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Derivation of gauge invariance from high-energy unitarity bounds on the S matrix*

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A systematic search is made for all renormalizable theories of heavy vector bosons. It is argued that in any renormalizable Lagrangian theory high-energy unitarity bounds should not be violated in perturbation theory (apart from logarithmic factors in the energy). This leads to the specific requirement of "tree unitarity": the N-particle S-matrix elements *in the tree approximation* must grow no more rapidly than E^{4-N} in the limit of high energy (E) at fixed, nonzero angles (i.e., at angles such that all invariants $p_i \cdot p_j$, $i \neq j$, grow like E^2). We have imposed this tree-unitarity criterion on the most general scalar, spinor, and vector Lagrangian with terms of mass dimension less than or equal to four; a certain class of nonpolynomial Lagrangians is also considered. It is proved that any such theory is tree-unitary if and only if it is equivalent under a point transformation to a spontaneously broken gauge theory, possibly modified by the addition of mass terms for vectors associated with invariant Abelian subgroups. Our result suggests that gauge theories are the only renormalizable theories of massive vector particles and that the existence of Lie groups of internal symmetries in particle physics can be traced to the requirement of renormalizability.

I. INTRODUCTION

The only systems of heavy vector bosons which are known to be renormalizable are spontaneously broken gauge theories[1] (SBGT's) and "conserved current" models. In an SBGT the field variables can always be chosen so that the Lagrangian is locally gauge-invariant. The vector bosons acquire mass through the mechanism of spontaneous symmetry breaking. Massless vector bosons have conserved source currents. On the other hand, "conserved current" models always contain at least one massive vector boson whose source current is conserved. Massive quantum electrodynamics (QED) is the simplest system of this type. The general prescription for constructing conserved-current models can be stated as follows: (1) Begin with a Lagrangian which is invariant under a nonsemisimple group of local gauge transformations (i.e., a group of transformations containing an invariant Abelian subgroup). (2) Arrange for spontaneous symmetry breaking (if any) such that the vacuum expectation values of the scalar fields are invariant under at least one invariant (single-parameter) Abelian subgroup (thus, at this stage the corresponding Abelian vector is massless and coupled to a conserved current). (3) Add an arbitrary mass term for the same Abelian vector. Notice that the Lagrangian is invariant under the entire group of global gauge transformations and under the semisimple subgroup of local gauge transformations.

Most massive vector Lagrangians are not renormalizable because the $k_\mu k_\nu$ term in the vector propagator induces "bad" high-energy behavior in the scattering amplitudes. Conserved-current and SBGT models are renormalizable because this "bad" high-energy behavior is vitiated by the symmetry of the vector couplings which multiply the "bad" $k_\mu k_\nu$ factors. For example, in conserved-current models the $k_\mu k_\nu$ factor is multiplied by and "eliminated" by the conserved source current of the massive Abelian vector. In the SBGT case massive vectors couple to source currents whose conservation is violated by spontaneous symmetry breaking. However, this nonconservation is systematically implemented so that the "bad" effects of $k_\mu k_\nu$ cancel out among the various diagrams contributing to any one S-matrix element. To summarize: In all heavy-vector-boson theories which are known to be renormalizable, the vector fields are coupled gauge invariantly; this gauge invariance seems to expedite the proof of renormalizability by removing "bad" high-energy behavior.

This leads to the following conjecture: Perhaps gauge invariance is a *necessary* property of any renormalizable theory of heavy vector particles. Such a connection would suggest that the origin of Lie groups of internal transformations in particle physics can be traced to the criterion of renormalizability. This paper describes a systematic search for renormalizable theories of heavy vector bosons; the results indicate that the above conjecture is true. First, we define a theory to be "tree-unitary" if the N-particle S-matrix elements in the tree approximation diverge no more rapidly than E^{4-N} in the high-energy limit (i.e., in the limit in which all angles are fixed and the over-all energy scale E increases to infinity). This means that S-matrix elements in the tree approximation "scale" at high energy; alternatively, tree unitarity can be considered to be a generalization of the usual "unitarity boundedness" criterion.[2] A dimensional argument makes it plausible that any perturbatively renormalizable theory must be tree-unitary. This connection is consistent with the fact that all known renormalizable models have been proved to be tree-unitary.[3] Roughly speaking, tree-unitary behavior is the most divergent high-energy behavior which is likely to be reproduced when trees are combined to form loop diagrams. If the tree approximation diverges more rapidly than E^{4-N}, the n-loop diagrams, which are higher and higher iterations of the trees, will diverge more and more rapidly and cannot be renormalized (with a finite number of counterterms). For this reason we believe that all theories which might be perturbatively renormalizable can be found by making a systematic search for all tree-unitary theories. Such a search was initiated but not completed in earlier publications by the present authors[4] and by others.[5,6] This paper[7] describes the first complete derivation of all tree-unitary models.

