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UNIFIED SUPERCONFORMAL GAUGE THEORIES *

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An action principle for a superconformal gauge theory based on the supergroup SU(2,2|N) is constructed using a graded Weyl-like geometry. The internal symmetry is U(N). The theory unifies gravity with other gauge interactions. The explicit gauging of conformal transformations is explored.

1. Introduction

The gauging of supersymmetries, just as that of ordinary symmetries, is geometric in nature [1]. In addition to the early super-Riemannian theories [2], supergravity [3] has recently been recast in geometric form [4,5]. A genuine unification of geometry and internal symmetry is expected in theories [6] that start from the superconformal SU(2,2|N) supersymmetries [7,8]. In ref. [6] we have described the algebraic structure of such theories. Here we construct a superconformal geometry and use it to write down action principles for our theory. We shall also consider the geometry of extended ordinary (i.e., not superconformal) supergravity. Our approach is rooted in superspace just like that of Wess, Zumino, Volkov et al. [4]. We shall briefly comment on conformal theories in an ordinary space setting à la MacDowell and Mansouri [5].

2. Superspace formalism and extended supergravity

We start from a superspace with 4 Bose space-time coordinates x^m ($m = 0, \dots, 3$) and $4N$ Fermi coordinates $\theta^{\mu\hat{\mu}}$ ($\mu = 1, \dots, 4$ is the Majorana spinor index, $\hat{\mu} = 1, \dots, N$ is the internal symmetry index). On the bundle of bases of this superspace we define [4] the solder form (supervierbein) $e^A = \mathrm{d}z^M e_M^A$. Here capital indices run over all 4 Bose and $4N$ Fermi values and the collective notation z^M is introduced for x^m and

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$\theta^{\mu\hat{\mu}}$. We next define a connexion form $\Phi_A{}^B = dz^M \Phi_{MA}{}^B$ valued not in the whole GL(4|4N) superalgebra (or we would be worse off than in super-Riemannian theory) but only in a subalgebra S thereof. The choice of S then determines the theory.

For the moment, with extended supergravity in mind, we choose S as the direct sum of the Lorentz and of the internal O(N) algebras, so that the connexion coefficients are given by

$$\Phi_{Ma}{}^b = \omega_{Ma}{}^b + S_{Ma}{}^b \ ,$$

$$\Phi_{M\alpha\hat{\alpha}}{}^{\beta\hat{\beta}} = \tfrac{1}{8}\omega_{Md}{}^e[\gamma^d, \gamma_e]_\alpha{}^\beta \delta_{\hat{\alpha}}{}^{\hat{\beta}} + A^i_M \Lambda^i_{\alpha\hat{\alpha}}{}^{\beta\hat{\beta}} + S_{M\alpha\hat{\alpha}}{}^{\beta\hat{\beta}} \ ,$$

$$\Phi_{Ma}{}^{\beta\hat{\beta}} = \Phi_{M\alpha\hat{\alpha}}{}^b = 0 \ , \tag{1}$$

with

$$\omega_{Mab} = -\omega_{Mba} \ ,$$

$$\Lambda^i_{\alpha\hat{\alpha}}{}^{\beta\hat{\beta}} = \delta_\alpha{}^\beta \lambda^{i}_{\hat{\alpha}}{}^{\hat{\beta}} \ , \qquad \lambda^{i}_{\hat{\alpha}}{}^{\hat{\beta}} = -\lambda^{i}_{\hat{\beta}}{}^{\hat{\alpha}} \ , \qquad i = 1, \ldots, \tfrac{1}{2}N(N-1) \ ,$$

$$S_{Ma}{}^b = S_{M\alpha\hat{\alpha}}{}^{\beta\hat{\beta}} = 0 \ . \tag{2}$$

The λ^i's generate O(N). The here vanishing and superfluous S's have been introduced solely for future convenience. The ω's and A's are functions of x^m and $\theta^{\mu\hat{\mu}}$. The covariant differentials of the solder and connexion forms yield [4] torsion and curvature forms T^A and $R_A{}^B$. The coefficients are, (notation: a in $(-1)^a$ means the grade of z^A)

$$T_{BC}{}^A = (-1)^{b(m+c)} e_C{}^M e_B{}^N \partial_N e_M{}^A - (-1)^{cm} e_B{}^M e_C{}^N \partial_N e_M{}^A$$
$$+ e_B{}^N \Phi_{N,C}{}^A - (-1)^{bc} e_C{}^N \Phi_{N,B}{}^A \ , \tag{3}$$

$$R_{DCA}{}^B = [(-1)^{d(m+c)} e_C{}^M e_D{}^N - (-1)^{cm} e_D{}^M e_C{}^N]$$
$$\times [\partial_N \Phi_{M,A}{}^B + (-1)^{n(m+a+e)} \Phi_{M,A}{}^E \Phi_{N,E}{}^B] \ .$$

