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Superplane Integrability and Three Dimensional Supergravity

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Abstract. The requirement of integrability in all light-like superplanes yields the proper kinematical constraints for the three-dimensional supergravity theory formulated in superspace. The theory with the Breitenlohner set of auxiliary fields is obtained.

1. Introduction

When supergravity [1] theories are formulated in superspace [2–5] a certain number of constraints has to be imposed on the torsion or curvature coefficients.

It has recently been suggested [6] that the so-called "kinematical" constraints could be viewed as integrability conditions on light-like superplanes. This simple geometrical ideal proved to work well both for Supersymmetric Yang Mills theories and for the pure $(N=1)$ supergravity theory. In this last case, the constraints obtained from Superplane Integrability were those corresponding to the Breitenlohner set of auxiliary fields [7].

In this paper, we show that superplane integrability also gives the correct kinematical constraints for the three dimensional supergravity formulated in superspace. Again we obtain a formulation with the Breitenlohner set of auxiliary fields.

The paper is organized as follows. In Sect. 2 we derive the kinematical constraints from superplane integrability. In Sect. 3 we solve these constraints in terms of the physical and auxiliary fields. The "Breitenlohner set" of auxiliary fields is found in Sect. 4 and contact is made with the auxiliary field structure presented in Sect. 3. A short discussion of the results is given in Sect. 5. Our notation and other useful formulae are collected in Appendix A, while in Appendix B a derivation is given of the vector super multiplet for two and three dimensional spacetimes.

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2. The Kinematical Constraints

As a Majorana spinor in three dimensions has two independent components, we embed the three dimensional spacetime in a five dimensional superspace.

The geometry of superspace is described [2] by the vielbein E_M^A and the Lie algebra valued connection $\Phi_{M,A}{}^B$. The local tangent space group is chosen [2] to be a Lorentz group operating in the usual way on the vector and spinor tangent space indices. Therefore, the infinitesimal generators $\hat{X}_A{}^B$ will have the form

$$\hat{X}_A{}^B = \left(\begin{array}{c|c} \varepsilon_a{}^{bc} L_c & 0 \\ \hline 0 & \frac{1}{2}(\gamma^c)_\alpha{}^\beta L_c \end{array}\right) . \tag{2.1}$$

Our index conventions are the same as in [6]. The conventions for the γ-matrices and other useful formulae are given in Appendix A. The inverse of the vielbein, $E_A{}^M$, gives the covariant derivatives $D_A \equiv E_A{}^M D_M$. Their (graded) bracked is given by the well-known formula [8]

$$[D_A, D_B] = -T_{AB}{}^C D_C + R_{AB},{}^{CD}\hat{X}_{DC} \tag{2.2}$$

where we introduced the torsion coefficients $T_{AB}{}^C$ and the curvature coefficients $R_{AB,CD}$.

The connections $\Phi_{M,A}{}^B$ and the curvature $R_{CD,A}{}^B$ are Lie algebra valued and can therefore be written as follows

$$\begin{aligned} &R_{CD,ab} \equiv \varepsilon_{ab}{}^e R_{CD,e} \\ &R_{CD,\alpha\beta} \equiv \tfrac{1}{2}(\gamma^e)_{\alpha\beta} R_{CD,e} \\ &\Phi_{M,ab} \equiv \varepsilon_{ab}{}^e \Phi_{M,e} \\ &\Phi_{M,\alpha\beta} \equiv \tfrac{1}{2}(\gamma^e)_{\alpha\beta}\Phi_{M,e} \end{aligned} \tag{2.3}$$

We now apply the idea of Superplane Integrability [6] to this case. Given any c-number Majorana spinor ρ^α we form a light-like vector r^a

$$r^a \equiv \rho^\alpha(\gamma^a)_\alpha{}^\beta \rho_\beta\,;\ r^a r_a = 0 \tag{2.4}$$

Corresponding to the null directions ρ^α and r^a we have a pair of tangent space directions

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$$D \equiv r^a D_a; \quad Q \equiv \rho^\alpha D_\alpha \tag{2.5}$$

(D_a and D_α are the components of D_A). Every Majorana spinor ρ^α determines through (2.5) a tangent "Superplane". Integrability on these superplanes gives [6]

$$[D_\alpha, D_\beta] = -(\gamma^a)_{\alpha\beta} D_a \tag{2.6}$$

for the graded bracket of two fermionic covariant derivatives.