We have studied a class of Lagrangians constructed from arbitrary numbers of scalar, spinor, and massive vector fields. This class includes the most general scalar, spinor, and vector field interaction with mass dimension less than or equal to four. A wide range of nonpolynomial Lagrangians is also considered. Imposing tree unitarity on all multiparticle amplitudes results in an infinite set of relations between the coupling constants and masses of the theory. The problem then is to find all solutions of this infinite set of conditions. This task is simplified by exploiting the great freedom which is associated with the possibility of changing field variables. It is known[8] that two Lagrangians which are related by a point transformation of field variables have the same S matrix; in particular, the S matrix in the tree approximation is independent of the field coordinates in terms of which the Lagrangian is expressed. This suggests that constraints on the S matrix, like tree unitarity, can be written as constraints on the Lagrangian, which are independent of the choice of field coordinates. In order to construct such field-coordinate-independent statements, we define a Riemannian geometry on the manifold of fields which represent scalar particles and longitudinal modes of vectors. It turns out that the infinite set of tree-unitarity conditions is equivalent to a finite set of covariant (in the Riemann sense) differential equations to be obeyed by certain tensor quantities of the Riemannian manifold associated with the Lagrangian. We have found all solutions of these covariant tree-unitarity conditions. The results are as follows: If a Lagrangian describes a tree-unitary S matrix, then there must be a choice of field variables which puts the Lagrangian into the form of an SBGT, possibly modified by the addition of mass terms for vectors associated with invariant Abelian subgroups. We also prove the converse, namely, the covariant equations are *sufficient* so that all of the above theories are, in fact, fully tree-unitary. Hence, our central conclusion: Any Lagrangian of our class describes a tree-unitary S matrix if and only if the Lagrangian can be transformed into a gauge theory (modulo Abelian vector mass terms). Notice that all of the resulting models are invariant under global (and, possibly, local) groups of internal transformations. The gauge-invariant Lagrangians can be viewed as a set of "standard" forms into which any tree-unitary Lagrangian can be transformed. In this sense the criterion of unitarily bounded high-energy behavior imposes internal symmetry on heavy-vector-boson interactions.

The set of all tree-unitary Lagrangians can be divided into SBGT, conserved-current, and "hybrid"[9] categories. The conserved-current and hybrid models correspond to SBGT's for nonsemisimple groups, modified by the addition of mass terms for vectors associated with invariant Abelian subgroups. For example, suppose the gauge group contains just one invariant Abelian subgroup, and the Lagrangian contains a mass term for the corresponding vector boson. In the conserved-current case that Abelian subgroup is not spontaneously broken; the corresponding massive Abelian vector couples to a conserved current. In a hybrid model the Abelian subgroup is spontaneously broken; then the Abelian vector can mix with other vectors and need not couple to a conserved current. These hybrid theories have not received much attention. Since they are unitarily bounded, they may represent a *new* class of renormalizable theories of heavy vector bosons.

