$

This allows a simple interpretation for the variational parameter γ. In fact $|\gamma|^2$ represents the probability of having an excited flux loop in a given plaquette.

The parameter γ will be determined as function of λ by minimization of the ground state energy. Before we go into this, we have to find the matrix elements for the "electric" and "magnetic" part of the Hamiltonian in the ground state $|\Omega\rangle$. We have for the "electric" part

$$\begin{aligned}\langle\Omega|\sigma_1(l)|\Omega\rangle &= \langle 0|\prod_{p'}{}'(\alpha^* + \gamma^*\sigma_3\sigma_3\sigma_3\sigma_3(p')) \\ &\cdot(\alpha^* + \gamma^*\sigma_3\sigma_3\sigma_3\sigma_3(p_1))(\alpha^* + \gamma^*\sigma_3\sigma_3\sigma_3\sigma_3(p_2)) \\ &\cdot\sigma_1(l)(\alpha + \gamma\sigma_3\sigma_3\sigma_3\sigma_3(p_1)(\alpha + \gamma\sigma_3\sigma_3\sigma_3\sigma_3(p_2)) \\ &\cdot\prod_{p'}{}'(\alpha + \gamma\sigma_3\sigma_3\sigma_3\sigma_3(p'))|0\rangle\end{aligned} \tag{3.5}$$

where p_1 and p_2 are the two plaquettes that share link l and π' means product over all plaquettes except for plaquettes p_1 and p_2. We get then

$$\begin{aligned}&\langle\Omega|\sigma_1(l)|\Omega\rangle = \langle 0|(\alpha^* + \gamma^*\sigma_3\sigma_3\sigma_3\sigma_3(p_1)) \\ &\cdot(\alpha^* + \gamma\sigma_3\sigma_3\sigma_3\sigma_3(p_2)) \\ &(\alpha - \gamma\sigma_3\sigma_3\sigma_3\sigma_3(p_1))(\alpha - \gamma\sigma_3\sigma_3\sigma_3\sigma_3(p_2))|0\rangle \\ &= (|\alpha|^2 - |\gamma|^2)^2 \\ &= (1 - 2|\gamma|^2)^2\end{aligned} \tag{3.6}$$

where use was made of the normalization condition, (3.4).

For the "magnetic" part, we have for a given plaquette p

$$\begin{aligned}&\langle\Omega|\sigma_3\sigma_3\sigma_3\sigma_3(p)|\Omega\rangle = \langle|\prod_{p'}{}'(\alpha^* + \gamma^*\sigma_3\sigma_3\sigma_3\sigma_3(p')) \\ &\cdot(\alpha^* + \gamma^*\sigma_3\sigma_3\sigma_3\sigma_3(p))(\sigma_3\sigma_3\sigma_3\sigma_3(p)) \\ &\cdot(\alpha + \gamma\sigma_3\sigma_3\sigma_3\sigma_3(p))\prod_{p''}{}'(\alpha + \gamma\sigma_3\sigma_3\sigma_3\sigma_3(p''))|0\rangle\end{aligned} \tag{3.7}$$

where π' means the product over all plaquettes except for plaquette p. Using the normalization condition (2.4) we get

$$\langle\Omega|\sigma_3\sigma_3\sigma_3\sigma_3(p)|\Omega\rangle = \alpha^*\gamma + \alpha\gamma^* = 2\,Re(\alpha\gamma^*) \tag{3.8}$$

The matrix elements, (3.6) and (3.8) will be used in the next section to get the ground state energy to be minimized. Now we will generalize these results to an arbitrary Z_N.

