# CONFORMAL AND SUPERCONFORMAL GRAVITY AND NON-LINEAR REPRESENTATIONS 

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#### Abstract

The method of non-linear representations, when applied to the (super) conformal group, is shown not to be equivalent to the usual (super) conformal gravity theories. However, the requirement of invariance under the full (super) conformal group transformations, uniquely leads to the usual theories.


## 1. Introduction

Gürsey and Marchildon [1] and Chang and Mansouri [2] proposed an approach to gravity and supergravity based on non-linear group and supergroup representations. Starting from the Poincaré (super Poincaré) group, and choosing $O(3,1)$ as the linear subgroup, they showed that in a particular gauge, an appropriate linear combination of their Lagrangian reproduced pure gravity (Poincaré supergravity).

As Gürsey and Marchildon noted for the case of Poincaré supergravity, the choice of a particular gauge makes the various possible Lagrangians non-invariant under transformations not contained in the linear subgroup. Requiring that, even in this gauge, the full Lagrangian be invariant under supersymmetry transformations leads to the usual Lagrangian for supergravity.

When these ideas are applied to the conformal (SU(2,2)) and superconformal ( $\mathrm{SU}(2,2 \mid 1)$ ) groups, the number of possible Lagrangians is, as we shall see, very large. Furthermore, the vanishing of the torsion does not follow from the field equations but has to be imposed as a constraint $[3,4]$.

In this paper we first show that if we start with the conformal group and take Lorentz $\times$ dilatations as the linear subgroup, then the most general Lagrangian written in the Gürsey and Marchildon gauge is the Weyl Lagrangian [5] **. Staying in

[^0]this gauge, we require, next, invariance under special conformal transformations. We obtain then uniquely the Lagrangian of refs. [3,4]. Finally, we show that when the same procedure is applied to the superconformal group $\operatorname{SU}(2,2 \mid 1)$, taking Lorentz $X$ dilatations $X$ chiral $U(1)$ as the linear subgroup, the Lagrangian of ref. [6] is obtained after requiring invariance under local special conformal transformation, their supersymmetric square roots, and local $Q$-supersymmetric transformations (the square roots of the momenta).

## 2. Conformal gravity

To each of the 15 generators of the conformal group $\operatorname{SU}(2,2)$, we associate a gauge field and a curvature in the following way:
$\begin{array}{lllll}\text { generator: } & P_{a}, & K_{a}, & M_{a b}, & D, \\ \text { gauge field: } & e^{a}{ }_{\mu}, & f^{a}{ }_{\mu}, & \omega_{\mu a b}, & B_{\mu}, \\ \text { curvature: } & R_{\mu \nu a}(P), & R_{\mu \nu a}(K), & R_{\mu \nu a b}(M), & R_{\mu \nu}(D),\end{array}$
where the latin (greek) indices refer to the tangent space (world) indices. We use the notation and conventions of ref. [4], but for convenience some of the formulae are summarized in appendix A.

We also impose the torsion free condition

$$
R_{\mu \nu a}(P)=0,
$$

which allows us to solve for the spin connexion in terms of $e^{a}{ }_{\mu}$ and $B_{\mu}$.
We now write down the Lagrangian that is obtained by the method of non-linear representations of the group $G=\operatorname{SU}(2,2)$, the linear subgroup being $H=$ Lorentz $X$ dilatations. As we are only interested in the Lagrangian in a particular gauge [1], we do not have to introduce a parametrization of the coset space $\mathrm{G} / \mathrm{H}$. The Lagrangian is then given by all terms invariant under H and general coordinate transformations, that are built out of the curvatures and the gauge fields corresponding to $\mathrm{G} / \mathrm{H}$. As we do not want the special conformal gauge field, $f^{a}{ }_{\mu}$, to propagate, we will not consider terms with $R_{\mu \nu a}(K)$. If, in addition, we require the Lagrangian to be parity conserving, we obtain

$$
\begin{align*}
\mathcal{L}= & e\left[a_{1}\left(-f R^{(0)}+2 f_{\mu \nu} R^{(0) \mu \nu}\right)+a_{2}\left(f^{2}-f_{\mu \nu} f^{\nu \mu}\right)+a_{3} f_{\mu \nu} f^{\nu \mu}+a_{4} f_{\mu \nu} f^{\mu \nu}\right. \\
& \left.+a_{5} f_{\mu \nu} R^{(0) \nu \mu}+a_{6} f_{\mu \nu} F^{\mu \nu}+a_{7} F_{\mu \nu} F^{\mu \nu}+a_{8} R_{\mu \nu a b}^{(0)} R^{(0) \mu \nu a b}\right] \tag{1}
\end{align*}
$$

where the $a_{i}$ 's are arbitrary constants and

$$
\begin{array}{lc}
f_{\mu \nu}=e^{a}{ }_{\mu} f_{a \nu}, \quad f=g^{\mu \nu} f_{\mu \nu}, \quad e=\operatorname{det}\left(e^{a}{ }_{\mu}\right), \\
R_{\mu \nu}^{(0)}=e^{b}{ }_{\nu} e^{a \rho} R_{\mu \rho a b}^{(0)}, \quad R^{(0)}=g^{\mu \nu} R_{\mu \nu}^{(0)} . \tag{2}
\end{array}
$$

This form for the Lagrangian is the most convenient for our calculations. One can easily check that other terms that would seem possible, like, for example, $\epsilon^{\mu \nu \rho \sigma} \epsilon^{a b c d} e_{c \rho} f_{d \sigma} R_{\mu \nu a b}(M)$, can be expressed as linear combinations of the terms already in (1).