This paper is organized as follows. Section II is a discussion of the connection between renormalizability and the specific constraint of tree unitarity. In Sec. III we introduce the general class of Lagrangians to be studied and record the Feynman rules. In Sec. IV we discuss the consequences of tree-unitary behavior for the four-point scattering amplitudes of vector and spinor particles in the theory. These are relations among coupling constants showing that the purely vector terms in the Lagrangian must have a Yang-Mills structure associated with some compact Lie group and that the vector-spinor terms must have a gauge-invariant (minimal) form under the same group. In Sec. V we perform a generalized Stückelberg-type decomposition of the massive vector fields; this amounts to an ordinary Stückelberg decomposition followed by an unspecified point transformation. The net effect is to replace the massive vector fields with "Stückelberg vectors" (having "good" propagators of the Feynman type) and an equal number of "Stückelberg scalars." The proof that the tree approximation for the S matrix is unaffected by such changes of variables is relegated to Appendix A. Next, the entire Lagrangian is shown to be invariant under a group of nonlinear gauge transformations; this gauge invariance is a direct consequence of the form of the Stückelberg decomposition. Point transformations of the (physical and Stückelberg) scalar fields are conveniently treated in terms of an invariant geometry of the Riemann type which is naturally induced on the manifold of all scalar fields. In particular, the nonlinear gauge invariance of the Lagrangian is translated into a set of covariant (i.e., field-coordinate-independent) differential constraints on the Lagrangian. In Sec. VI a second set of covariant equations is derived by imposing tree unitarity on multiparticle amplitudes with scalar particles in the in and out states. In Sec. VII we show that the full set of covariant equations, which we have derived from tree unitarity, implies that the Lagrangian is equivalent under a point transformation to an SBGT, conserved-current, or hybrid Lagrangian. These results are summarized and discussed in Sec. VIII. Some of the lengthier derivations are carried out in appendixes. In Appendix A we prove the equivalence theorem stating that the S matrix is invariant under a generalized Stückelberg decomposition and a point transformation of the scalar fields. Appendix B contains the proof of a lemma used in Sec. VI to obtain covariant equations from the tree unitarity of scalar interactions: The necessary and sufficient condition that a purely scalar Lagrangian of the form $\frac{1}{2} g_{pq}(\pi)\,\partial_\mu \pi^p \partial^\mu \pi^q$ have a vanishing T matrix is that the metric $g_{pq}(\pi)$ is flat.

II. TREE UNITARITY AND RENORMALIZABILITY

The criterion of tree unitarity is central to our work. In this section we give a precise definition of this criterion. We also discuss the reasons why we believe that a quantum field theory based on a classical Lagrangian (i.e., a "conventional" perturbative quantum field theory) is meaningful only if it is tree-unitary. In particular, tree unitarity may be a *necessary* condition for renormalizability.

The S matrix can be expressed in terms of a T matrix[10] by

$$\langle f|S|i\rangle = 1 + i(2\pi)^4\,\delta^4(P'-P)\,N_i\,N_f\langle f|T|i\rangle\,.$$

Here, P, P' are the initial and final four-momenta, and N_i, N_f are products of the normalization factors for the initial and final particles $[(2E_i)^{-1/2}$ for each particle$]$. These "invariant" T-matrix elements satisfy the unitarity relation

$$\operatorname{Im}\langle f|T|i\rangle = \tfrac{1}{2}\sum_n (2\pi)^{4-3n}\int \frac{d^3k_1}{2k_{10}}\cdots\frac{d^3k_n}{2k_{n0}}\,\delta^4\Big(\sum k_i - P\Big)\langle k_1\cdots k_n|T|f\rangle^*\langle k_1\cdots k_n|T|i\rangle\,.$$

The summation is understood to run over all possible sets of intermediate particles and their helicities.

Let $T_{N-n,n}$ denote a T matrix for n incoming particles with four-momenta $p_1, \ldots, p_n$ and helicities $\lambda_1, \ldots, \lambda_n$ and $N-n$ outgoing particles with four-momenta $-p_{n+1}, \ldots, -p_N$ and helicities $\lambda_{n+1}, \ldots, \lambda_N$. In the center-of-momentum frame choose *fixed* values for the incoming variables

$$\left\{ \hat{p}_i \equiv \frac{\vec{p}_i}{|\vec{p}_i|},\ \frac{|\vec{p}_i|}{|\vec{p}_j|},\ \lambda_i \text{ for } 1 \leq i, j \leq n \right\}$$

and for the analogous outgoing variables. For given values of these "fixed variables" each four-momentum p_i grows as E as the total center-of-momentum energy (E) approaches infinity. A field theory will be called *tree-unitary* if in the tree approximation all amplitudes $T_{N-n,n}$ grow at most like E^{4-N} as $E \to \infty$; in this limit the "fixed variables" are taken to have values such that all kinematical invariants of the type

$$(p_{i_1} + p_{i_2} + \cdots + p_{i_r})^2, \quad 2 \leq r \leq N-2 \tag{1}$$

grow like E^2. Roughly speaking, we are considering the limit in which all squared momentum transfers and subenergies grow like E^2.