$Z_N (N > 2)$

Our trial ground states for Z_N have a small difference depending on whether N is even or odd. We take

$$\begin{aligned}|\Omega\rangle &= \prod_{\text{plaq}}\left(\alpha + \sum_{r=1}^{[N/2]} \frac{\gamma_r}{\sqrt{2}}(B_N^r + B_N^{r*})\right)|0\rangle;\ N\ odd \\ |\Omega\rangle &= \prod_{\text{plaq}}\left(\alpha + \sum_{r=1}^{N/2-1} \frac{\gamma_2}{\sqrt{2}}(B_N^r + B_N^{r*}) + \gamma_{N/2}B_N^{N/2}\right)|0\rangle; \\ &N\ even\end{aligned} \tag{3.9}$$

where $B_N^r \equiv (U_N U_N U_N^* U_N^*)^r$ is the $\mathbf{r}$ power of the oriented product of the group elements around a plaquette. One can easily see that this definition coincides with the one given before for $N = 2$.

The normalization condition $\langle\Omega|\Omega\rangle = 1$ gives now

$$|\alpha|^2 + \sum_{r=1}^{[N/2]} |\gamma_r|^2 = 1 \tag{3.10}$$

both for N odd and N even.

The electric matrix elements $\langle\Omega|(P_N(l) + P_N^+(l))|\Omega\rangle$ can be easily calculated if we notice that for each plaquette that contains the link l we have (we take the case N odd for simplicity)

$$P_N^+(l)\left(\alpha + \sum_{r=1}^{[N/2]} \frac{\gamma_r}{\sqrt{2}}(B_N^r + B_N^{r*})\right)|0\rangle$$
$$= \left(\alpha + \sum_{r=1}^{[N/2]} \frac{\gamma_r}{\sqrt{2}}(e^{i\delta r} B_N^r + e^{-i\delta r} B_N^{r*})\right)|0\rangle \quad (3.11)$$

Using this result one can easily get

$$\tfrac{1}{2}\langle\Omega|(P_N^+(l) + P_N(l))|\Omega\rangle = \left(|\alpha|^2 + \sum_{r=1}^{[N/2]} |\gamma_r|^2 \cos(\delta r)\right)^2$$
$$= \left(1 = \sum_{r=1}^{[N/2]} |\gamma_r|^2 K_r\right)^2 \quad (3.12)$$

where use was made of the normalization (3.10), and we have defined

$$K_r \equiv 1 - \cos(\delta r) \quad (3.13)$$

For the magnetic energy, we begin by noticing that the product by $(B_N + B_N^*)$ amount to a redefinition of the α and $\gamma_r's$ of that plaquette. In fact, for a given plaquette we have (N odd)

$$\tfrac{1}{2}(B_N + B_N^*)\left(\alpha + \sum_{r=1}^{[N/2]} \frac{\gamma_r}{\sqrt{2}}(B_N^r + B_N^{r*})\right)$$
$$= \left(\hat{\alpha} + \sum_{r=1}^{[N/2]} \frac{\hat{\gamma}_r}{\sqrt{2}}(B_N^r + B_N^{r*})\right) \quad (3.14)$$

where

$$\hat{\alpha} = \frac{1}{\sqrt{2}}\gamma_1$$
$$\hat{\gamma}_1 = \frac{1}{\sqrt{3}}\alpha + \tfrac{1}{2}\gamma_2$$
$$\vdots$$
$$\hat{\gamma}_{[N/2]-1} = \tfrac{1}{2}\gamma_{[N/2]-2} + \tfrac{1}{2}\gamma_{[N/2]}$$
$$\hat{\gamma}_{[N/2]} = \tfrac{1}{2}\gamma_{[N/2]-1} + \tfrac{1}{2}\gamma_{[N/2]} \quad (3.15)$$

For N even we have a similar result:

$$\tfrac{1}{2}(B_N + B_N^*)\left(\alpha + \sum_{r=1}^{N/2-1} \frac{\gamma_r}{\sqrt{2}}(B_N^r + B_N^{r*}) + \gamma_{N/2} B_N^{N/2}\right)$$
$$= \left(\hat{\alpha} + \sum_{r=1}^{N/2-1} \frac{\hat{\gamma}_r}{\sqrt{2}}(B_N^r + B_N^{r*}) + \hat{\gamma}_{N/2} B_N^{N/2}\right) \quad (3.16)$$

where

$$\hat{\alpha} = \frac{1}{\sqrt{2}}\gamma_1$$
$$\hat{\gamma}_1 = \frac{1}{\sqrt{2}}\alpha + \tfrac{1}{2}\gamma_2$$
$$\vdots$$
$$\hat{\gamma}_{N/2-1} = \tfrac{1}{2}\gamma_{N/2-2} + \frac{1}{\sqrt{2}}\gamma_{N/2}$$
$$\hat{\gamma}_{N/2} = \frac{1}{\sqrt{2}}\gamma_{N/2-1} \quad (3.17)$$