Varying the Lagrangian (1) we get the field equation for the non-propagating $f_{\mu \nu}$

$$
\begin{align*}
& -a_{1} g_{\mu \nu} R^{(0)}+2 a_{1} R_{\mu \nu}^{(0)}+a_{5} R_{\nu \mu}^{(0)}+a_{6} F_{\mu \nu}+2 a_{2} f g_{\mu \nu} \\
& \quad+2\left(a_{3}-a_{2}\right) f_{\nu \mu}+2 a_{4} f_{\mu \nu}=0 \tag{3}
\end{align*}
$$

It is convenient to separate $f_{\mu \nu}$ in its symmetric and antisymmetric parts

$$
\begin{equation*}
s_{\mu \nu}=\frac{1}{2}\left(f_{\mu \nu}+f_{\nu \mu}\right), \quad a_{\mu \nu}=\frac{1}{2}\left(f_{\mu \nu}-f_{\nu \mu}\right) . \tag{4}
\end{equation*}
$$

Then the solution of (3) is

$$
\begin{align*}
& a_{\mu \nu}=\frac{1}{2} \alpha_{1}\left(R_{\mu \nu}^{(0)}-R_{\nu \mu}^{(0)}\right), \\
& s_{\mu \nu}=\frac{1}{2} \beta_{1}\left(R_{\mu \nu}^{(0)}+R_{\nu \mu}^{(0)}\right)+\beta_{2} g_{\mu \nu} R^{(0)}, \\
& f=g^{\mu \nu} f_{\mu \nu}=\beta_{0} R^{(0)}, \tag{5}
\end{align*}
$$

where

$$
\begin{array}{ll}
\alpha_{1}=-\frac{2 a_{1}-a_{5}+a_{6}}{2\left(a_{2}+a_{4}-a_{3}\right)}, & \beta_{0}=\frac{2 a_{1}-a_{5}}{6 a_{2}+2 a_{3}+2 a_{4}}, \\
\beta_{1}=-\frac{2 a_{1}+a_{5}}{2\left(-a_{2}+a_{3}+a_{4}\right)}, & \beta_{2}=\frac{a_{1}-2 a_{2} \beta_{0}}{2\left(-a_{2}+a_{3}+a_{4}\right)} . \tag{6}
\end{array}
$$

When we substitute (5) into (1) we get the Lagrangian

$$
\begin{equation*}
\mathcal{L}_{\mathrm{W}}=e\left[a R^{(0) 2}+b R_{\mu \nu}^{(0)} R^{(0) \mu \nu}+c F_{\mu \nu} F^{\mu \nu}\right] \tag{7}
\end{equation*}
$$

where $a, b$ and $c$ are arbitrary and the $R_{\mu \nu a b}^{(0)} R^{(0) \mu \nu a b}$ term was dropped since by the Bach identity [7] it can be written as a linear combination of $\left(R^{(0)}\right)^{2}$ and $R_{\mu \nu}^{(0)} R^{(0) \mu \nu}$ modulo an exact divergence. This Lagrangian is invariant under Lorentz, general coordinate and local dilatations transformations, but for arbitrary $a, b$ and $c$ it is not invariant under special conformal transformations. It is equivalent to the Lagrangian of Weyl [5].

To have invariance under the full conformal group, we require then invariance under the special conformal transformations. It is easier to do this with the Lagrangian at the level of eq. (1), that is, before substituting back for $f_{\mu \nu}$ as given in (5).

Under a transformation with parameters $\epsilon^{A}$ (the index $A$ runs over 15 values) the curvatures transform as

$$
\begin{equation*}
\delta\left(\epsilon^{A}\right) R_{\mu \nu}^{B}=f_{C}^{B} R_{\mu \nu}^{C} \epsilon^{A}, \tag{8}
\end{equation*}
$$

where $f_{C A}^{B}$ are the structure constants of the group. Therefore, under a special conformal transformation we have $\left(R_{\mu \nu}^{a}(P)=0\right)$

$$
\begin{equation*}
\delta^{K} R_{\mu \nu a b}(M)=\delta^{K} R_{\mu \nu}(D)=0 \tag{9}
\end{equation*}
$$

From the expressions for the curvatures given in appendix A , we can easily derive

$$
\begin{align*}
& \delta^{K} R_{\mu \nu}^{(0)}=\left(-2 g_{\mu \nu} \delta^{K} f-4 \delta^{K} f_{\nu \mu}\right) \\
& \delta^{K} R^{(0)}=-12 \delta^{K} f \\
& \delta^{K} F_{\mu \nu}=+2 \delta^{K} f_{\mu \nu}-2 \delta^{K} f_{\nu \mu} \tag{10}
\end{align*}
$$

By requiring consistence between the conformal transformation properties of the left- and right-hand sides of eqs. (5) we find

$$
\begin{equation*}
\alpha_{1}=-\beta_{1}=\frac{1}{4}, \quad \beta_{2}=\frac{1}{24}, \quad \beta_{0}=-\frac{1}{12} . \tag{11}
\end{equation*}
$$

From eqs. (5), (6) and (11) we get then

$$
\begin{equation*}
f_{\mu \nu}=-\frac{1}{4}\left(R_{\nu \mu}^{(0)}-\frac{1}{6} g_{\mu \nu} R^{(0)}\right), \tag{12a}
\end{equation*}
$$

and

$$
\begin{equation*}
4 a_{1}+a_{2}=a_{5}, \quad a_{3}+a_{4}=3 a_{5}, \quad a_{4}-a_{3}=a_{5}-2 a_{6} . \tag{12b}
\end{equation*}
$$

