# GEOMETRY OF SUPERSPACE CONSTRAINTS 

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#### Abstract

The requirement of integrability in all light-like superplanes yields the proper constrants for ordinary ( $N=1$ ) and extended ( $N=2,4$ ) supersymmetric Yang-Mills theorres (SSYM) and for ordinary ( $N=1$ ) supergravity. It is suggested that this geometrical principle yields the proper constraints for all ordinary and extended SSYM and supergravity theories in all space-tıme dimensions where such theories exist.


## 1. Introduction

When supergravity [1] is formulated in superspace [2-5], the need arises to impose constraints on the geometrical objects of the theory. Recently there has been some work [6, 19], on the origin of these so-called "kinematical" constraints.

The idea of superplane integrability [6] is to identify the kinematical constraints with integrability conditions on light-like superplanes. This idea is closely related with previous work by Witten [7] who derived the supersymmetric Yang-MillsDirac field equations from similar integrability conditions. It is mainly a geometrical idea and differs from the approach of ref. [19].

In this paper we will strengthen the relation by showing that the kinematical constraints needed to formulate the supersymmetric Yang-Mills (SSYM) theories in superspace can also be obtained from superplane integrability. We will also show that for supergravity theories the constraints obtained from superplane integrability, are those corresponding to the Breitenlohner set of auxiliary fields. We must, however, remark that both for SSYM and for supergravity, superplane integrability only give us the kinematical constraints. A lagrangian is still needed to obtain the dynamics.

The paper is organized as follows. In sect. 2 superplane integrability is explained. In sect. 3 we apply it to SSYM theories and in sect. 4 to supergravity theories. Finally

[^0]in sect. 5 we discuss our results and briefly comment on other approaches to the problem.

## 2. Superplane integrability

Here we review the superplane integrability proposed in [6]. We begin with some notation. As usual, we collect the four Bose coordinates $x^{m}$ and the four Fermi coordinates $\theta^{\mu}$ (Majorana spinor) into a set of eight superspace coordinates $Z^{M}$ ( $M=0,1, \ldots 7$ ). Lower case letters from the latin (greek) alphabet will describe vector (spinor) indices. Letters from the beginning (end) of both alphabets will be used for tangent space (world) indices. Capital letters will be used to describe both vector and spinor indices $[A=(a, \alpha), a=0, \ldots, 3, \alpha=1, \ldots, 4 ; M=(m, \mu), m=$ $0, \ldots 3, \mu=1, \ldots, 4]$.

The geometry of superspace is described [1] by the vielbein $E_{M}{ }^{A}$, and the Lie algebra valued connection $\phi_{M, A}{ }^{B}$. The local tangent space group is chosen [1] to be a Lorentz group operating in the usual way on the vector and spinor tangent space indices. Therefore the infinitesimal generators of the tangent space Lie algebra $\hat{X}_{A}{ }^{B}$, will have the form

$$
\hat{\boldsymbol{X}}_{A}{ }^{B}=\left(\begin{array}{c:c}
\hat{\boldsymbol{X}}_{a}{ }^{a} & 0  \tag{2.1}\\
\hdashline 0 & \frac{1}{2}\left(\boldsymbol{\Sigma}^{c d}\right)_{\alpha}{ }^{\boldsymbol{\beta}} \bar{X}_{c d}
\end{array}\right) .
$$

The inverse of the vielbein, $E_{A}{ }^{M}$, gives the covariant derivatives $D_{A} \equiv E_{A}{ }^{M} D_{M}$. Mathematically they correspond to the basic horizonatal vector fields of the superbundle and they obey the well-known relation [10]

$$
\begin{equation*}
\left[D_{A}, D_{B}\right]=-T_{A B}{ }^{C} D_{C}+R_{A B}{ }^{C D} \hat{X}_{D C} \tag{2.2}
\end{equation*}
$$

where we introduced the torsion coefficients $T_{A B}{ }^{C}$ and the curvature coefficients, $\boldsymbol{R}_{A B, C D}$. In eq. (2.2), and in the following, the bracket is to be understood as a graded bracket, which is equal to a commutator in all cases except when both bracketed quantities are fermionic, in which case it is an anticommutator. Our notation is given in the appendix.