With these results (3.14–17) one can easily obtain the magnetic energy matrix elements

$$\tfrac{1}{2}\langle\Omega|(B_N + B_N^*)|\Omega\rangle = \alpha^*\hat{\alpha} + \sum_{r=1}^{[N/2]} \gamma_r^*\hat{\gamma}_r \quad (3.18)$$

The parameters $\gamma_r (r = 1, \ldots [N/2])$ are to be obtained through the minimization of the ground state energy. More specifically we define a function F_N in the following way

$$\langle\Omega|H_N|\Omega\rangle \equiv \frac{1}{a_{\text{links}}} \sum F_N(\gamma, \lambda) \quad (3.19)$$

Minimization of $F_N(\gamma, \lambda)$ will determine the functions $\gamma_r = \gamma_r(\lambda)$. Substitution of $\gamma_r(\lambda)$ into $F_N(\gamma_r, \lambda)$ will give the ground state energy functional $F_N^{\text{Min}}(\lambda)$. Phase transitions will be revealed by discontinuities in the derivatives of $F_N^{\text{Min}}(\lambda)$.

4. Results for Z_2, Z_3 and Z_4

Z_2

Using the results of the previous section the function to be minimized is

$$F_2 = -(\alpha^2 - \gamma^2)^2 - \lambda\alpha\gamma \quad (4.1)$$

where we used the result, that can be easily proved, that the parameters α, γ are to be taken real to get the minimum of F_2. Due to the normalization condition, it is better to choose another parametrization given by an angle θ,

$$\alpha = \cos\theta/2; \gamma = \sin\theta/2 \quad (4.2)$$

In terms of this parameter θ, the function F_2 is

$$F_2 = -\cos^2\theta - \frac{\lambda}{2}\sin\theta \quad (4.3)$$

The minimum of F_2 depends on the value of λ. We get a minimum for the following values of θ:

$$\begin{cases} \sin\theta = \dfrac{\lambda}{4} & \text{for } \lambda \leqq 4 \\ \theta = \dfrac{\pi}{2} & \text{for } \lambda \geqq 4 \end{cases} \quad (4.4)$$

which in turn give for the ground state energy per link, $F_2^{\text{Min}}(\lambda)$ the following result

$$F_2^{\text{Min}}(\lambda) = \begin{cases} -1 - \dfrac{\lambda^2}{16} & \lambda \leqq 4 \\ -\dfrac{\lambda}{2} & \lambda \geqq 4 \end{cases} \quad (4.5)$$

The function $F_2^{\min}(\lambda)$ and its first derivative $dF_2^{\min}/d\lambda$ are represented in Fig. 1. We see that both $F_2^{\min}(\lambda)$ and its first derivative are continuous, but the second derivative has a discontinuity at $\lambda_c^{(2)} = 4$, indicating