Now we calculate the variation of $\mathcal{L}$ in eq. (1) under $K$ transformations (special conformal transformations). We do not have to vary $f_{\mu \nu}$ because the coefficient of $\delta^{K} f_{\mu \nu}$ is the field equation for $f_{\mu \nu}$ and it vanishes identically. We obtain (remember $\delta^{K} e^{a}{ }_{\mu}=0$ )

$$
\begin{align*}
\delta^{K} \mathcal{L}= & e\left[-a_{1} f \delta^{K} R^{(0)}+2 a_{1} f_{\mu \nu} \delta^{K} R^{(0) \mu \nu}\right. \\
& \left.+a_{5} f_{\mu \nu} \delta^{K} R^{(0) \nu \mu}+a_{6} f_{\mu \nu} \delta^{K} F^{\mu \nu}+2 a_{7} F_{\mu \nu} \delta^{K} F^{\mu \nu}\right] \\
= & e\left[-a_{1} R^{(0)} \delta^{K} f+\left(2 a_{1}+4 a_{7}+\frac{1}{2} a_{6}\right) R^{(0) \mu \nu} \delta^{K} f_{\mu \nu}\right. \\
& \left.+\left(a_{5}-4 a_{7}-\frac{1}{2} a_{6}\right) R^{(0) \nu \mu} \delta^{K} f_{\mu \nu}\right] \tag{13}
\end{align*}
$$

where use of (10) and (12a) was made. Using the result (see appendix B)

$$
\begin{equation*}
e\left[-R^{(0)} \delta^{K} f+2 R_{\mu \nu}^{(0)} \delta^{K} f^{\mu \nu}\right]=\text { exact divergence } \tag{14}
\end{equation*}
$$

we get

$$
\begin{equation*}
a_{5}=0, \quad a_{6}=-8 a_{7}, \tag{15a}
\end{equation*}
$$

and from (12a) and (15a) we obtain finally

$$
\begin{equation*}
a_{3}=-a_{4}=a_{6}=-8 a_{7}, \quad a_{2}=-4 a_{1} \tag{15b}
\end{equation*}
$$

Eqs. (12a) and ( 15 b ) then imply

$$
a_{3} f_{\mu \nu} f^{\nu \mu}+a_{4} f_{\mu \nu} f^{\mu \nu}+a_{6} f_{\mu \nu} F^{\mu \nu}+a_{7} F_{\mu \nu} F^{\mu \nu}=0
$$

and therefore the $\mathcal{L}$ reduces to

$$
\begin{align*}
\mathcal{L} & =e a_{1}\left[-f R^{(0)}+2 f_{\mu \nu} R^{(0) \mu \nu}-4 f^{2}+4 f_{\mu \nu} f^{\nu \mu}\right] \\
& =-e \frac{1}{4} a_{1}\left(R_{\mu \nu}^{(0)} R^{(0) \nu \mu}-\frac{1}{3} R^{(0) 2}\right) \\
& =-\frac{1}{32} a_{1} \epsilon^{\mu \nu \rho \sigma} \epsilon^{a b c d} R_{\mu \nu a b}(M) R_{\rho \sigma c d}(M) \tag{16}
\end{align*}
$$

which is just the Lagrangian proposed in refs. [3,4].

## 3. Conformal supergravity

The superconformal group [8] * $\operatorname{SU}(2,2 \mid 1)$ has 24 generators. We use the notation and conventions of ref. [6]. For convenience, the formulae for the curvatures and transformation laws are summarized in appendix A. The correspondence among generators, gauge fields and curvatures is as follows:
generator: $\quad P_{a}, \quad K_{a}, \quad M_{a b}, \quad Q_{\alpha}, \quad S_{\alpha}, \quad D, \quad A$,
gauge field: $e_{\mu}^{a}, \quad f^{a}{ }_{\mu}, \quad \omega_{\mu a b}, \quad \bar{\psi}_{\mu}^{\alpha}, \quad \bar{\phi}_{\mu}^{\alpha} \quad B_{\mu}, \quad A_{\mu}$,
curvatures: $R_{\mu \nu}^{a}(P), R_{\mu \nu}^{a}(K), R_{\mu \nu}^{a b}(M), R_{\mu \nu}^{\alpha}(Q), R_{\mu \nu}^{\alpha}(S), R_{\mu \nu}(D), R_{\mu \nu}(A)$.
As before, we are imposing the constraint

$$
\begin{equation*}
R_{\mu \nu a}(P)=0, \tag{17}
\end{equation*}
$$

which allows to solve for $\omega_{\mu a b}(e, B)$, but consistency requires also [6], a constraint on the curvatures associated with the $Q$-supersymmetry, namely

$$
\begin{equation*}
R_{\mu \nu}(Q) \gamma^{\mu}=0 \tag{18}
\end{equation*}
$$

This constraint allows to solve for $\phi_{\mu}$. It also implies [6], a self-duality type of relation on $R_{\mu \nu}(Q)$,

$$
\begin{equation*}
R_{\mu \nu}(Q)+\frac{1}{2} \tilde{R}_{\mu \nu}(Q) \gamma_{5}=0 \tag{19}
\end{equation*}
$$

where

$$
\tilde{R}_{\mu \nu}(Q) \equiv e \epsilon_{\mu \nu \rho \sigma} R^{\rho \sigma}(Q)
$$

Now we take as the linear subgroup $H=$ Lorentz $X$ dilatations $X$ chiral $U(1)$, and the most general Lagrangian invariant under this subgroup of $\mathrm{G}=\mathrm{SU}(2,2 \mid 1)$ and built out of the $R_{\mu \nu}^{a b}(M), R_{\mu \nu}(D), R_{\mu \nu}(A), R_{\mu \nu}^{\alpha}(Q)$ and $R_{\mu \nu}^{\alpha}(S)$ curvatures and $e^{a}{ }_{\mu}$, $f^{a}{ }_{\mu}, \bar{\phi}_{\mu}$ and $\bar{\psi}_{\mu}$ gauge fields has the form

$$
\mathcal{L}=e a_{1}\left(-f \hat{R}+2 f_{\mu \nu} \hat{R}^{\mu \nu}(M)\right)+e a_{2}\left(f^{2}-f_{\mu \nu} f^{\nu \mu}\right)+e a_{3} f_{\mu \nu} f^{\mu \nu}
$$