Comparison between (2.2) and (2.6) implies the following kinematical constraints

$$T_{\alpha\beta}{}^c = (\gamma^c)_{\alpha\beta} \tag{2.7a}$$
$$T_{\alpha\beta}{}^\gamma = 0 \tag{2.7b}$$
$$R_{\alpha\beta,c} = 0 \tag{2.7c}$$

One can use constraints (2.7 a–c) plus the Bianchi identities (see Appendix A) to get further relations for other torsion and curvature components. A straightforward calculation gives

$$T_{a\beta}{}^D = (\gamma_a)_\beta{}^\alpha G_\alpha{}^D \tag{2.8a}$$
$$R_{a\beta,d} = (\gamma_a)_\beta{}^\alpha H_{\alpha,d} \tag{2.8b}$$

where $G_\alpha{}^D$ and $H_{\alpha,d}$ are arbitrary superfields.

As we do not obtain $T_{a\beta}{}^c = 0$, we are not going to get the minimal component theory as in [9]. In fact we will show that the theory that corresponds to the choice of constraints (2.7) is supergravity with the Breitenlohner set of auxiliary fields.

3. Solving the Constraints

The set of constraints (2.7 a–c) can be solved. By this we mean that some of the components of the vielbein $E_M{}^A$ (or its inverse $E_A{}^M$) and of the connection $\Phi_{A,b}(\Phi_{A,b} \equiv E_A{}^M \Phi_{M,b})$ can be written in terms of other components of the vielbein and/or connection, thus reducing the number of independent components. We will do this in two different ways.

We begin by looking at (2.7b). Using the expressions of Appendix A we can write

$$0 = -C_{\alpha\beta}{}^\gamma - \Phi_{(\alpha,\beta)}{}^\gamma \tag{3.1}$$

where

$$C_{\alpha\beta}{}^\gamma = \partial_{(\alpha} E_{\beta)}{}^M E_M{}^\gamma \tag{3.2}$$

Now, one can write (3.1) in the form

$$\tfrac{1}{2}\Phi_{\alpha,d}(\gamma^d)_\beta{}^\gamma + \tfrac{1}{2}\Phi_{\beta,d}(\gamma^d)_\alpha{}^\gamma = -C_{\alpha\beta}{}^\gamma.$$

This last expression can be easily solved for $\Phi_{\alpha,d}$ with the result

$$\Phi_{\alpha,d} = -\tfrac{1}{2}(\gamma_d)_\gamma{}^\beta C_{\alpha\beta}{}^\gamma - \tfrac{1}{2}(\gamma_d)_\alpha{}^\beta C_{\beta\delta}{}^\delta \tag{3.3}$$

This solves constraint (2.7b) by expressing $\Phi_{\alpha,d}$ as a function of the vielbein. Now we look at (2.7c). It can be written as

$$0 = \partial_{(\alpha}\Phi_{\beta),c}{}^d + (\gamma^e)_{\alpha\beta}\Phi_{e,c}{}^d + \Phi_{(\alpha,\beta)}{}^\gamma \Phi_{\gamma,c}{}^d + \Phi_{(\alpha,c}{}^e \Phi_{\beta),e}{}^d \tag{3.4}$$

A straightforward calculation gives

$$\Phi_{a,b} = (\gamma_a)^{\alpha\beta}\left[\partial_\alpha \Phi_{\beta,d} + \tfrac{1}{2}(\gamma^d)_\beta{}^\gamma \Phi_{\alpha,d}\Phi_{\gamma,b} - 2\Phi_{\alpha,c}\Phi_{\beta,d}\varepsilon^{cd}{}_b\right] \tag{3.5}$$

Therefore (2.7 b–c) completely determine the connection $\Phi_{A,d}$ in terms of the vielbein (and its inverse). The vielbein itself is not completely arbitrary because constraint (2.7a) implies

$$(\gamma^c)_{\alpha\beta} = -\partial_{(\alpha} E_{\beta)}{}^M E_M{}^c \tag{3.6}$$