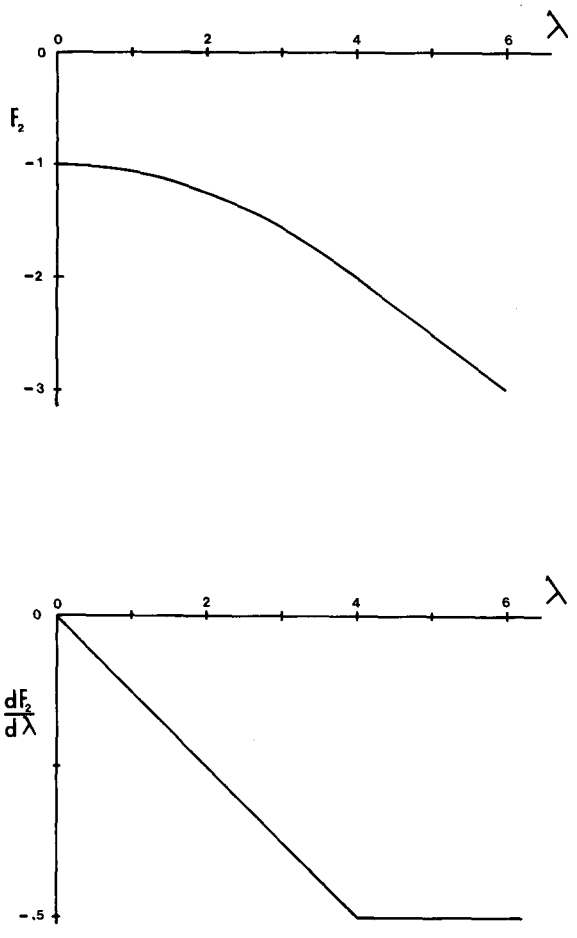

Fig. 1. The ground state energy functional $F_2^{\mathrm{Min}}(\lambda)$ and its first derivative for Z_2 gauge theory in $d = 1 + 2$ dimensions

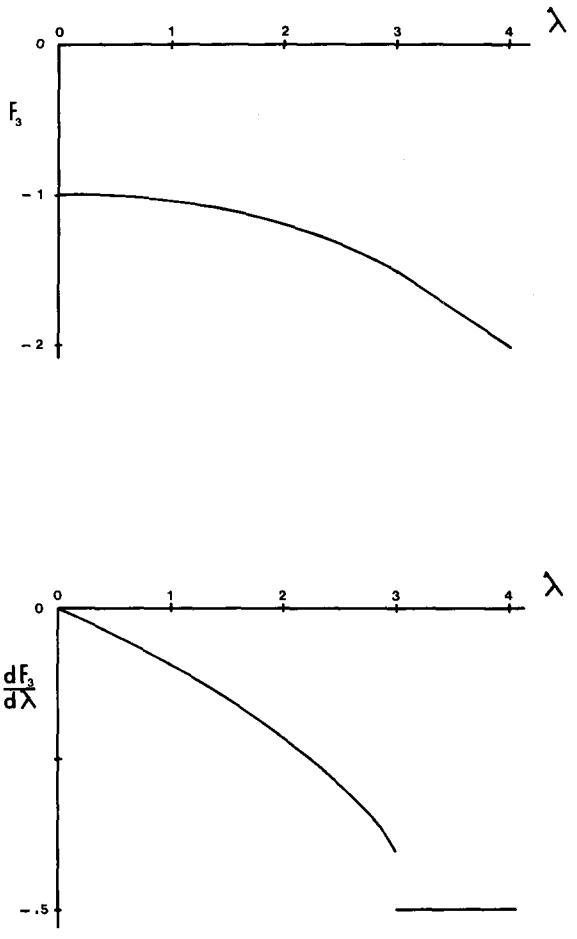

Fig. 2. The ground state energy functional $F_3^{\mathrm{Min}}(\lambda)$ and its first derivative for the Z_3 gauge theory in $d = 1 + 2$ dimensions

a second order phase transition at this value of λ. This is in good agreement with known results either from Monte Carlo [9] or from other analytical calculations [10, 11]. We remark that our results are very similar to those of another variational calculation [10] and were obtained with much less effort.

Z_3

The ground state energy per link to be minimized is given by

$$F_3 = -(1 - \tfrac{3}{2}x)^2 - \frac{\lambda}{2}(\sqrt{2}\sqrt{x - x^2} + \tfrac{1}{2}x) \tag{4.6}$$

where we have defined the variable x by

$$x = \gamma^2 \tag{4.7}$$

The minimum condition can be used to get the function $x = x(\lambda)$. This can be done analytically but as $x(\lambda)$ is one root of a cubic algebraic equation the expression it is a bit complicated. We prefer to present our results in graphic form. In Figs. 2a and b we show, respectively, the ground state functional $F_3^{\mathrm{Min}}(\lambda)$ and its first derivative $dF_3^{\mathrm{Min}}(\lambda)/d\lambda$. We see that there is a discontinuity in the derivative revealing a first order phase transition at $\lambda_c^{(3)} = 3$. The nature of the transition it is in good agreement with Monte Carlo results [9, 12] that show that Z_3 has a peculiar first order phase transition.