The origin of this simple asymptotic boundedness condition is essentially the notion that high-energy unitarity bounds should not be grossly violated in perturbation theory. To see this, consider an amplitude $T_{N-2,2}$ for two particles going into $N-2$ particles. For any Lagrangian theory a tree graph for this process has the asymptotic form

$$(T_{N-2,2})_{\text{tree}} \underset{E\to\infty}{\sim} E^{\beta} \times (\text{function of "fixed variables"}),$$

the exponent β being independent of the values of the "fixed variables" unless these values are "exceptional," i.e., such that some invariant of the type (1) does not grow like E^2. This is because then the squared four-momentum of some propagator may not grow like E^2. Now the *partial-wave* amplitudes $T^J_{N-2,2}$ are obtained from $T_{N-2,2}$ by an angular integration over the $\{\hat{p}_i\}$; therefore, in the limit $E \to \infty$ we have

$$(T^J_{N-2,2})_{\text{tree}} \propto E^{\beta}$$

apart from factors of $\ln E$ which may arise from integration over a neighborhood of "exceptional" values of the "fixed variables." [The deviations from the power behavior are at most logarithmic because the singularities in the variables of Eq. (1) are *simple* poles.] Unitarity, however, demands that the *exact* $T^J_{N-2,2}$ satisfies the inequality

$$\int d\Omega_{N-2} |T^J_{N-2,2}|^2 < \text{Im}\, T^J_{2,2} < \text{constant}.$$

Notice that the phase space Ω_{N-2} grows like E^{2N-8}. Consequently, if we require that the tree approximation does not violate this bound except by factors of $\ln E$, we must have "tree unitarity": $2N - 8 + 2\beta \leq 0$ or $\beta \leq 4 - N$. Observe that for $N=4$ this is just the Kinoshita-Loeffel-Martin bound,[11] $T_{2,2} \lesssim (\ln E)^{3/2}$, applied to trees.

It is important to bear in mind that in non-tree-unitary theories the violations of unitarity at high energy must become *worse* in the one-loop approximation. This is because the imaginary part of an amplitude in the one-loop approximation is completely determined by the tree approximation. Thus, for example, if $(T^J_{2,2})_{\text{tree}} \propto E^{\beta}$, with $\beta > 0$, then except for factors of $\ln E$

$$\text{Im}(T^J_{2,2})_{1\text{-loop}} \propto (\text{2-particle phase space})\, |(T^J_{2,2})_{\text{tree}}|^2 \propto E^{2\beta}.$$

Evidently, non-tree-unitary theories increasingly violate unitarity in higher-loop approximations.

Our basic objection to non-tree-unitary theories is their lack of perturbative unitarity at high energy. Tree unitarity is a property of the S matrix, and we have not been able to relate it to renormalizability in a rigorous way. But we can at least make it plausible that tree unitarity is *necessary* for a renormalizable theory by constructing an off-mass-shell version of the above argument. Suppose, for example, that we are dealing with a non-tree-unitary theory, and $(T_{2,2})_{\text{tree}} \propto E^2$. Then the corresponding amputated *off*-shell Green's function in the tree approximation grows at least like ρ^2 as the external *four*-momenta p_i approach infinity according to $p_i = \rho p_i'$, $\rho \to \infty$, $p_i' =$ fixed. This behavior leads to a quartic divergence in the one-loop expression and requires a *new* counterterm growing like ρ^4. Continuation of this (admittedly superficial) argument to higher-loop approximations shows that there must be an ever-increasing number of new counterterms which cannot be multiplicatively absorbed by the terms of the original Lagrangian. Therefore, it seems that non-tree-unitary models are not renormalizable. No such difficulty is forced on us by a tree-unitary theory: On the mass shell $(T_{2,2})_{\text{tree}} \sim O(1)$, and one can envisage a choice of field variables such that the off-shell amplitude behaves like ρ^0 in the tree approximation. As a consequence, the one-loop integral can diverge only logarithmically, and a counterterm behaving like ρ^0 is sufficient. Nothing *a priori* prevents such a counterterm from being multiplicatively absorbed in the original Lagrangian. Thus, it may not be a coincidence that all *known* renormalizable theories are tree-unitary since tree unitarity may be a necessary property of any renormalizable model.