Kronecker's δ^B_A and the quantities

$$Q^{AB} = \begin{cases} C^{\alpha\beta}\delta^{\hat{\alpha}\hat{\beta}} & \text{for } A = \alpha\hat{\alpha}, \quad B = \beta\hat{\beta} \ , \\ 0 & \text{otherwise} \ , \end{cases} \tag{4}$$

(C = charge conjugation matrix) are invariant tensors. These invariant tensors together with the curvature and torsion tensors (3) can be used to construct invariant actions. As an illustration, we consider the simple invariant action (a generalization of the $N = 1$ ansatz of Volkov et al. [4])

$$A = \int \mathrm{d}^{4+4N} z \, \det(e_M{}^A)(-1)^{b+bc} Q^{CA} R_{ABC}{}^B \, . \tag{5}$$

It leads to field equations which for $N = 1$ after some straightforward algebra and suppressing the (for $N = 1$) irrelevant internal index take the form:

$$T_{\beta\gamma}{}^A (\Sigma^{ef})^{\beta\gamma} = 0 \, , \qquad (-1)^b T_{\alpha B}{}^B = 0 \, , \qquad R_{\alpha B}{}^{\alpha\gamma} = 0 \, , \tag{6}$$

where $(\Sigma^{ef})^{\beta\gamma} = (1/2i) C^{\beta\alpha} (\sigma^{ef})_\alpha{}^\gamma$, with C the charge conjugation matrix. These equations are compatible with those from which Wess and Zumino [4] derived ordinary supergravity.

For $N > 1$ it seems paradoxical that we should have gauged internal $O(N)$ symmetry without a cosmological term. This is due to an unorthodox contraction of the de-Sitter superalgebra. One usually associates the internal $O(N)$ with the de-Sitter like $OSp(N|4)$ superalgebra and then contracts $OSp(N|4)$ to get the physical algebra. The contraction proceeds as follows (see the first paper of ref. [7]): multiply all Fermi charges by $\sqrt{\lambda}$ and the de-Sitter boosts by λ, thereby the latter become translations and together with the Lorentz generators span the Poincaré algebra. Now, this still leaves open the contraction factor of the internal $O(N)$ charges. Were one to contract these with the same factor λ as the de-Sitter boosts, they would become central charges and the theory would have a decent flat space limit according to the theorem of Haag et al. [8]. On the other hand, if one chooses *not* to contract the internal $O(N)$ charges (i.e., to multiply them with $\lambda^0 = 1$ rather than with λ) one gets a superalgebra containing the direct sum of the Poincaré and $O(N)$ algebras as its Bose part. The internal charges are thus $O(N)$ and not central. The theory again has an admissible [8] flat limit. With this contraction the cosmological term vanishes as can be cross-checked e.g., for $N = 2$ by explicitly contracting the $OSp(2|4)$ Lagrangian of ref. [9]. It is this type of theory * that is covered by the ansatz (1), (2). It is an interesting problem to see whether this formalism can be adapted for full $OSp(N|4)$ gauging. For now, we prefer to go on to superconformal theories.

3. Superconformal theories

As was pointed out in ref. [6], the step from supergravity to superconformal theory is analogous to that from Einstein gravity to Weyl theory [10]. In Einstein theory the connexion is valued in the Lorentz algebra, whereas in Weyl theory it is valued in the direct sum of the Lorentz and Weyl-dilatation algebras. No connexion co-

* See note added in proof.

efficients (i.e., gauge fields) are introduced for the conformal boosts (for an alternative see sect. 4). Similarly, in the graded case we leave conformal boosts and their fermionic "square roots" the superconformal boosts ungauged and value the connexion in a subalgebra of the superconformal superalgebra SU(2,2|N): the direct sum of the Lorentz, Weyl-dilatation and internal U(N)-symmetry algebras. The latter has the N^2 generators [6]

$$H^i_{\alpha\hat{\alpha}}{}^{\beta\hat{\beta}} = \begin{cases} \delta_\alpha{}^\beta \lambda^{i}_{\hat{\alpha}}{}^{\hat{\beta}} & \text{if } \lambda^i \text{ is an antisymmetric matrix} \\ (\gamma_5)_\alpha{}^\beta \lambda^i_{\hat{\alpha}}{}^{\hat{\beta}} & \text{if } \lambda^i \text{ is a symmetric matrix .} \end{cases} \tag{7}$$

Here λ^i provide a basis for the hermitean $N \times N$ matrices. The H's without a γ_5 generate the O(N) subgroup of U(N).