Z_4

For $N = 4$ we have for the ground state energy per link

$$F_4 = -(1 - x_1 - 2x_2)^2 - \frac{\lambda}{2}\sqrt{2}(\sqrt{x_1}\sqrt{1 - x_1 - x_2} + \sqrt{x_1 x_2}) \tag{4.8}$$

where we defined $x_1 = \gamma_1^2$ and $x_2 = \gamma_2^2$.

Minimization can be done easily and one gets

$$\begin{cases} \sqrt{x_1} = \dfrac{1}{\sqrt{2}}\dfrac{\lambda}{4} \\ \sqrt{x_2} = \dfrac{1}{2}\left(1 - \sqrt{1 - \dfrac{\lambda^2}{16}}\right) \end{cases} \quad \text{for } \lambda \leqq 4 \tag{4.9}$$

and

$$\begin{cases} \sqrt{x_1} = \dfrac{1}{\sqrt{2}} \\ \qquad\qquad \text{for } \lambda \geqq 4 \\ \sqrt{x_2} = \frac{1}{2} \end{cases} \tag{4.10}$$

Substitution of these values in (4.8) gives

$$F_4^{\text{Min}}(\lambda) \begin{cases} -1 - \dfrac{\lambda^2}{16} & \lambda \leqq 4 \\ -\dfrac{\lambda}{2} & \lambda \geqq 4 \end{cases} \tag{4.11}$$

Which coincides exactly with $F_2^{\text{Min}}(\lambda)$. Therefore we have a second order phase transition at $\lambda_c^{(4)} = 4$. This is in agreement with a known relation between the Z_2 and Z_4 gauge models in $d = 1 + 2$ dimensions. In fact it is known that for this case there is a duality between Z_N gauge models and Z_N spin systems [13]. It has been shown [14] that in $d = 1 + 2$ the Z_4 spin theory factorizes into two independent spin systems and this gives by duality a relation between the Z_2 and Z_4 gauge theories. In the Apendice we shown that even with a simpler one parameter ansätze, we can still recover the same relation between Z_2 and Z_4.

5. Results for $Z_N (N>4)$

The general expression of the ground state energy functional is

$$F_N = -\left(1 - \sum_{r=1}^{[N/2]} K_r \gamma_r^2\right)^2 - \frac{\lambda}{2}\left(\alpha\hat{\alpha} + \sum_{r=1}^{[N/2]} \gamma_r \hat{\gamma}_r\right) \tag{5.1}$$

with all quantities as defined in Sect. 2. In Fig. 3, we show the results for a typical case, $N = 6$. In fact the results all have the same structure and can be summarized as follows:

i) There is a phase transition for each N.

ii) This phase transition is always second order, that is the second derivative of $F_N^{\text{Min}}(\lambda)$ is discontinuous.

iii) The critical value of λ at which the phase transition takes place can be given by a very simple expression.

$$\lambda_c^{(N)} \frac{4}{1 - \cos(2\pi/N)} \tag{5.2}$$

valid for $N \geq 4$.

These results are in very good agreement with what is known for Z_N is $d = 1 + 2$ dimensions. In fact for these theories it is expected a second order phase transition [9]. It is also known [7] that for large N we get $\lambda_c \sim N^2$ which is in agreement with the simple result (5.2).