[^1]\[

$$
\begin{align*}
& +e a_{4} f_{\mu \nu} f^{\nu \mu}+e a_{5} f_{\mu \nu} \hat{R}^{\nu \mu}(M)+e a_{6} f_{\mu \nu} \hat{R}^{\mu \nu}(M)+e a_{7} f_{\mu \nu} \hat{R}^{\mu \nu}(D) \\
& +e a_{8} f_{\mu \nu} \widetilde{R}^{\mu \nu}(A)+e a_{9} f \bar{\psi}_{\mu} \sigma^{\mu \nu} \phi_{\nu}+e a_{10} f_{\mu \nu} \bar{\psi}_{\rho} \sigma^{\rho \nu} \phi^{\mu} \\
& +e a_{11} f_{\mu \nu} \bar{\psi}^{\mu} \sigma^{\nu \rho} \phi_{\rho}+e a_{12} f_{\mu \nu} \bar{\psi}^{\nu} \sigma^{\mu \rho} \phi_{\rho}+e a_{13} f_{\mu \nu} \bar{\psi}_{\rho} \sigma^{\rho \mu} \phi^{\nu} \\
& +e a_{14} f_{\mu \nu} \bar{\psi}_{\rho} \sigma^{\mu \nu} \phi^{\rho}+e a_{15} f_{\mu \nu} \bar{\psi}^{\mu} \phi^{\nu}+e a_{16} f_{\mu \nu} \bar{\psi}^{\nu} \phi^{\mu} \\
& +e a_{17} f \bar{\psi}_{\mu} \phi^{\mu}+a_{18} f_{\mu \nu} \epsilon^{\mu \nu \rho \sigma} \bar{\psi}_{\rho} \gamma_{5} \phi_{\sigma}+e a_{19} f^{\mu \nu} R_{\rho \nu}(Q) \gamma_{\mu} \psi^{\rho} \\
& +e a_{20} \hat{R}_{\mu \nu}(D) \hat{R}^{\mu \nu}(D)+e a_{21} \hat{R}_{\mu \nu}(D) \widetilde{R}^{\mu \nu}(A)+e a_{22} R_{\mu \nu}(A) R^{\mu \nu}(A) \\
& +e a_{23} \hat{R}_{\mu \nu}(D) \bar{\psi}^{\mu} \phi^{\nu}+a_{24} \epsilon^{\mu \nu \rho \sigma} \hat{R}_{\mu \nu}(D) \bar{\psi}_{\rho} \gamma_{5} \phi_{\sigma} \\
& +e a_{25} \hat{R}_{\mu \nu}(D) \bar{\psi}_{\rho} \sigma^{\rho \mu} \phi^{\nu}+a_{26} \hat{R}_{\mu \nu}(D) \bar{\psi}^{\mu} \sigma^{\nu \rho} \phi_{\rho} \\
& +e a_{27} \hat{R}_{\mu \nu}(D) \bar{\psi}^{\rho} \sigma^{\mu \nu} \phi_{\rho}+a_{28} \epsilon^{\mu \nu \rho \sigma} \epsilon^{a b c d} \hat{R}_{\mu \nu a b} \hat{R}_{\rho \sigma c d} \\
& +e a_{29} \hat{R}_{\mu \nu a b} \bar{\psi}^{a} \sigma^{\mu \nu} \phi^{b}+e a_{30} \hat{R}_{\mu \nu a b} \bar{\psi}^{\mu} \sigma^{a b} \phi^{\nu} \\
& +e a_{31} \hat{R}_{\mu \nu}(M) \bar{\psi}^{\mu} \phi^{\nu}+e a_{32} \hat{R}_{\mu \nu} \bar{\psi}^{\nu} \phi^{\mu}+e a_{33} \hat{R} \bar{\psi}_{\mu} \phi^{\mu} \\
& +e a_{34} \hat{R} \bar{\psi}_{\mu} \sigma^{\mu \nu} \phi_{\nu}+e a_{35} \hat{R}_{\mu \nu}(M) \bar{\psi}_{\rho} \sigma^{\rho \nu} \phi^{\mu}+e a_{36} \hat{R}_{\mu \nu}(M) \bar{\psi}_{\rho} \sigma^{\rho \mu} \phi^{\nu} \\
& +e a_{37} \hat{R}_{\mu \nu}(M) \bar{\psi}^{\mu} \sigma^{\nu \rho} \phi_{\rho}+e a_{38} \hat{R}_{\mu \nu}(M) \bar{\psi}^{\nu} \sigma^{\mu \rho} \phi_{\rho}+e a_{39} \hat{R}_{\mu \nu}(M) \bar{\psi}_{\rho} \sigma^{\mu \nu} \phi^{\rho} \\
& +e a_{40} \tilde{R}_{\mu \nu}(A) \bar{\psi}^{\mu} \phi^{\nu}+e a_{41} R_{\mu \nu}(A) \bar{\psi}^{\mu} \gamma_{5} \phi^{\nu}+a_{42} \epsilon^{\mu \nu \rho \sigma} R_{\mu \nu}(Q) \gamma_{5} \hat{\bar{R}}_{\rho \sigma}(S) \\
& +a_{43} \epsilon^{\mu \nu \rho \sigma} \hat{R}_{\mu \nu}(S) \gamma_{5} \gamma_{\sigma} \phi_{\rho}+e a_{44} \hat{R}_{\mu \nu}(S) \gamma^{\mu} \phi^{\nu}+e a_{45} R_{\rho \nu}(Q) \gamma_{\mu} \psi^{\rho} \bar{\psi}^{\mu} \phi^{\nu} \\
& +e a_{46} R_{\rho \nu}(Q) \gamma_{\mu} \psi^{\rho} \bar{\psi}^{\nu} \phi^{\mu}+e a_{47} R_{\rho \nu}(Q) \gamma_{\mu} \psi^{\rho} \bar{\psi}^{\mu} \sigma^{\nu \rho} \phi_{\rho} \\
& +e a_{48} R_{\rho \mu}(Q) \gamma_{\nu} \psi^{\rho} \bar{\psi}^{\mu} \sigma^{\nu \rho} \phi_{\rho}+e a_{49} R_{\rho \nu}(Q) \gamma_{\mu} \psi^{\rho} \bar{\psi}_{\rho} \sigma^{\rho \mu} \phi^{\nu} \\
& +e a_{50} R_{\rho \mu}(Q) \gamma_{\nu} \psi^{\rho} \bar{\psi}_{\rho} \sigma^{\rho \mu} \phi^{\nu}+e a_{51} R_{\rho \mu}(Q) \gamma_{\nu} \psi^{\rho} \bar{\psi}_{\rho} \sigma^{\mu \nu} \phi^{\rho} \\
& +a_{52} \epsilon^{\omega \nu \rho \sigma} \hat{R}_{\mu \nu}(M) \bar{\psi}_{\rho} \gamma_{5} \phi_{\sigma} \\
& + \text { non-derivative terms . } \tag{20}
\end{align*}
$$
\]