The connexion coefficients $\Phi_{MA}{}^B$ now have again the form (1) but with

$$\begin{aligned} &\omega_{Mab} = -\omega_{Mba} , \\ &\Lambda^i_{\alpha\hat{\alpha}}{}^{\beta\hat{\beta}} = H^i_{\alpha\hat{\alpha}}{}^{\beta\hat{\beta}} , \\ &S_{Ma}{}^b = D_M \delta_a{}^b , \qquad S_{M\alpha\hat{\alpha}}{}^{\beta\hat{\beta}} = \tfrac{1}{2} D_M \delta_\alpha{}^\beta \delta_{\hat{\alpha}}{}^{\hat{\beta}} , \end{aligned} \tag{2a}$$

replacing eq. (2) (here H^i are those defined by eqs. (7)). The superfield D_M is the generalization of Weyl's gauge field. The formulae (3) still define torsion and curvature. To generalize the invariant tensor Q^{AB} one may replace the factor $\delta^{\hat{\alpha}\hat{\beta}}$ on the right-hand side of eq. (4) by a matrix $R^{\hat{\alpha}\hat{\beta}}$. Invariance under internal SU(N) gauge transformations then requires for the traceless λ^i's the matrix equations $R\lambda^i + \lambda^{iT}R = 0$. For $N = 2$ this specifies $R = i\sigma_2$, but for $N \geqslant 3$ such an R-matrix does not exist (R-reflection is an *outer* automorphism). Thus internal SU(N) gauge symmetry cannot be insured *this way* (we shall presently give a method for insuring U(N) symmetry!) except for $N = 2$, and even then full U(2) symmetry fails. We may nevertheless use the invariant tensor (4) to increase the gauge symmetry at least to the extent of Weyl dilatations. Indeed if instead of (2) and (2a) we consider the intermediate case

$$\begin{aligned} &\omega_{Mab} = -\omega_{Mba} , \\ &\Lambda^i_{\alpha\hat{\alpha}}{}^{\beta\hat{\beta}} = \text{as in eq. (2)} , \\ &S_{Ma}{}^b = D_M \delta_a{}^b , \qquad S_{M\alpha\hat{\alpha}}{}^{\beta\hat{\beta}} = \tfrac{1}{2} D_M \delta_\alpha{}^\beta \delta_{\hat{\alpha}}{}^{\hat{\beta}} , \end{aligned} \tag{2b}$$

then Q^{AB} of eq. (4) is a tensor of Weyl dilatation weight 1 (the transformation of tensors under gauge transformations follows [4] from the prototypes

$$\delta v^A = v^B X_B{}^A , \qquad \delta v_A = -X_A{}^B v_B ;$$

Weyl dilatations correspond to

$$X_b{}^a = \xi\delta_b{}^a\,, \qquad X_{\beta\hat\beta}{}^{\alpha\hat\alpha} = \tfrac{1}{2}\xi\delta_\beta{}^\alpha\delta_{\hat\beta}{}^{\hat\alpha}\,,$$

and all other components of X vanishing).

Constructing the curvature tensor $R_{CDA}{}^B$ from the connexion (2b), we find that $\bar R = (-1)^{b+bc}Q^{CA}\bar R_{ABC}{}^B$ (with Q given by (4)) is a scalar of weight 1. Since $\det(e^A_M)$ has weight $2(2-N)$, the product of $\bar R$ with a suitable Jordan-Brans-Dicke factor [4] can serve as a lagrangian for this modest intermediate case (for $N = 2$ even SU(2) can be achieved by replacing $\delta^{\hat\alpha\hat\beta}$ by $\epsilon^{\hat\alpha\hat\beta}$ in $\Lambda^i_{\alpha\hat\alpha}{}^{\beta\hat\beta}$).