We succeeded in reducing the independent superfields to just the vielbein $E_M{}^A$. However, the vielbein still contains far too many fields. We can use its gauge freedom

$$\delta E_M{}^A = \xi^N \partial_N E_M{}^A + \partial_M(\xi^N) E_N{}^A + E_M{}^B X_B{}^A,$$

where $X_B{}^A(z)$ are the parameters of the tangent space Lorentz transformations and $\xi^M(z)$ are the parameters of general coordinate transformations in superspace, to gauge away most of the superfluous fields appearing in the vielbein. This can be done while still preserving normal (three dimensional) spacetime, general coordinate transformations of parameter $f^m(x)$, supersymmetry transformations of parameter $\varepsilon^\mu(x)$ and local Lorentz transformations of parameter $\varepsilon^{abc} l_c(x)$, where $f^m(x), \varepsilon^\mu(x)$ and $l_c(x)$ are the θ-independent pieces of the superfields $\xi^m(z), \xi^\mu(z)$ and $L_c(z)$ respectively. After doing this, the fields that could not be eliminated from $E_M{}^A$ should be the physical fields, plus the auxiliary fields.

Instead of carrying out this programme, we prefer to make contact with the work of Gates [10] and use some of his results. To this end, he will solve the constraints (2.7 a–c) in a different although equivalent way.

We begin by solving (2.7c) exactly as before. We obtain again (3.5), that is, the connection $\Phi_{a,b}$ can be expressed in terms of $\Phi_{\alpha,b}$ and $E_\alpha{}^M(\partial_\alpha \equiv E_\alpha{}^M \partial_M)$. Knowing $\Phi_{\alpha,b}$ and $E_\alpha{}^M$ one can construct the spinorial covariant derivative

$$D_\alpha = E_\alpha{}^M \partial_M - \tfrac{1}{2}\Phi_{\alpha,cd}\hat{X}^{cd} = E_\alpha{}^M \partial_M - \tfrac{1}{2}\Phi_{\alpha,b}\varepsilon_{cd}{}^b \hat{X}^{cd} \tag{3.7}$$

Now, constraints (2.7 a–c) imply (see 2.6)

$$[D_\alpha, D_\beta] = -(\gamma^c)_{\alpha\beta} D_c \tag{3.8}$$

Equations (3.7) and (3.8) imply then the relation

$$-(\gamma^c)_{\alpha\beta} E_c{}^N = E_{(\alpha}{}^M \partial_M E_{\beta)}{}^N + \tfrac{1}{2}\Phi_{\beta,b}(\gamma^b)_\alpha{}^\delta E_\delta{}^N + \tfrac{1}{2}\Phi_{\alpha,b}(\gamma^b)_\beta{}^\delta E_\delta{}^N \tag{3.9}$$

which can be solved for $E_c{}^N$

$$E_c{}^N = (\gamma_c)^{\alpha\beta}\left[E_\alpha{}^M \partial_M E_\beta{}^N + \tfrac{1}{2}\Phi_{\alpha,b}(\gamma^b)_\beta{}^\gamma E_\gamma^N\right] \tag{3.10}$$

We conclude, therefore, that instead of taking $E_A{}^M$ as independent fields, we can take $E_\alpha{}^M$ and $\Phi_{\alpha,b}$ and express E_a^M and $\Phi_{a,b}$ as functions of them through (3.5) and (3.10).

As it was shown by Gates [10], we can use the gauge invariance of the theory to write $E_\alpha{}^M$ and $\Phi_{\alpha,b}$ in the form

$$E_\alpha{}^m = -\tfrac{1}{2}(\theta\gamma^a)_\alpha e_a{}^m + \tfrac{1}{4}\theta\theta\hat{\chi}_\alpha{}^m \qquad (3.11a)$$
$$E_\alpha{}^\mu = \delta_\alpha{}^\mu - \tfrac{1}{2}(\theta\gamma^a)_\alpha \psi_a{}^\mu + \tfrac{1}{4}\theta\theta\hat{\Phi}_\alpha{}^\mu \qquad (3.11b)$$
$$\Phi_{\alpha,a} = -\tfrac{1}{2}(\theta\gamma^b)_\alpha \omega_{b,a} + \tfrac{1}{4}\theta\theta\hat{\xi}_{\alpha,a} \qquad (3.11c)$$