In the last equation, non-derivative terms refer to all possible terms constructed out of two $\phi$ 's and two $\psi^{\prime}$ s, e.g., $e \bar{\psi}_{\mu} \phi_{\nu} \bar{\psi}^{\mu} \phi^{\nu}$. The terms including $f_{\mu \nu}$ were explicitly written in (20). A caret ( ${ }^{\wedge}$ ) on the curvature, means the curvature with all terms that contain $f_{\mu \nu}$ set equal to zero.

As before, we can solve the algebraic equation for $f_{\mu \nu}$. Requiring this equation to be consistent with the transformation properties of the various fields, we get the following relations:

From K-transformations

$$
\begin{align*}
& 4 a_{1}+a_{2}=a_{5}+a_{6} \\
& a_{3}-a_{4}=a_{5}-3 a_{6}+2 a_{7}, \\
& a_{3}+a_{4}=3\left(a_{5}+a_{6}\right), \\
& a_{9}=a_{10}=a_{11}=a_{12}=a_{13}=a_{14}=0, \\
& a_{15}=a_{16} \tag{21a}
\end{align*}
$$

From S-transformations

$$
\begin{align*}
& 2 a_{1}-a_{5}+a_{6}+a_{7}+4 i a_{8}+4 a_{19}=0 \\
& a_{15}=a_{16}=a_{17}=a_{18}=0 \tag{21b}
\end{align*}
$$

With these relations we can write

$$
\begin{align*}
f_{\mu \nu} & =-\frac{1}{4}\left(\hat{R}_{\nu \mu}(M)-\frac{1}{6} g_{\mu \nu} \hat{R}(M)\right)+i \alpha_{2} \tilde{R}_{\mu \nu}(A)+\left(\alpha_{1}+\beta_{1}\right) R_{\rho \nu}(Q) \gamma_{\mu} \psi^{\rho} \\
& +\left(\beta_{1}-\alpha_{1}\right) R_{\rho \mu}(Q) \gamma_{\nu} \psi^{\rho}+\alpha_{3}\left(2 \hat{R}_{\mu \nu}(D)+\hat{R}_{\mu \nu}(M)-\hat{R}_{\nu \mu}(M)\right) \tag{22}
\end{align*}
$$

where

$$
\begin{array}{ll}
\alpha_{1}=-\frac{a_{19}}{2\left(2 a_{2}+2 a_{3}-2 a_{4}\right)}, & \beta_{1}=-\frac{a_{19}}{8\left(2 a_{1}+a_{5}+a_{6}\right)} \\
\alpha_{2}=\frac{i a_{8}}{2 a_{2}+2 a_{3}-2 a_{4}}, & \alpha_{3}=\frac{-a_{7}}{2\left(2 a_{2}+2 a_{3}-2 a_{4}\right)} \tag{23}
\end{array}
$$

From (21b) we can derive

$$
\begin{equation*}
\frac{1}{4}+4 \alpha_{3}-4 \alpha_{2}+8 \alpha_{1}=0 \tag{24}
\end{equation*}
$$

so there are only three independent parameters among the $\alpha_{i}, \beta_{i}$, but, certainly at this stage, still more $a_{i}$ 's.

To determine further parameters, we require invariance of $\mathcal{L}$, eq. (20), under special conformal transformations. As in the conformal group case, we do not have to vary $f_{\mu \nu}$ (the variation being multiplied by its field equation that vanishes identically). When this is done we obtain:

$$
\begin{array}{lll}
a_{1}=-32 a_{28}, & a_{2}=-4 a_{1}, & a_{8}=4 a_{21} \\
a_{19}+4 a_{42}=0, & a_{24}=a_{52}, & a_{45}=a_{46} \\
a_{41}, a_{22} \text { undetermined }, & a_{3}=-a_{4}=a_{7}=8 a_{20} \tag{25}
\end{array}
$$

all the others vanishing.