III. INTERACTION LAGRANGIAN AND FEYNMAN RULES

In this section we define the class of Lagrangian field theories to be subjected to the criterion of tree unitarity. These theories involve any number of Hermitian scalar fields ϕ_k (mass μ_k), spinor fields ψ_i (mass m_i), and Hermitian vector fields W_a (mass $M_a > 0$). The associated propagators in momentum space are[10]

$$\langle T(\phi_k \phi_l)\rangle = \frac{i\delta_{kl}}{k^2-\mu_k^2}\ ,\quad \langle T(\bar\psi_{i\alpha}\psi_{j\beta})\rangle = \left(\frac{i\delta_{ij}}{\not k - m_i}\right)_{\alpha\beta}\ ,$$
$$\langle T(W_{a\mu}W_{b\nu})\rangle = -i\delta_{ab}\left(\frac{g_{\mu\nu}-k_\mu k_\nu/M_a^2}{k^2-M_a^2}\right)\ . \tag{2}$$

External spinor and vector particles are associated with the usual wave functions, $u_{i\alpha}(k)$ and $\epsilon_{a\mu}(k)$; note that the longitudinal vector wave function has the form

$$\epsilon_a^\mu(k) = \frac{1}{M_a}\left(\,|\vec k|\,,\ \frac{k_0}{|\vec k|}\,\vec k\right)\ . \tag{3}$$

We consider all interactions whose vertices are described by the cubic and higher terms in the following Lagrangian:

$$\begin{aligned}\mathcal{L}(W,\psi_R,\psi_L,\phi) = &-\tfrac14(\partial_\mu W_{a\nu}-\partial_\nu W_{a\mu})^2 - \epsilon^{\mu\nu\lambda\rho}(A_{abc}\,\partial_\rho W_{a\mu}W_{b\nu}W_{c\lambda} + B_{abcd}W_{a\mu}W_{b\nu}W_{c\lambda}W_{d\rho})\\ &- C_{abc}\,\partial_\nu W_{a\mu}W_b^\mu W_c^\nu - D_{abcd}W_{a\mu}W_b^\mu W_{c\nu}W_d^\nu + \bar\psi_R i(\not\partial + i\not W_a R^{(a)})\psi_R + \bar\psi_L i(\not\partial + i\not W_a L^{(a)})\psi_L\\ &+\tfrac12\partial_\mu\phi_k\partial^\mu\phi_l P_{kl}(\phi) + W_{a\mu}W_b^\mu F_{ab}(\phi) - W_{a\mu}\partial^\mu\phi_k G_{ak}(\phi) - V(\phi) - \bar\psi_L H(\phi)\psi_R - \bar\psi_R H^\dagger(\phi)\psi_L\ .\end{aligned} \tag{4}$$

Here, A_{abc}, B_{abcd}, C_{abc}, and D_{abcd} are real constants such that

$$A_{abc} = -A_{acb}\ ,$$
$$B_{abcd} = \text{totally antisymmetric},$$
$$D_{abcd} = D_{bacd} = D_{cdab}\ .$$

$R^{(a)}$ and $L^{(a)}$ are any constant Hermitian matrices on the space of (suppressed) internal spin indices. $P_{kl}(\phi)$, $F_{ab}(\phi)$, $G_{ak}(\phi)$, and $V(\phi)$ denote real power series in ϕ and satisfy

$$F_{ab}(0) = \tfrac12 M_a^2\delta_{ab}\ ,\quad F_{ab}(\phi) = F_{ba}(\phi)\ ,$$
$$G_{ak}(0) = 0\ ,\quad V(\phi) = \tfrac12\mu_k^2\phi_k^2 + O(\phi^3)\ , \tag{5a}$$
$$P_{kl}(0) = \delta_{kl}\ ,\quad P_{kl}(\phi) = P_{lk}(\phi)\ .$$

$H(\phi)$ is a complex matrix in the space of internal spinor indices and obeys

$$H_{ij}(0) = m_i\delta_{ij}\ .$$

It should be emphasized that $\mathcal{L}$ is nothing more than a summary of the vertices in the Feynman rules; it is not necessarily meant to describe a canonically quantized operator field theory. The functions $P_{kl}(\phi)$, $F_{ab}(\phi)$, $G_{ak}(\phi)$, $V(\phi)$, and $H(\phi)$ denote (possibly infinite) series of powers of ϕ, which need not converge at any point. In other words, each term in these series (but not the sum of terms) has a well-defined meaning as a vertex part.