Given any c-number Majorana spinor $\rho^{\alpha}$, we can form a light-like vector $r^{a}$ :

$$
\begin{equation*}
r^{a} \equiv \rho^{\alpha}\left(\gamma^{a}\right)_{\alpha}^{\beta} \rho_{\beta}, \quad r^{a} r_{a}=0 \tag{2.3}
\end{equation*}
$$

Corresponding to the null directions $\rho^{\alpha}, r^{a}$, we have a pair of tangent space directions

$$
\begin{equation*}
D \equiv r^{a} D_{a}, \quad Q \equiv \rho^{\alpha} D_{\alpha} \tag{2.4}
\end{equation*}
$$

$D$ being Bose and $Q \operatorname{Fermi}\left(D_{a}\right.$ and $D_{\alpha}$ are the components of $\left.D_{A}\right)$. Every Majorana spinor $\rho^{\alpha}$ determines, through (2.4), a tangent "superplane". Integrability in each superplane requires the algebra of $D$ and $Q$, to be the same as in the absence of gauge fields, i.e. for flat superspace,

$$
\begin{equation*}
[Q, Q]=D \tag{2.5a}
\end{equation*}
$$

$$
\begin{align*}
& {[D, Q]=0,}  \tag{2.5b}\\
& {[D, D]=0 .} \tag{2.5c}
\end{align*}
$$

Eq. (2.5c) is an identity, and eq. (2.5b) follows from (2.5a) and the Jacobi identity $[Q,[Q, Q]]=0$. Having to hold for arbitrary $\rho^{\alpha}$, eq. (2.5a) implies

$$
\begin{equation*}
\left[D_{\alpha}, D_{\beta}\right]=-\left(\gamma^{a}\right)_{\alpha \beta} D_{a} . \tag{2.6}
\end{equation*}
$$

This simple result can be generalized first to accommodate a cosmological term and then to extended supersymmetries when the superspace has $4 N$ fermionic coordinates $\theta^{i \mu}(\mu=1, \ldots, 4 ; i=1, \ldots, N)$.

To include a cosmological term, we note that starting from the Majorana spinor $\rho^{\alpha}$ we can construct not only a null vector, $r^{a}$, but also a null antisymmetric tensor

$$
\begin{equation*}
t^{a b}=\rho^{\alpha}\left(\Sigma^{a b}\right)_{\alpha}^{\beta} \rho_{\beta}, \quad t^{a b} t_{a b}=0 \tag{2.7}
\end{equation*}
$$

Now replace the tangent direction $D$ by

$$
\begin{equation*}
\bar{D} \equiv r^{a} D_{a}+\frac{1}{2 \lambda} t^{a b} \hat{X}_{a b}, \tag{2.8}
\end{equation*}
$$

where $\hat{X}_{a b}$ are the Lorentz generator directions in tangent space and $\lambda$ a cosmological radius. Now we use the algebra of (2.5) for $\bar{D}$ and instead of (2.6) we get

$$
\left[D_{\alpha}, D_{\beta}\right]=-\left(\gamma^{a}\right)_{\alpha \beta} D_{a}-\frac{1}{2 \lambda}\left(\Sigma^{a b}\right)_{\alpha \beta} \hat{X}_{a b}
$$