where

$$\hat{\chi}_\alpha{}^m = \chi_\alpha{}^m - \tfrac{1}{2}(\gamma^a\gamma^b\psi_a)_\alpha e_b{}^m \qquad (3.12a)$$
$$\hat{\Phi}_\alpha{}^\mu = \Phi_\alpha{}^\mu + \hat{\chi}_\alpha{}^b\psi_b{}^\mu - \tfrac{1}{2}(\gamma^a\gamma^b)_\alpha{}^\mu\omega_{a,b} \qquad (3.12b)$$
$$\hat{\xi}_{\alpha,b} = \xi_{\alpha,b} + \hat{\chi}_\alpha{}^a\omega_{a,b} \qquad (3.12c)$$

In these expressions $e_a{}^M$ is the "dreibein", ψ_m^α the Rarita Schwinger field, $\omega_{b,a}$ the Lorentz connection in the three dimensional spacetime and χ_α^m, $\Phi_\alpha{}^M$ and $\xi_{a,b}$ are the auxiliary fields. The remaining gauge freedom is that of general coordinate transformations of parameter f^m, local supersymmetry transformations of parameter ε^μ and local Lorentz transformations of parameter $\varepsilon^{abc} l_c$ where f^m, ε^μ and l_c are the θ independent components of the superfields $\xi^m(z)$, $\xi^\mu(z)$ and $L_c(z)$. All the other components of $\xi^M(z)$ and $L_c(z)$ were used to gauge away the superfluous fields in $E_\alpha{}^M$ and $\Phi_{\alpha,d}$.

Using (3.5) and (3.10) one can express $E_a{}^M$ and $\Phi_{a,b}$ in terms of $E_\alpha{}^M$ and $\Phi_{\alpha,s}$, and therefore any quantity appearing in the theory can be expressed in terms of the physical and auxiliary fields.

4. The Breitenlohner Set of Auxiliary Fields

Our next task is to show that the auxiliary fields obtained with the constraints given by Superplane Integrability correspond to the so-called "Breitenlohner set" [7]. To this end one has to find the auxiliary fields that constitute the Breitenlohner set for three dimensional supergravity. To do this, one needs to know the vector supermultiplet in three dimensions. In Appendix B we find the vector supermultiplet, both for two and three dimensional spacetimes embeded, respectively, in four and five dimensional superspaces. The result for the three dimensional spacetime is that the vector supermultiplet consists only of a vector field and a Majorana spinor, no scalar field is needed to close the algebra. Therefore Breitenlohner's argument [7], will lead us to an auxiliary field structure consisting of the following fields: $\chi_\alpha{}^b$, $\Phi_\alpha{}^\beta$ and $\xi_{\alpha,b}$.

We have used the same notation for these auxiliary fields and those of (3.12) because, as it was shown in [10], the supersymmetry algebra closes, in the Breitenlohner sense, on those fields, allowing, therefore, the identification of $\chi_\alpha{}^b$, $\Phi_\alpha{}^\beta$ and $\xi_{\alpha,b}$ with the Breitenlohner set. The last thing we want to show, is how these auxiliary fields relate to the arbitrary superfields $G_\alpha{}^D$ and $H_{\alpha,d}$ introduced in (2.8) giving the most general solution for the torsion $T_{\alpha\beta}{}^D$ and curvature $R_{\alpha\beta,d}$ components. As we know [9] that for three dimensional supergravity without auxiliary fields these components vanish, they should be related to the auxiliary fields introduced above. In fact, the auxiliary fields are the θ-independent components of those superfields as we now show.