To compare our results with known data, we can take the large N limit. Then we can have two limiting

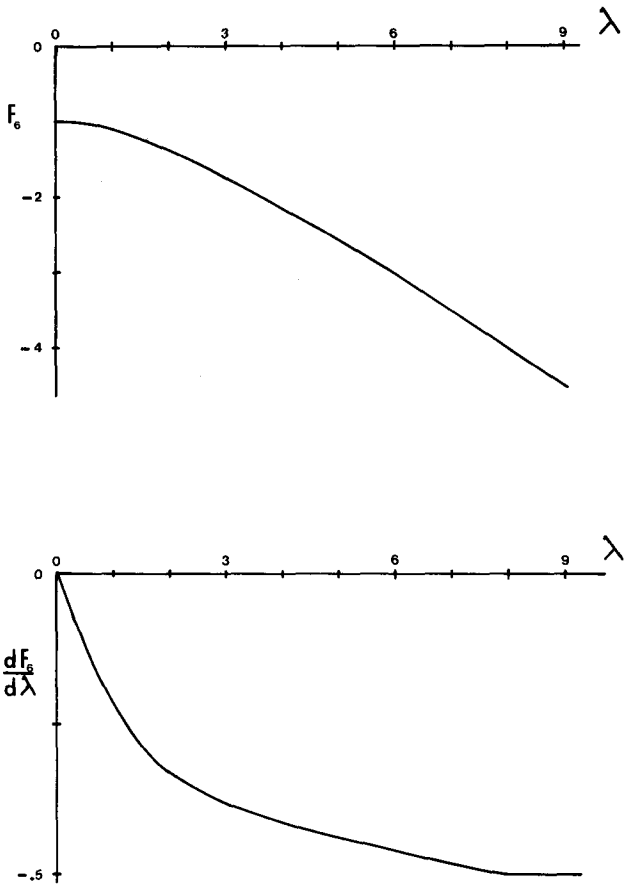

Fig. 3. The ground state energy functional $F_N^{\text{Min}}(\lambda)$ and its first derivative for Z_6 gauge theory in $d = 1 + 2$ dimensions

situations:

i) $\lambda \to \infty$, but the ratio $K^2 = (4\pi^2)(\lambda/N^2)$ remains finite. Then the Z_N gauge theory is dual to a $X - Y$ spin model [7] defined by the Hamiltonian

$$H_D = \sum_{\langle pq \rangle} [1 - \cos(\phi_p - \phi_q)] + \tfrac{1}{2} K^2 \sum_p L_p^2 \tag{5.3}$$

As it is well known [15], this model has two phases: At $K^2 \ll 1$ the system is in an ordered phase while at $K^2 \gg 1$ we have a disordered phase. This is in agreement with our results. In fact for large N we have from (5.2)

$$\lambda_c^{(N)} \simeq \frac{2}{\pi^2} N^2 \tag{5.4}$$

and therefore

$$K^2 = 8 \frac{\lambda}{\lambda_c^{(N)}} \tag{5.5}$$

Thus for $\lambda \gg \lambda_c$ we have $K^2 \gg 1$ corresponding to a disordered phase in the spin model and to an ordered phase in the dual Z_N gauge theory, which is precisely our result.

For $\lambda \ll \lambda_c$ one should have $K^2 \ll 1$ corresponding to an ordered phase in the $X - Y$ model and to a

disordered phase in the Z_N gauge theory. Our variational approach predicts then a value of K^2 at which the phase transition takes place in the $X - Y$ model. This is for $K_c^2 \sim 8$.

ii) $\lambda \to 0$ but the ratio $g^2 = 1/\beta = (2\pi/N)(1/\sqrt{\lambda})$ remains finite. Then the Z_N gauge theory ($N \to \infty$) is equivalent to PQED [7]. As the only critical value for the Z_N gauge theory grows with N^2 it is clear that in the g^2 variable the critical value goes to zero. Therefore there exists only one phase in PQED in $d = 1 + 2$ dimensions extending from the strong coupling to the weak coupling. This is in agreement with known results [16].

To compare quantitatively the critical values $\lambda_c^{(N)}$ with the Monte Carlo $\beta_c^{(N)}$ [9], seems to be difficult because the Monte Carlo simulations were done in the Euclidean formulation while our results are obtained in the Hamiltonian formalism.