There are several ways to generalize the idea to extended supersymmetries, depending on how one defines $r^{a}$ in this case. Our definition is chosen to provide the correct supersymmetric extended Yang-Mills results, as will be shown in sect. 3. It is more transparent to use the two-component spinor notations. We will follow the conventions of [11]. Then

$$
\begin{gather*}
Z^{M} \equiv\left(x^{m}, \theta_{t}^{\mu}, \bar{\theta}^{\mu_{l}}\right), \quad D_{A} \equiv\left(D_{a}, D_{\alpha}^{t}, \bar{D}_{\alpha t}\right) \\
\rho^{\alpha} \equiv\left(\rho^{\alpha}, \bar{\rho}^{\alpha}\right), \quad r^{a}=-\rho^{\alpha}\left(\sigma^{a}\right)_{\alpha \beta} \bar{\rho}^{\beta}, \quad r^{a} r_{a}=0 ; \tag{2.9}
\end{gather*}
$$

the tangent space null directions will then be

$$
\begin{equation*}
D=r^{a} D_{a}, \quad Q^{i}=\rho^{\alpha} D_{\alpha}^{t}, \quad \bar{Q}_{i}=\bar{\rho}^{\beta} \bar{D}_{\beta l} \tag{2.10}
\end{equation*}
$$

Integrability in each superplane requires the algebra of $D, Q^{\prime}$ and $\bar{Q}_{J}$ to be the same as in the absence of gauge fields, that is

$$
\begin{align*}
{\left[Q^{\prime}, Q^{\prime}\right] } & =\left[\bar{Q}_{1}, \bar{Q}_{I}\right]=0,  \tag{2.11a}\\
{\left[Q^{\prime}, \bar{Q}_{I}\right] } & =\delta_{I}^{l} D,  \tag{2.11b}\\
{\left[Q^{\prime}, D\right] } & =\left[\bar{Q}_{3}, D\right]=0,  \tag{2.11c}\\
{[D, D] } & =0 \tag{2.11~d}
\end{align*}
$$

Having to hold for arbitrary $\rho^{\alpha}, \bar{\rho}^{\alpha}$, eqs. (2.11a) and (2.11b) imply

$$
\begin{gather*}
{\left[D_{\alpha \imath}, \bar{D}_{\beta_{l}}\right]+\left[\bar{D}_{\beta,}, \bar{D}_{\alpha J}\right]=0} \\
{\left[D_{\alpha}^{\iota}, D_{\beta}^{\prime}\right]+\left[D_{\beta}^{\iota}, D_{\alpha}^{\iota}\right]=0}  \tag{2.12}\\
{\left[D_{\alpha}^{l}, \bar{D}_{\beta_{J}}\right]=-\delta_{l}^{l}\left(\sigma^{a}\right)_{\alpha \beta} D_{a}}
\end{gather*}
$$

In the following sections, we will apply these results both to supersymmetric Yang-Mills and supergravity theories and show that the kinematic constraints can be obtained from (2.6), (2.6) or (2.12).

## 3. Supersymmetric Yang-Mills theories

Consider a $n$-parameter Lie Group $G$ with generators $Y_{\hat{i}}(\hat{\imath}=1, \ldots, n)$. Then the bracket of two covariant derivatives is given by

$$
\begin{equation*}
\left[D_{A}, D_{B}\right]=-T_{A B}{ }^{c} D_{C}-F_{A B}{ }^{i} Y_{\hat{\imath}} \tag{3.1}
\end{equation*}
$$

where $T_{A B}{ }^{C}$ is the flat superspace torsion and $F_{A B}{ }^{i}$ are the (super) field strengths or curvatures. We will treat the $N=1$ and $N>1$ cases separately.

## $3.1 \quad N=1$ SSYM

Superplane integrability (SI) gives for this case [eq. (2.6)]

$$
\begin{gather*}
T_{\alpha \beta}^{c}=\left(\gamma^{c}\right)_{\alpha \beta}, \quad T_{\alpha \beta}^{\gamma}=0  \tag{3.2a}\\
F_{\alpha \beta}{ }^{\hat{t}}=0 \tag{3.2b}
\end{gather*}
$$

therefore, we obtain torsion constraints which are consistent with the superspace being flat and a constraint on the spinorial components of the superfield strengths. Eq. (3.2b) is precisely the "kinematic" constraint necessary to formulate the $N=1$ SSYM theory in superspace [12].