To do this one needs to calculate $E_a{}^M$ and $\Phi_{a,b}$ through (3.5) and (3.10). But as we want to evaluate $T_{\alpha\beta}{}^D$ and $R_{\alpha\beta,d}$ only to the lowest order (θ-independent), it is only necessary to calculate E_a^M and $\Phi_{a,b}$ up to a given order. The necessary results are

$$E_a{}^m = e_a{}^m + \tfrac{1}{2}(\theta\gamma^c\psi_a)e_c{}^m - \tfrac{1}{2}(\theta\gamma_a\chi^m) + \overset{(2)}{E}_a{}^m \qquad (4.1a)$$
$$E_a{}^\mu = \psi_a{}^\mu + \tfrac{1}{2}(\theta\gamma^d\psi_a)\psi_d{}^\mu - \tfrac{1}{2}(\theta\gamma_a\chi^b)\psi_b^\mu + \tfrac{1}{2}(\theta\gamma^b)^\mu\omega_{a,b} - \tfrac{1}{2}(\theta\gamma_a\Phi)^\mu + \overset{(2)}{E}_a{}^\mu \qquad (4.2a)$$

and

$$\Phi_{a,d} = \omega_{a,d} - \tfrac{1}{2}(\theta\gamma_a)^\alpha\xi_{\alpha,b} - \tfrac{1}{2}(\theta\gamma_a\chi^c)\omega_{c,d} + \tfrac{1}{2}(\theta\gamma^c\psi_a)\omega_{c,b} + \overset{(2)}{\Phi}_{a,d} \qquad (4.2b)$$

where we have used the obvious notation $\overset{(i)}{E}_a{}^M$ to indicate a term with the ith power of θ^a.

The quantities to be evaluated are

$$\overset{(0)}{T}_{\alpha\beta}{}^d = -\overset{(0)}{C}_{\alpha\beta}{}^d + \tfrac{1}{2}\overset{(0)}{\Phi}_{\beta,b}\varepsilon_a{}^{db} \qquad (4.3a)$$
$$\overset{(0)}{T}_{\alpha\beta}{}^\delta = -\overset{(0)}{C}_{\alpha\beta}{}^\delta - \tfrac{1}{2}\overset{(0)}{\Phi}_{a,b}(\gamma^b)_\beta{}^\delta \qquad (4.3b)$$
$$\overset{(0)}{R}_{\alpha\beta,d} = \overset{(0)}{E}_\alpha{}^\mu\partial_\mu\overset{(1)}{\Phi}_{\beta,d} - \overset{(0)}{E}_\beta{}^\mu\partial_\mu\overset{(1)}{\Phi}_{a,d} - \overset{(0)}{C}_{\alpha\beta}{}^e\overset{(0)}{\Phi}_{e,d} \qquad (4.3c)$$

Using (3.11) and (4.2) a straightforward calculation gives then

$$\overset{(0)}{T}_{\alpha\beta}{}^d = -\tfrac{1}{2}(\gamma_a)_\beta{}^\alpha\chi_\alpha{}^d \qquad (4.4a)$$
$$\overset{(0)}{T}_{\alpha\beta}{}^\delta = -\tfrac{1}{2}(\gamma_a)_\beta{}^\alpha\Phi_\alpha{}^\delta \qquad (4.4b)$$
$$\overset{(0)}{R}_{\alpha\beta,d} = \tfrac{1}{2}(\gamma_a)_\beta{}^\alpha\xi_{\alpha,d} \qquad (4.4c)$$

thus showing that the θ-independent components of the superfields $G_\alpha{}^D$ and $H_{\alpha,d}$ given in (2.8) are, up to normalization, the auxiliary fields that close the algebra in the Breitenlohner sense.

5. Conclusions

We have shown how superplane integrability [6] can be used to derive the kinematical constraints needed to formulate three dimensional supergravity in superspace. As it was the case for supergravity in a four dimensional spacetime [6], the kinematical constraints obtained from superplane integrability give origin to formulations with the Breitenlohner [7] rather than the minimal set of auxiliary fields.

It is an interesting open question to find out if the simple geometrical idea of Superplane Integrability also gives the correct kinematical constraints for supergravity theories in higher dimensionality spacetimes. The case of the 11-dimension spacetime [11] would be particularly interesting.