These Feynman rules can be used to construct the Green's functions

$$\begin{aligned}G[\phi_k(k_1)\cdots\psi_{i\alpha}(k_2)\cdots W_{a\mu}(k_3)\cdots](2\pi)^4\delta^4(k_1+k_2+k_3+\cdots)\\ \equiv \int dx_1\,dx_2\,dx_3\cdots\exp[-i(k_1x_1+k_2x_2+k_3x_3+\cdots)]\\ \times\langle 0|T^*\{(k_1^2-\mu_k^2)\phi_k(x_1)\cdots[(\not k_2-m_i)\psi_i(x_2)]_\alpha\cdots(k_3^2-M_a^2)W_{a\mu}(x_3)\cdots\}|0\rangle\ .\end{aligned} \tag{6}$$

We shall use the symbol $\mathfrak{M}(\phi_k\cdots\psi_{i\alpha}\cdots W_{a\mu}\cdots)$ to denote the above Green's function when all external lines are on the mass shell. Every T-matrix element of Sec. II is given (up to a multiplicative constant) by an amplitude of the form $\mathfrak{M}(\phi_k\cdots\bar u_i\psi_i\cdots\times\epsilon_a\cdot W_a\cdots)$. Our problem is to characterize all tree-unitary S matrices constructed from the above prescription. Notice that tree unitarity is a purely on-mass-shell property which bounds the high-energy behavior of the $\mathfrak{M}$ amplitudes only; no explicit constraints are placed on the off-shell Green's functions. Since the mass dimension of the N-field $\mathfrak{M}$ amplitude is $4-N$, $\mathfrak{M}$ is tree-unitary if and only if it scales at high energy. However, only the sum of all diagrams (not individual diagrams) in $\mathfrak{M}$ needs to display this scaling behavior.

Our class of Lagrangians is quite general. As a special case [$P_{kl}(\phi)$ is constant in ϕ, F_{ab} = quadratic, G_{ak} = linear, V = quartic, H = linear] it includes the most general Hermitian, Lorentz-invariant interaction with mass dimension less than or equal to four. By allowing the quantities P_{kl}, F_{ab}, G_{ak}, V, and H to be arbitrary power series in the scalar fields we have *a priori* included a wide class of nonpolynomial interactions. In fact, it turns out that tree unitarity excludes all terms of dimension higher than four within our class of Lagrangians; more precisely, tree unitarity implies that high-dimension terms can be transformed away by a change of variables.

Our class of interactions could be enlarged in two ways. First, it is possible to include interactions which contain higher powers of W_μ, $\partial_\nu W_\mu$, or $\partial_\mu \phi$. Such couplings must have dimension greater than four. It may be that high powers of W_μ, $\partial_\nu W_\mu$, and $\partial_\mu \phi$ (like high powers of ϕ) either violate tree unitarity or can be transformed away. On the other hand, there is a small possibility of genuinely new tree-unitary models of this type, in which the bad high-energy behavior of the high-dimension interactions cancels out in any S-matrix element. A second way of broadening the scope of this work is to include massless vectors in the particle spectrum. There is some reason to believe that this will not lead to the discovery of any new tree-unitary interactions. For example, in an earlier publication[4] we found all tree-unitary models with a particle spectrum of one *massless* vector, three massive vectors, and one scalar. The result is that the only tree-unitary interactions of this type are equivalent to the well-known[1] Weinberg [SU(2)⊗U(1)], Georgi-Glashow [SO(3)], or Higgs [U(1)] SBGT's.