## $3.2 N \geqslant 2$ SSYM

Superplane integrability, eqs. (2.12) and (3.1) give

$$
\begin{align*}
& T_{\alpha \beta_{1}}{ }^{c}=\delta_{I}^{t}\left(\sigma^{c}\right)_{\alpha \beta},  \tag{3.3a}\\
& F_{\alpha \beta}^{y \hat{i}}=M_{[\alpha \beta]}^{[1]] \hat{t}},  \tag{3.3b}\\
& F_{\alpha i \beta]}^{i}=N_{[\alpha \beta][y]}{ }^{\hat{i}},  \tag{3.3c}\\
& F_{\alpha \beta l}^{t}{ }^{r}=0 . \tag{3.3d}
\end{align*}
$$

Eq. (3.3a) gives correctly the non-vanishing component of the flat supertorsion. Eqs. (3.3b)-(3.3d) are a set of constraints on the Fermi-Fermi components of the
superfield strengths. These are precisely the constraints needed to formulate the $N=2$ [11] and $N=4$ [13] SSYM theories in superspace. Eqs. (3.3b)-(3.3d) are not, however, the full content of the $N>1$ theories. They are simply "kinematical" constraints that express geometrical quantities in terms of the propagating and auxiliary fields of the theory. We still need a lagrangian to get the field equations for these fields [11,13]. We believe the same results hold for $N=3$ (there are no $N>4$ SSYM in 4 space-time dimensions).

## 4. Superspace supergravity theories

## 4.1. $\operatorname{SUPERGRAVITY}(N=1)$

Superplane integrability, eq. (2.6), and the definition of the bracket of two covariant derivatives, eq. (2.2), imply

$$
\begin{gather*}
T_{\alpha \beta}^{c}=\left(\gamma^{c}\right)_{\alpha \beta}, \quad T_{\alpha \beta}{ }^{\gamma}=0  \tag{4.1}\\
R_{\alpha \beta, C D}=0 \tag{4.2}
\end{gather*}
$$

These constraints are weaker than those of Wess and Zumino [1], even those for the minimal auxiliary set of fields. Therefore, we expect to obtain an SG formulation with more than the minimal set. It was shown by Gates and Shapiro [14] and by Taylor et al. [15] that constraints (4.1), (4.2) imply a formulation of SG with the Breitenlohner set of auxiliary fields. Just as for SSYM theories, one still needs a lagrangian to get the equations of motion.

This result can also be obtained in a different way. It is straightforward to show that the Bianchi identities, plus the above constraints imply

$$
\begin{align*}
T_{e \alpha}^{d} & =\left(\gamma_{e} \chi^{d}\right)_{\alpha},  \tag{4.3a}\\
T_{e \alpha}{ }^{\delta} & =\left(\gamma_{e} M\right)_{\alpha}{ }^{\delta},  \tag{4.3b}\\
R_{e \alpha}{ }^{c d} & =\left(\gamma_{e} \chi^{c d}\right)_{\alpha}, \tag{4.3c}
\end{align*}
$$

where $\chi_{\beta}^{d}, M_{\alpha}{ }^{\beta}$ and $\chi_{\beta}^{c d}$ are arbitrary superfields. Now if we compare eqs. (4.3) with table 7 of Brink et al. [3], we see that they are precisely the superspace equations of motion for supergravity with the Breitenlohner set of auxiliary fields. The inclusion of a cosmological term (finite de Sitter radius) presents no problem. As said above, eq. (2.6) should be replaced by (2.6') and therefore we get instead of eq. (4.2), the new result

$$
R_{\alpha \beta, c d}=\frac{1}{\lambda}\left(\Sigma_{c d}\right)_{\alpha \beta},
$$

in agreement with the results of Brink et al. [3].

### 4.2 EXTENDED SUPERGRAVITY $(N>1)$

The generalization to $N>1$ can be made along two different lines. We can use an extended superspace $\left[Z^{M} \equiv\left(x^{m}, \theta^{\mu i}\right)\right]$, as we did for SSYM theories, or we can begin with the $N=1$ theory in a higher dimensionality superspace, that is, with Bose dimension $d_{\mathrm{B}}>4$. One would then compactify the extra Bose dimensions, and end up with an $N>1$ theory for $d_{\mathrm{B}}=4,[16]$. Specially interesting would be the $d_{\mathrm{B}}=11$ case, [17].