Despite this fact, if we take the expression

$$\beta = \sqrt{\lambda}\frac{N}{2\pi} \tag{5.6}$$

outside its range of validity, that is, for large N but *not* small λ, our results for $\lambda_c^{(N)}$ and those of $\beta_c^{(N)}$ from [9] agree within 15%. Because we are taking (5.6) outside the domain where it was derived, this agreement may just be an accident.

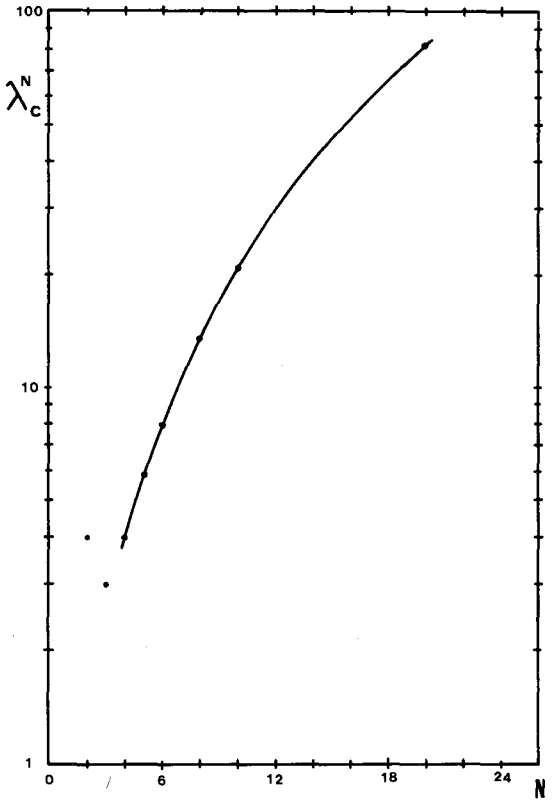

Fig. 4. The critical values $\lambda_c^{(N)}$ for Z_N gauge theories in $d = 1 + 2$ dimensions as a function of N. The curve represents the function $4/(1 - \cos(2\pi/N))$

6. Summary

We have presented the results of a variational calculation for Z_N gauge theories in $d = 1 + 2$ dimensions, showing that there is a single order-disorder transition at a finite value of the coupling λ. The critical values λ_c^N are plotted in Fig. 4, λ_c^N moves to $\lambda = \infty$ as N^2 for large N. We have shown that this implies that in $d = 1 + 2$ dimensions the $U(1)$ theory has only one phase. We have also shown that our results are in agreement with what it is expected, for large N, from the duality between Z_N gauge theories and the $X-Y$ spin model, that is known to have a two phase structure

Appendix

Comparison between Z_2 and Z_4 gauge theories. We want to show that the Z_2 and Z_4 gauge theories are related, having the same ground state energy functional $F_{2,4}^{\text{Min}}(\lambda)$ (for $d = 1 + 2$ dimensions).

To do this we choose an ansätze of type (2.9) but just with one variational parameter. We write

$$|\Omega\rangle = \prod_{\text{plaq}} \exp\left\{\frac{\lambda}{4}[B_N + B_N^*]\right\} \times \frac{1}{\sqrt{Z(\gamma)}} \tag{A.1}$$

where B_N have the same meaning as before. This form is a particular case of (2.9) for a well defined relation among the various $\gamma_r's$.

The function $Z(\gamma)$ is defined by the normalization condition $\langle\Omega|\Omega\rangle = 1$, and for $d = 1 + 2$ we get easily

$$Z(\gamma) = I_0(\gamma) + 2\sum_{k=1}^{\infty} I_{kN}(\gamma). \tag{A.2}$$

where $I_i(\gamma)$ are the modified Bessel functions of i^{th} order.