It is useful to discuss the high-energy behavior of the Feynman rules. In general, we shall say that a propagator, wave function, or vertex is "good" if that diagram part "scales" at high energy For instance, a propagator is good in the limit $E \to \infty$ if it behaves like E^{-2} for bosons (scalar or vector) and like E^{-1} for spinors. A wave function is good if it behaves like E^0 for bosons of any spin and like $E^{1/2}$ for spinors. Good vertices are those with dimension less than or equal to four. It is clear that any *single* diagram, constructed from good (i.e., individually scaling) parts, must also scale and, therefore, be tree-unitary. Now, all of the (W, ψ, ϕ) Feynman rules are good except for bad vector propagators ($\to k_\mu k_\nu/k^2 \propto E^0$), bad longitudinal vector wave functions ($\to k_\mu \propto E$), and bad vertices from the higher-order terms in $P_{kl}(\phi)$, $F_{ab}(\phi)$, $G_{ak}(\phi)$, $V(\phi)$, and $H(\phi)$. Even if the bad vertices are ignored, a single diagram, containing longitudinal vector modes, will usually display bad high-energy behavior (i.e., nonscaling behavior, behavior more divergent than E^{4-N}). Thus, a massive vector theory will be tree-unitary only if the bad k_μ factors are "eliminated" in individual diagrams or cancel out in the sum of all diagrams for each scattering process. This can happen only if the vector-boson source current, which multiplies each k_μ factor, is conserved or at least constrained in some way. Indeed, it turns out that the vector-boson source currents satisfy the tree-unitarity constraints if and only if the vector bosons are coupled according to a gauge-invariant prescription.

IV. TREE UNITARITY OF VECTOR AND VECTOR-SPINOR FOUR-POINT AMPLITUDES

We begin our search for tree-unitary Lagrangians by considering, in this section, the four-point trees with only vector and spinor external particles. A straightforward but tedious calculation[4,5] shows that the scattering amplitudes for $W_a W_b \to W_c W_d$ and $W_a \psi_i \to W_b \psi_j$ are unitarily bounded (by E^0) only if

$$A_{abc} = 0\ , \tag{7a}$$

$$B_{abcd} = 0\ , \tag{7b}$$

$$C_{abc} = \text{totally antisymmetric}, \tag{7c}$$

$$C_{abe}C_{cde} - C_{ace}C_{bde} - C_{ade}C_{cbe} = 0\ , \tag{7d}$$

$$8\,D_{abcd} = C_{ace}C_{bde} + C_{ade}C_{bce}\ ; \tag{7e}$$

$$[R^{(a)}, R^{(b)}] = iC_{abc}R^{(c)}\ , \tag{8a}$$

$$[L^{(a)}, L^{(b)}] = iC_{abc}L^{(c)}\ . \tag{8b}$$

Equation (7d), the Jacobi identity, implies that the C_{abc} are the structure constants of a Lie algebra. Equation (7c) shows that the Lie algebra corresponds to a Lie group (G) with the form $G = \mathrm{U}(1) \otimes \cdots \otimes \mathrm{U}(1) \otimes S$, where S is a compact, semisimple subgroup. Thus, the purely vectorial vertices are determined in terms of C_{abc} by Eqs. (7a), (7b), and (7e); as a result the corresponding terms of the Lagrangian must have the gauge-invariant Yang-Mills[12] form, corresponding to the group G. Equations (8a) and (8b) show that the matrices $R^{(a)}$ and $L^{(a)}$ form representations of the Lie algebra on the spaces of internal indices of ψ_R and ψ_L, respectively. This means that the purely vector-spinor interactions must have the (minimal) gauge-invariant form. So far, tree unitarity implies that the Lagrangian of Eq. (4) looks like

$$\mathcal{L}(W,\psi_R,\psi_L,\phi) = -\tfrac{1}{4}(\partial_\mu W_{a\nu} - \partial_\nu W_{a\mu} - C_{abc}W_{b\mu}W_{c\nu})^2 + \bar\psi_R\, i(\not\partial + i \not W_a R^{(a)})\psi_R + \bar\psi_L\, i(\not\partial + i\not W_a L^{(a)})\psi_L$$
$$+\tfrac{1}{2}\partial_\mu\phi_k\partial^\mu\phi_l P_{kl}(\phi) + W_{a\mu}W^\mu_b F_{ab}(\phi) - W_{a\mu}\partial^\mu\phi_k G_{ak}(\phi) - V(\phi) - \bar\psi_L H(\phi)\psi_R - \bar\psi_R H^\dagger(\phi)\psi_L \ . \quad (9)$$

Notice that the gauge-invariant structure of the purely vector and vector-spinor couplings lies very near the surface; it is a direct consequence of unitarily bounded high-energy behavior at the four-point level. In principle one could go on to derive an infinite string of relations between coupling constants and masses by setting equal to zero the coefficients of all powers of E higher than E^{4-N} for all N-particle amplitudes ($N = 5, 6, \ldots$). This straightforward procedure is forbiddingly laborious.[4,5] Instead we have been able to derive the complete set of relations by a more elegant method, to be described in Sec. VI.