The Lagrangian can therefore be written in the form

$$
\begin{align*}
\mathcal{L}= & a_{28} \epsilon^{\mu \nu \rho \sigma} \epsilon^{a b c d} R_{\mu \nu a b}(M) R_{\rho \sigma c d}(M)+a_{42} \epsilon^{\mu \nu \rho \sigma} R_{\mu \nu}(Q) \gamma_{5} \bar{R}_{\rho \sigma}(S) \\
& +e a_{21} R_{\mu \nu}(D) \tilde{R}^{\mu \nu}(A)+e a_{22} R_{\mu \nu}(A) R^{\mu \nu}(A)+e a_{20} R_{\mu \nu}(D) R^{\mu \nu}(D) \\
& +e a_{41} R_{\mu \nu}(A) \bar{\psi}^{\mu} \gamma_{5} \phi^{\nu}+a_{24} \epsilon^{\mu \nu \rho \sigma} \hat{R}_{\mu \nu}(D) \bar{\psi}_{\rho} \gamma_{5} \phi_{\sigma} \\
& +a_{24} \epsilon^{\mu \nu \rho \sigma} \hat{R}_{\mu \nu}(M) \bar{\psi}_{\rho} \gamma_{5} \phi_{\sigma}+e a_{45} R_{\rho \nu}(Q) \gamma_{\mu} \psi^{\rho}\left(\bar{\psi}^{\mu} \phi^{\nu}+\bar{\psi}^{\nu} \phi^{\mu}\right) . \tag{26}
\end{align*}
$$

While performing these transformation on the Lagrangian we did not consider the non-derivative terms (products of two $\psi$ 's and two $\phi$ 's) because they could not cancel the curvature terms. It is not difficult to write down the most general linear combination of such terms. We omit the details, but it is then easy to show that invariance under special conformal transformations alone, requires all these terms to vanish.

We now require invariance under $S$-transformations (the square-roots of the special conformal transformations). We get then

$$
\begin{array}{ll}
8 a_{28}+a_{42}=0, & a_{20}+2 i a_{21}-a_{42}=0 \\
2 i a_{21}-4 a_{22}+3 a_{42}=0, & a_{24}=a_{41}=a_{45}=0 \tag{27}
\end{array}
$$

Thus,

$$
\begin{align*}
\mathscr{L}= & a_{28} \epsilon^{\mu \nu \rho \sigma} \epsilon^{a b c d} R_{\mu \nu a b}(M) R_{\rho \sigma c d}(M)-8 a_{28} \epsilon^{\mu \nu \rho \sigma} R_{\mu \nu}(Q) \gamma_{5} \bar{R}_{\rho \sigma}(S) \\
& +e a_{21} R_{\mu \nu}(D) \widetilde{R}^{\mu \nu}(A)+e a_{22} R_{\mu \nu}(A) R^{\mu \nu}(A)+e a_{20} R_{\mu \nu}(D) R^{\mu \nu}(D) . \tag{28}
\end{align*}
$$

The four coefficients in (28) are not all independent but satisfy the two relations

$$
\begin{equation*}
a_{20}+2 i a_{21}+8 a_{28}=0, \quad 2 i a_{21}-4 a_{22}+24 a_{28}=0 \tag{29}
\end{equation*}
$$

The Lagrangian in eq. (28) is invariant under $M, D, K, A, S$ and general coordinate transformations. To have invariance under the full superconformal group we still have to require invariance under $Q$-supersymmetry (the square roots of the momenta). This will give one more relation

$$
\begin{equation*}
a_{20}=0, \tag{30}
\end{equation*}
$$

and all the coefficients are determined (except for an overall constant). We get then the final Lagrangian

$$
\begin{align*}
\mathcal{L}= & a_{28}\left\{\epsilon ^ { \mu \nu \rho \sigma } \left[\epsilon^{a b o d} R_{\mu \nu a b}(M) R_{\rho \sigma c d}(M)-8 R_{\mu \nu}(Q) \gamma_{5} \bar{R}_{\rho \sigma}(S)\right.\right. \\
& \left.\left.+4 i R_{\mu \nu}(A) R_{\rho \sigma}(D)\right]-8 R_{\mu \nu}(A) R^{\mu \nu}(A)\right\}, \tag{31}
\end{align*}
$$

which is precisely the Lagrangian proposed in ref. [6].

## 4. Conclusions

We have shown that the method of non-linear representations when applied to the conformal group $\operatorname{SU}(2,2)$, the linear subgroup being Lorentz $\times$ dilatations gives, in the Gürsey and Marchildon gauge [1], the Weyl Lagrangian [5]. We had to further require invariance under special conformal transformations before we were uniquely led to the Lagrangian proposed in [3] and [4].

When we apply the same method to the superconformal group $\operatorname{SU}(2,2 \mid 1)$, the linear subgroup being Lorentz $X$ dilatations $X$ chiral $U(1)$, we obtain a fairly complicated Lagrangian. If we then require invariance under the special conformal transformations, their square roots ( $S$-transformations) and $Q$-supersymmetry transformations we are led to the Lagrangian of ref. [6].

We close by noting that the more general non-linear realizations of refs. [1,2] are not equivalent to the simple conformal and superconformal theories. The precise content of these much richer "non-linear" theories is yet to be clarified.

Following the completion of this work, we received a preprint by Dr. L. Marchildon [9] on the non-linear realizations of the superconformal group. The emphasis of his paper is very different from ours and there is little overlap in results with us.

I am very grateful to Professor Peter Freund for suggesting this problem to me, as well as for numerous discussions, suggestions and a critical reading of the manuscript.

## Appendix A

## (a) The curvatures

We give below the formulae for the curvatures of the superconformal group:

$$
\begin{align*}
& R_{\mu \nu a b}(M)=R_{\mu \nu a b}^{(0)}-2\left[\left(e_{a \mu} f_{b \nu}-a \leftrightarrow b\right)-\mu \leftrightarrow \nu\right]-\bar{\psi}_{\mu} \sigma_{a b} \phi_{\nu}+\bar{\psi}_{\nu} \sigma_{a b} \phi_{\mu}, \\
& R_{\mu \nu}(D)=-\partial_{\mu} B_{\nu}+\bar{\psi}_{\mu} \phi_{\nu}+2 f_{\mu \nu}-(\mu \leftrightarrow \nu), \\
& R_{\mu \nu a}(P)=-\partial_{\mu} e_{a \nu}+\omega_{\mu a}^{b} e_{b \nu}+\frac{1}{4} \bar{\psi}_{\mu} \gamma_{a} \psi_{\nu}+e_{a \mu} B_{\nu}-(\mu \leftrightarrow \nu), \\
& R_{\mu \nu a}(K)=-\partial_{\mu} f_{a \nu}+\omega_{\mu a}^{b} f_{b \nu}-\frac{1}{4} \bar{\phi}_{\mu} \gamma_{a} \phi_{\nu}-f_{a \mu} B_{\nu}-(\mu \leftrightarrow \nu), \\
& R_{\mu \nu}(A)=-\partial_{\mu} A_{\nu}-i \bar{\psi}_{\mu} \gamma_{5} \phi_{\nu}-(\mu \leftrightarrow \nu), \\
& R_{\mu \nu}^{\alpha}(Q)=\left(D_{\nu} \bar{\psi}_{\mu}+\bar{\phi}_{\mu} \gamma_{\nu}+\frac{1}{2} B_{\nu} \bar{\psi}_{\mu}-\frac{3}{4} i A_{\nu} \bar{\psi}_{\mu} \gamma_{5}\right)^{\alpha}-(\mu \leftrightarrow \nu), \\
& R_{\mu \nu}^{\alpha}(S)=\left(D_{\nu} \bar{\phi}_{\mu}-\bar{\psi}_{\mu} \gamma_{a} f^{a}{ }_{\nu}-\frac{1}{2} B_{\nu} \bar{\phi}_{\mu}+\frac{3}{4} i A_{\nu} \bar{\phi}_{\mu} \gamma_{5}\right)^{\alpha}-(\mu \leftrightarrow \nu), \tag{A.1}
\end{align*}
$$

where

$$
\begin{align*}
& R_{\mu \nu a b}^{(0)}=-\partial_{\mu} \omega_{\nu a b}+\omega_{\mu a}^{c} \omega_{\nu c b}-(\mu \leftrightarrow \nu) \\
& F_{\mu \nu}=\partial_{\mu} B_{\nu}-\partial_{\nu} B_{\mu} \\
& D_{\nu} \psi_{\mu}=\left(\partial_{\nu}-\frac{1}{2} \sigma^{a b} \omega_{\nu a b}\right) \psi_{\mu} \tag{A.2}
\end{align*}
$$

To obtain the curvatures for the conformal group, set $A_{\mu}=\psi_{\mu}=\phi_{\mu}=0$ in the previous formulae.

## (b) Transformation laws

$K$-transformations (parameter $\xi^{a}$ )

$$
\begin{align*}
& \delta^{K} R_{\mu \nu}^{a}(P)=\delta^{K} R_{\mu \nu}(Q)=\delta^{K} R_{\mu \nu}(A)=\delta^{K} R_{\mu \nu}(D)=\delta^{K} R_{\mu \nu a b}(M)=0, \\
& \delta^{K} R_{\mu \nu}(S)=R_{\mu \nu}(Q) \gamma \cdot \xi, \\
& \delta^{K} e_{\mu}^{a}=\delta^{K} \bar{\psi}_{\mu}=\delta^{K} A_{\mu}=0, \\
& \delta^{K} B_{\mu}=-2 \xi_{\mu}, \quad \delta^{K} \bar{\phi}_{\mu}=\bar{\psi}_{\mu} \gamma \cdot \xi . \tag{A.3}
\end{align*}
$$

$S$-transformations (parameter $\lambda^{\alpha}$ )

$$
\begin{align*}
& \delta^{S} R_{\mu \nu}^{a}(P)=\delta^{S} R_{\mu \nu}(Q)=0, \\
& \delta^{S} R_{\mu \nu}(A)=-\frac{1}{2} i \tilde{R}_{\mu \nu}(Q) \lambda, \\
& \delta^{S} R_{\mu \nu}(D)=\delta^{S} R_{\mu \nu}(M)=-\frac{1}{2} R_{\mu \nu}(Q) \lambda, \\
& \delta^{S} e_{\mu}^{a}=0, \quad \delta^{S} \bar{\psi}_{\mu}=\bar{\lambda} \gamma_{\mu}, \quad \delta^{S} A_{\mu}=i \bar{\psi}_{\mu} \gamma_{5} \lambda, \\
& \delta^{S} B_{\mu}=-\frac{1}{2} \bar{\psi}_{\mu} \lambda, \quad \delta^{S} f^{a}{ }_{\mu}=\frac{1}{2} \bar{\lambda} \gamma^{a} \phi_{\mu}, \\
& \delta^{S} \phi_{\mu}=\left(\partial_{\mu}-\frac{1}{2} \sigma_{a b} \omega_{\mu}^{a b}\right) \lambda-\frac{1}{2} B_{\mu} \lambda+\frac{3}{4} i \gamma_{5} \lambda A_{\mu} . \tag{A.4}
\end{align*}
$$

Q-transformations (parameter $\epsilon^{\alpha}$ )

$$
\begin{aligned}
& \delta^{Q} R_{\mu \nu a b}(M)=-\bar{\epsilon} \sigma_{a b} \bar{R}_{\mu \nu}(S), \\
& \delta^{Q} R_{\mu \nu}(D)=\frac{1}{2} \bar{\epsilon} \overline{R_{\mu \nu}}(S) \\
& \delta^{Q} R_{\mu \nu}(A)=-i \bar{\epsilon} \gamma_{5} \bar{R}_{\mu \nu}(S) \\
& \delta^{Q} R_{\mu \nu}(Q)=\frac{1}{2} \bar{\epsilon} R_{\mu \nu}^{a}(D)-\frac{3}{4} i \bar{\epsilon} \gamma_{5} R_{\mu \nu}(A)+\frac{1}{2} \bar{\epsilon} \sigma_{a b} R_{\mu \nu a b}(M),
\end{aligned}
$$

$$
\begin{align*}
& \delta^{Q} R_{\mu \nu}(S)=-\bar{\epsilon} \gamma_{a} R_{\mu \nu}(K) \\
& \delta^{Q} e_{\mu}^{a}=\frac{1}{2} \bar{\epsilon} \gamma^{a} \psi_{\mu} \tag{A.5}
\end{align*}
$$