Appendix A

a) Three Dimensional Spacetime

Our metric convention is

$$\eta_{ab} = \operatorname{diag}(-,+,+); \quad \gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}\mathbb{1} \tag{A.1}$$

We choose a representation where

$$\gamma^a \equiv (i\sigma_2, \sigma_3, \sigma_1) \tag{A.2}$$

$$\text{“}\gamma_5\text{”} \equiv \gamma_0\gamma_1\gamma_2 = \mathbb{1} \tag{A.3}$$

Other useful expressions are

$$\gamma^a\gamma^b = \eta^{ab}\mathbb{1} - \varepsilon^{abc}\gamma_c; \varepsilon^{012} = +1$$
$$\Sigma^{ab} \equiv \tfrac{1}{4}[\gamma^a, \gamma^b] = -\tfrac{1}{2}\varepsilon^{abc}\gamma_c \tag{A.4}$$

the full set of Dirac matrices is

$$\mathbb{1}, \gamma^a; \quad a = 0, 1, 2 \tag{A.5}$$

Spinor indices are raised and lowered with the charge conjugation matrix

$$C \equiv \gamma^0; \quad C^{-1} = C^T = -C \tag{A.6}$$

It is useful then, to define a spinorial metric

$$\eta_{\alpha\beta} \equiv C_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv \eta^{\alpha\beta} \tag{A.7}$$

and to use Chang's sum convention[12]:

$$Q^A \equiv \eta^{AB} Q_B = (-1)^b Q_B \eta^{BA}$$
$$Q_A \equiv Q^B \eta_{BA} = (-1)^b \eta_{AB} Q^B \tag{A.8}$$

where

$$\eta^{AB} = \left(\begin{array}{c|c} \eta^{ab} & 0 \\ \hline 0 & \eta^{\alpha\beta} \end{array}\right); \quad \eta^{AB} = (-1)^{ab}\eta^{BA} \tag{A.9}$$

b) Superspace Curvature and Torsion
The torsion coefficients are given by

$$T_{AB}{}^C \equiv -C_{AB}{}^C - \Phi_{[A,B]}{}^C \tag{A.10}$$

where

$$C_{AB}{}^C \equiv \partial_{[A} E_{B]}{}^M E_M{}^C; \partial_A \equiv E_A{}^M \partial_M \tag{A.11}$$

the curvature coefficients are given by

$$R_{AB,C}{}^D \equiv \partial_{[A}\Phi_{B],C}{}^D - C_{AB}{}^E \Phi_{E,C}{}^D$$
$$+ \Phi_{[A.C}{}^E \Phi_{B]E}{}^D \tag{A.12}$$

the Bianchi identities are

$$D_A T_{BC}{}^D + T_{AB}{}^E T_{EC}{}^D + R_{AB,C}{}^D + \text{g.c.p.} = 0 \tag{A.13}$$
$$D_A R_{BC,D}{}^F + T_{AB}{}^E R_{EC,D}{}^F + \text{g.c.p.} = 0 \tag{A.14}$$

g.c.p. = graded cyclic permutation.

Appendix B

In this appendix we will use the superspace approach to study the vector supermultiplet in two and three dimensions. A globally supersymmetric superspace lagrangian will be given.

a) Three Dimensional Spacetimes
In this case we have a five dimensional superspace. Following [13], we will call vector supermultiplet, the superfield that contains spin 1 as the maximum spin of its component fields.

The most general such supermultiplet is

$$V_\alpha = \chi_\alpha + (\theta\gamma^m)_\alpha V_m + \theta_\alpha S + \theta\theta\psi_\alpha \tag{B.1}$$

where χ_α and ψ_α are Majorana spinors, V_m is a vector and S is a scalar field. We want the vector field to have gauge transformations and we will define them by

$$\delta V_\alpha \equiv D_\alpha V \tag{B.2}$$

where

$$D_\alpha = \partial_\alpha - \frac{i}{2}(\theta\gamma^m)_\alpha \partial_m \tag{B.3}$$

and Λ is a scalar superfield

$$\Lambda = A + \theta^\alpha \xi_\alpha + \theta\theta F \tag{B.4}$$

Although our choice (B.2) is different from that for the four dimensional case [13], it is going to work in a similar way.