To get the ground state energy one needs to know the matrix elements for the electric and magnetic part of the Hamiltonian. We have for the magnetic part for a given plaquette p

$$\begin{aligned}&\tfrac{1}{2}\langle\Omega|(B_N(p) + \text{h.c})|\Omega\rangle \\ &= \langle 0|\prod_{p' \neq p}\frac{1}{Z(\gamma)} e^{(\gamma/2)(B_N(p') + \text{h.c})} \\ &\cdot\frac{(B_N(p) + \text{h.c.})}{2Z(\gamma)} e^{(\gamma/2)(B_N(p) + \text{h.c.})}|0\rangle = \frac{1}{Z(\gamma)}\frac{dZ}{d\gamma}\end{aligned} \tag{A.3}$$

For the electric matrix elements one uses the commutation relations (2.4) to get

$$\begin{aligned}&P_N(l)\prod_p e^{(\gamma/4)(B_N(p) + \text{h.c.})} \\ &= \prod_{\substack{p \\ l \notin p}} e^{(\gamma/4)(B_N(p) + \text{h.c.})} \prod_{\substack{p' \\ l \in p'}} e^{(\gamma/4)(e^{-i\delta}B_N(p) + \text{h.c.})} P_N(l)\end{aligned}$$

and

$$P_N^+(l)\prod_p e^{(\gamma/4)(B_N(p)+\text{h.c.})}$$
$$= \prod_{\substack{p \\ l\notin p}} e^{(\gamma/4)(B_N(p)+\text{h.c.})} \prod_{\substack{p' \\ l\in p'}} e^{(\gamma/4)(e^{i\delta}B_N+\text{h.c.})} P_N^+(l) \qquad (A.4)$$

with these expressions one finally gets

$$\tfrac{1}{2}\langle\Omega|(P_N(l)+P_N^+(l))|\Omega\rangle$$
$$= \langle 0|\prod_{\substack{p \\ l\notin p}} \frac{e^{(\gamma/2)(B_N+\text{h.c})}}{Z(\gamma)} \prod_{\substack{p' \\ l\in p'}} \frac{e^{(\gamma/2)|W|(Q_N+Q_N^+)}}{Z(\gamma)} |0\rangle$$
$$= \left(\frac{E(\gamma)}{Z(\gamma)}\right)^2 \qquad (A.5)$$

where

$$E(\gamma) = I_0(\gamma|W|) + 2\sum_{k=1}^{\infty}(-1)^k I_{KN}(\gamma|W|) \qquad (A.6)$$

and $W = \cos\delta/2$, $\delta = 2\pi/N$.

Therefore with the one parameter ansätze of (A.1) one gets for all N the following expression to be minimized

$$F_N = -\left(\frac{E(\gamma)}{Z(\gamma)}\right)^2 - \frac{\lambda}{2}\frac{1}{Z(\gamma)}\frac{dZ}{d\gamma} \qquad (A.7)$$

One can check this expression for the cases $N = 2$ and $N = 3$ where it should give expressions equivalent to (3.1) and (3.6) because for these values of N the ansätze (2.9) is also a one parameter ansätze. This can easily be done using the properties of the modified Bessel functions.

We do not give the details but the method is the same that we are going to use for $N = 4$. For this value of N we get

$$Z(\gamma) = I_0(\gamma) + 2I_4(\gamma) + 2I_8(\gamma) + \cdots$$
$$= \frac{1+\cosh\gamma}{2} = \cosh^2(\gamma/2) \qquad (A.8)$$

and

$$E(\gamma) = I_0(\gamma/\sqrt{2}) - 2I_4(\gamma/\sqrt{2}) + 2I_8(\gamma/\sqrt{2}) + \cdots$$
$$= \cosh(\gamma/2) \qquad (A.9)$$

where we have used some properties of the modified Bessel functions [17]. We therefore get

$$F_4 = -\left(\frac{1}{\cosh(\gamma/2)}\right)^2 - \frac{\lambda}{2}\tanh\gamma/2 \qquad (A.10)$$

A simple change of variables

$$\tanh\gamma/2 = \text{Sin}\,\theta \qquad (A.11)$$

gives

$$\frac{1}{\cosh^2(\gamma/2)} = 1 - \tanh^2\gamma/2 = 1 - \sin^2\theta = \cos^2\theta \qquad (A.12)$$

and we get for F_4 in this one parameter approximation exactly the same expression (3.3) that one gets for Z_2 Minimization will therefore give the same result.

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