To deal with the full superconformal case (2a) we circumvent the necessity for invariant tensors by introducing new superfields ψ^{AB} transforming like a graded antisymmetric tensor density of weight $2(N-2)$.

$$\psi^{AB} = -(-1)^{ab}\psi^{BA}\,, \qquad \delta\psi^{AB} = \psi^{CB}X_C{}^A + \psi_C{}^B - \mathrm{Tr}(X)\psi^{AB}\,.$$

The action (μ is a real parameter):

$$A = \int \mathrm{d}^{4+4N}z\,\det(e_M{}^A)((-1)^{b+bc}\psi^{CA}R_{ABC}{}^B + (-1)^a\mu D_A\psi^{BC}T_{CB}{}^A) \quad (7)$$

is invariant under the x- and θ-dependent Lorentz, Weyl and internal U(N) transformations. The specific tensorial nature of ψ allows the appearance of the second term in eq. (7) which plays the role of a kinetic term for ψ, a pure kinetic term being unavailable on account of the absence of invariant tensors. The A^i_M's now involve U(N) Yang-Mills fields. Yet the variation of the action (7) leads to second order derivatives even in this "first order" formalism. This can be corrected by using an auxiliary vectorial superfield χ^A of Weyl-weight $N-2$ instead of ψ^{AB}. Then an invariant action is (ν_i are real parameters):

$$A = \int \mathrm{d}^{4+4N}z\,\det(e_M{}^A)[(-1)^{b+bc}\chi^C\chi^A R_{ABC}{}^B$$
$$+ (-1)^a\nu_1 D_A\chi^B D_B\chi^A + (-1)^{a+b}\nu_2 D_A\chi^A D_B\chi^B]\,. \quad (7b)$$

The superfields χ and ψ are graded tensorial counterparts of the scalar field used in Dirac's approach to Weyl theory [10].

The actions (7a) provide examples of superconformal gauge theories with internal U(N) symmetry.

4. The problem of conformal gauge fields

As in Weyl's ungraded theory, in our superconformal theory we gauged Weyl dilatations but not conformal and superconformal boosts. Conventional wisdom has it that conformal boosts are ungaugeable because the conformal current – the source of the envisioned conformal gauge field   is explicitly x-dependent and its appearance in the theory would wreck its translational invariance. We want to reanalyze here briefly this argument in the light of recent work by McDowell and Mansouri [5] (MM). For simplicity, we restrict our discussion to the ungraded case. We consider a principal fibre bundle with the conformal group O(4,2) as a fibre and a 4-dimensional space-time manifold at its base. We thus have gauge fields (connexion coefficients) corresponding to translations, Lorentz transformations, dilatations *and* conformal boosts. The O(4,2) structure constants then define a curvature tensor $R^A_{\mu\nu}$ ($\mu, \nu = 0, \dots, 3, A = 1, \dots 15$ is a conformal index) in terms of the gauge fields. Generalizing MM we postulate the action

$$A_C = \int \mathrm{d}^4 x \, \epsilon^{\mu\nu\rho\sigma} R^A_{\mu\nu} R^B_{\rho\sigma} M_{AB} \,, \tag{8a}$$

where parity conservation again forces the MM choice

$$M_{AB} = \begin{cases} \epsilon_{aa'bb'} & \text{for } A = aa',\ B = bb' \text{ Lorentz boosts}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{8b}$$

After separating out the topological invariant part (exact divergence in the integrand) we again get first order field equations for the gauge fields (the dilatation gauge field drops out altogether for the action (8), although it could be made to appear at the expense of parity conservation). This theory is manifestly translationally invariant (x does not appear explicitly in the lagrangian). However, to prove its local conformal invariance, one needs constraints that, unlike the de-Sitter case treated in MM, do not follow from the field equations. This is the reason why in the graded case we used directly superspace methods rather than the very elegant MM method. As has been noted [9], the latter loses much of its geometrical simplicity already in the case of extended supergravity.

Following the completion of this work, Professor P. van Nieuwenhuizen kindly informed us that he, P.K. Townsend and M. Kaku have also considered the matters covered in the last section of this paper.

Note added in proof

The superspace action (5) has zero cosmological term and was interpreted in sect. 2 as corresponding to an unusual (Haag-Łopuszanski-Sohnius admissible) contraction of OSp(N|4) gauge theory. It was found that e.g., the OSp(2|4) theory [9] looses its cosmological term when contracted this way. It may happen though, that after contraction, the OSp(N|4) gauge transformations themselves become singular so that in the limit of full contraction the theory is not gauge-invariant. Alternatively, when reduced from superspace to ordinary space the action may develop a super-cosmological term as in the uncontracted OSp(N|4) theories of the last paper of ref. [3] and of refs. [4,9]. The action principle (5) was considered in sect. 2 as a preliminary illustration of our methods. In view of these possibilities it now appears that this theory may be interesting in its own right and we are further investigating it. One of us (P.G.O.F.) would like to thank the participants of the Aspen Workshop on Supergravity, especially Drs. M. Kaku and P. van Nieuwenhuizen for their stimulating remarks.

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