## Appendix B

We want to determine the coefficients $a, b$ and $c$ such that the quantity

$$
\begin{equation*}
J=a R^{(0)} \delta^{K} f+b R^{(0) \mu \nu} \delta^{K} f_{\mu \nu}+c F^{\mu \nu} \delta^{K} f_{\mu \nu} \tag{B.1a}
\end{equation*}
$$

is an exact divergence

$$
\begin{equation*}
J=V_{; \mu}^{\mu} \tag{B.1b}
\end{equation*}
$$

Notice that we do not consider a term $R^{(0) \nu \mu} \delta^{k} f_{\mu \nu}$ because that would amount to redefining $b$ and $c$ above. We need the following relations

$$
\begin{align*}
& R^{(0)}=R+6 B_{; \mu}^{\mu}+6 B_{\mu} B^{\mu} \\
& R_{\mu \nu}^{(0)}=R_{\mu \nu}+2 B_{\nu ; \mu}+g_{\mu \nu} B_{; \sigma}^{\sigma}+2\left(g_{\mu \nu} B^{2}-B_{\mu} B_{\nu}\right), \\
& \delta^{K} f_{\mu \nu}=\xi_{\mu ; \nu}-\xi_{\nu} B_{\mu}-\xi_{\mu} B_{\nu}+g_{\mu \nu} B \cdot \xi \tag{B.2}
\end{align*}
$$

where $\xi_{a}$ is the parameter of the special conformal transformations. Using (B.1), (B.2) we get

$$
\begin{align*}
J= & V_{; \mu}^{\mu}+\left(-a R^{(0) ; \mu}-b R_{; \rho}^{(0) \mu \rho}+c F_{; \rho}^{\rho \mu}\right) \xi_{\mu} \\
& +\left(a R^{(0)} g^{\mu \nu}+b R^{(0) \mu \nu}+c F^{\mu \nu}\right)\left(g_{\mu \nu} B \cdot \xi-\xi_{\mu} B_{\nu}-\xi_{\nu} B_{\mu}\right), \tag{B.3}
\end{align*}
$$

where

$$
\begin{equation*}
V^{\mu}=\left(a R^{(0)} \xi^{\mu}+b R^{(0) \nu \mu} \xi_{\nu}+c F^{\nu \mu} \xi_{\nu}\right) \tag{B.4}
\end{equation*}
$$

We now prove that all the terms in the parentheses either cancel or give an exact divergence. We separate them according to the different powers of $B_{\mu}$ and analyse each set separately.

Zero B's

$$
\begin{align*}
J^{(0)} & =\left(-a R^{; \mu}-b R_{; \rho}^{\rho \mu}\right) \xi_{\mu} \\
& =\left[\left(-a-\frac{1}{2} b\right) R^{; \mu}\right] \xi_{\mu} \tag{B.5}
\end{align*}
$$

where we have used

$$
R_{; \rho}^{\rho \mu}=\frac{1}{2} R^{; \mu}
$$

We get, therefore, the condition

$$
\begin{equation*}
b+2 a=0 \tag{B.6}
\end{equation*}
$$

## Linear terms

$$
\begin{aligned}
J^{(1)} & =\left[-6 a B_{; \sigma}^{\sigma} ; \mu\right. \\
& +\left(2 a B^{\nu ; \mu} ; \nu-b B_{; \sigma}^{\sigma} ; \mu+c\left(B^{\rho ; \mu} ; \rho-B^{\mu ; \rho} ; \rho\right)\right. \\
& =\left[-3(2 a+b) B^{\mu}-2 b R^{\rho \mu} B_{\rho}\right] \sigma_{\mu} ; \mu+(2 a+b) R B^{\mu} \\
& \left.+c\left(B^{\rho ; \mu} ; \rho-B_{; \rho}^{\mu ; \rho} ; \rho\right)\right] \xi^{\mu}=0, \quad \bmod (\text { exact divergence }),
\end{aligned}
$$

which implies

$$
\begin{equation*}
2 a+b=0, \quad c=0 \tag{B.7}
\end{equation*}
$$

Similar calculations for the quadratic and cubic terms give the same relations (B.6) and (B.7). Therefore,

$$
\begin{equation*}
J=e\left[a R^{(0)} \delta^{K} f-2 a R^{(0) \mu \nu} \delta^{K} f_{\mu \nu}\right]=\text { exact divergence }, \tag{B.8}
\end{equation*}
$$

as used in the text.

## References

[1] F. Gürsey and L. Marchildon, Phys. Rev. D17 (1978) 2038.
[2] L.N. Chang and F. Mansouri, Yale reports No. COO-3075-187, COO-3075-189 (1978).
[3] J. Crispim-Romāo, A. Ferber and P.G.O. Freund, Nucl. Phys. B126 (1977) 429.
[4] M. Kaku, P.K. Townsend and P. van Nieuwenhuizen, Phys. Lett. 69B (1977) 304.
[5] H. Weyl, Raum, Zeit, Materie, 5th ed. (Springer, Berlin, 1923).
[5a] P.G.O. Freund, Ann. of Phys. 84 (1974) 440.
[6] M. Kaku, P.K. Townsend and P. van Nieuwenhuizen, Phys. Rev. D17 (1978) 3179.
[7] R. Bach, Math. Z. 9 (1921) 110.
[8] J. Wess and B. Zumino, Nucl. Phys. B70 (1974) 39.
[8a] P.G.O. Freund and I. Kaplansky, J. Math. Phys. 17 (1976) 228.
[9] L. Marchildon, Yale report No. COO-3075-202 (1978).


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    ** See also, ref. [5a].

[^1]:    * See also, ref. [8a].