An obvious difficulty from the outset, is that we do not expect to get minimal theories; that is, theories will the minimal set of auxiliary fields. One will have to find, in both approaches, the corresponding "Breitenlohner" sets and then compare the results with what one gets from superplane integrability. Work along these lines is in progress. Here we note that one of us (J.C.-R.) has derived from SI the constraints corresponding to Breitenlohner auxiliary fields for $N=1$ supergravity in threedimensional space-time ( $d_{\mathrm{B}}=3$ ).

## 5. Conclusions

We have shown how superplane integrability, a simple geometrical idea, can be used to derive the kinematical constraints needed to formulate supersymmetric theories in superspace. SI works equally well for SSYM theories and for supergravity theories. We emphasize, however, that neither SSYM nor supergravity can be "derived" from SI. In both cases one still needs a lagrangian to obtain the dynamics. Nevertheless, a simple geometrical idea provides us with all the kinematical constraints for all these theories. It is interesting that in all the cases studied above, the constraints with the Breitenlohner [9] rather than the minimal set of auxiliary fields appear. It is an interesting open question why this is the case. Moreover, it would be interesting to know whether the constraints with other sets of auxiliary fields can also be derived from geometrical ideas. As our idea is mainly geometrical it differs quite substantially from the interesting approach proposed in ref. [19].

## Appendix

Our metric convention is

$$
\begin{equation*}
\eta_{a b}=\operatorname{diag}(-+++), \quad \gamma_{a} \gamma_{b}+\gamma_{b} \gamma_{a}=2 \eta_{a b} \mathbb{1} . \tag{A.1}
\end{equation*}
$$

We use the definitions

$$
\begin{equation*}
\Sigma_{a b} \equiv \frac{1}{4}\left[\gamma_{a}, \gamma_{b}\right], \quad \varepsilon^{0123}=+1 \tag{A.2}
\end{equation*}
$$

The charge conjugation matrix is

$$
\begin{equation*}
C=\gamma^{0}, \quad C^{-1}=C^{\mathrm{T}}=-C \tag{A.3}
\end{equation*}
$$

Spinor indices are raised and lowered by the spinorial metric $\eta_{\alpha \beta}$ :

$$
\begin{gather*}
\eta_{\alpha \beta}=C_{\alpha \beta}=\left(\begin{array}{cc}
0 & -\mathbb{1} \\
\mathbb{1} & 0
\end{array}\right), \quad \eta^{\alpha \beta}=\eta_{\alpha \beta},  \tag{A.4}\\
\eta_{A B} \equiv\left(\begin{array}{c:c}
\eta_{a b} & 0 \\
\hdashline 0 & \eta_{\alpha \beta}
\end{array}\right), \quad \eta_{A B}=(-1)^{a b} \eta_{B A} . \tag{A.5}
\end{gather*}
$$

Chang's sum convention [18] is used throughout.
The torsion coefficients are given by

$$
\begin{equation*}
T_{A B}^{C}=-C_{A B}^{C}-\phi_{[A, B]}{ }^{C}, \tag{A.6}
\end{equation*}
$$

where

$$
\begin{equation*}
C_{A B}^{C} \equiv \partial_{[A} E_{B]}^{M} E_{M}^{C}, \quad \partial_{A} \equiv E_{A}^{M} \partial_{M} \tag{A.7}
\end{equation*}
$$

the curvature coefficients are given by

$$
\begin{equation*}
R_{A B, C}{ }^{D} \equiv \partial_{[A} \phi_{B], C}{ }^{D}-C_{A B}{ }^{E} \phi_{E, C}{ }^{D}+\phi_{[A, C}{ }^{E} \phi_{B], E}{ }^{D} . \tag{A.8}
\end{equation*}
$$

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