We can use the gauge transformations (B.2) to gauge away some of the components of V_α. We have

$$D_\alpha \Lambda = \xi_\alpha + 2\theta_\alpha F - \frac{i}{2}(\theta\gamma^m)_\alpha \partial_m A$$
$$-\frac{i}{2}(\theta\gamma^m)_\alpha \theta^\beta \partial_m \xi_\beta \tag{B.5}$$

Comparing (B.1) and (B.5) we see that the choice $\xi_\alpha = -\chi_\alpha$ and $F = -\frac{1}{2}S$ will gauge away χ_α and S while redefining ψ_α and giving V_m the usual abelian gauge transformations

$$\delta V_m(x) = -\frac{i}{2}\partial_m A(x) \tag{B.6}$$

We can therefore choose a “Wess and Zumino” gauge where

$$V_\alpha{}^{wz} = (\theta\gamma^m)_\alpha V_m + \theta\theta\psi_\alpha \tag{B.7}$$

We conclude then that the vector supermultiplet in $d_B = 3$ consists only of a vector field and a Majorana spinor, no scalar field is needed to close the algebra. This can also be explicitly checked using the global supersymmetry transformation laws

$$\delta V_m = i\varepsilon\gamma_m\psi$$
$$\delta\psi_\alpha = -(\Sigma_{mn}\varepsilon)_\alpha F^{mn} \tag{B.8}$$

We can now construct a superfield that contains the field strength $F_{mn} = \partial_m A_n - \partial_n A_m$. It is given by

$$W_\alpha \equiv D^\beta D_\alpha V_\beta \tag{B.9}$$

and it gauge invariant ($D^\beta D_\alpha D_\beta = 0$). A straightforward calculation gives

$$W_\alpha = -2\psi_\alpha + \frac{i}{2}\varepsilon^{mnr}(\theta\gamma_r)_\alpha F_{mn} + \frac{i}{2}\theta\theta(\gamma\cdot\partial\psi)_\alpha \tag{B.10}$$

and the last component of $\frac{1}{2}W^\alpha W_\alpha$ will give the correct lagrangian

$$\tfrac{1}{2}W^\alpha W_\alpha\big|_{\theta\theta} = -\frac{i}{2}\bar{\psi}\gamma\cdot\partial\psi - \tfrac{1}{4}F_{mn}F^{mn} = \mathscr{L}(x) \qquad \text{(B.11)}$$

b) Two Dimensional Spacetime
Repeating the same steps, we have now

$$V_\alpha = \chi_\alpha + (\theta\gamma^m)_\alpha V_m + \theta_\alpha S + (\theta\gamma_5)_\alpha P + \theta\theta\psi_\alpha \qquad \text{(B.12)}$$

therefore we have an additional pseudoscalar P. The gauge function being the same

$$\Lambda = A + \theta^\alpha\xi_\alpha + \theta\theta F \qquad \text{(B.13)}$$

We can only gauge away χ_α and S, not P. Therefore the vector supermultiplet in the "Wess and Zumino" gauge is

$$V_\alpha^{wz} = (\theta\gamma^m)_\alpha V_m + (\theta\gamma_5)_\alpha P + \theta\theta\psi_\alpha \qquad \text{(B.14)}$$

In a similar way

$$W_\alpha \equiv D^\beta D_\alpha V_\beta = -2\psi_\alpha - i(\theta\gamma_m)_\alpha\varepsilon^{mn}\partial_n P + \frac{i}{2}\varepsilon^{mn}(\theta\gamma_5)_\alpha F_{mn} + \frac{i}{2}\theta\theta(\gamma\cdot\partial\psi)_\alpha \qquad \text{(B.15)}$$

and

$$\tfrac{1}{2}W^\alpha W_\alpha\big|_{\theta\theta} = -\frac{i}{2}\bar{\psi}\gamma\cdot\partial\psi - \tfrac{1}{2}(\partial_n P)^2 - \tfrac{1}{4}F_{mn}F^{mn} \qquad \text{(B.16)}$$

We have now a pseudoscalar field but it is not an auxiliary field. This is to be expected, because the number of degrees of freedom has to balance between fermions and bosons, and for that we need a propagating scalar field

$$B = (d-2) + 1 = 1$$
$$F = 2^{d/2-1} = 2^\circ = 1 \qquad \text{(B.17)}$$

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