The structure of the underlying group G is most apparent in a "Cartesian" basis, which separates the generators of the semisimple component from those of the invariant Abelian subgroups. First, define $\gamma_{ab} \equiv C_{ade}C_{bde}$. Since γ_{ab} is real, symmetric, and non-negative, it is diagonalized by an orthogonal matrix $\hat O_{ab}$; i.e., $(\hat O^{-1}\gamma\hat O)_{ab} = \gamma_a\delta_{ab}$ where $\gamma_a \geq 0$. If $\gamma_a = 0$, the index a is called Abelian; if $\gamma_a > 0$, a is called semisimple. Let f_{abc} denote the structure constants in the Cartesian basis: $f_{abc} \equiv C_{a'b'c'}\hat O_{a'a}\hat O_{b'b}\hat O_{c'c}$. It follows that $\gamma_a\delta_{ab} = f_{ade}f_{bde}$. Therefore, if a is Abelian, then $f_{abc} = 0$ for all b and c, and a labels the generator of an invariant Abelian subgroup. On the other hand, when a, b, and c are semisimple indices, the f_{abc} are the structure constants of the semisimple subgroup S. Let t_a denote the imaginary, antisymmetric matrices defined by $(t_a)_{bc} \equiv -if_{abc}$. It follows from the properties of f_{abc} that $\mathrm{Tr}\{t_a t_b\} = \gamma_a\delta_{ab}$ and that $[t_a, t_b] = if_{abc}t_c$. Thus, the t_a generate the adjoint representation of G in a Cartesian basis. Notice that if a is Abelian, then $t_a = 0$. On the other hand, for semisimple a the matrices t_a are linearly independent.

V. STÜCKELBERG VARIABLES

A. Stückelberg rules

Tree unitarity led directly to the gauge invariance of the purely vector and vector-spinor couplings. It was easy to recognize this gauge invariance because tree unitarity implies that the physical vector and spinor fields ($W_{a\mu}, \psi_{Ri}, \psi_{Li}$) form basis states for representations of the gauge group G. It turns out that tree unitarity does not imply that the physical scalar fields (ϕ_k) couple in any simple way or form a complete basis for a representation of the gauge group.[4,5] Rather, as might be expected from experience with SBGT's, tree unitarity will impose a simple form on $\mathcal{L}$ only when $\mathcal{L}$ is expressed in terms of new field variables, which represent the physical scalar and longitudinal vector degrees of freedom. Specifically, tree unitarity will require that these new scalar field variables form a complete basis for a linear representation of G and couple via gauge-invariant vertices of dimension less than or equal to four. In other words, the entire Lagrangian has a "hidden" symmetry which is unveiled only when the longitudinal modes of vectors are *explicitly* described by the vector field formalism. Stückelberg[13] has given such a prescription for describing a heavy vector particle, which is equivalent in every way to the spin-1 formalism used in Sec. III. Each massive vector field ($W_{a\mu}$) is replaced by a combination of a "Stückelberg vector field" ($A_{a\mu}$) and a Stückelberg scalar field (σ_a); the latter explicitly describes the longitudinal mode of the massive vector. In this section we adopt the Stückelberg representation for vector fields. No new physics is introduced; *the new formalism is simply a different kinematical representation* (a change in Feynman rules), *which does not change the S matrix.* The Stückelberg prescription provides a convenient language, in which the tree-unitarity constraints on scattering amplitudes are expressed as simple statements.

Define an orthogonal matrix function $U(\sigma)$ and another matrix function $Q(\sigma)$ by

$$U(\sigma) \equiv e^{-i\sigma\cdot t}\,,\qquad Q(\sigma) \equiv \frac{U-1}{-i\sigma\cdot t}\ . \qquad (10)$$

After transforming to the Cartesian basis, the exact decomposition rule is

$$\hat O^{-1}{}_{ae}W_{e\mu} \equiv U(\sigma)_{ae}A_{e\mu} + Q(\sigma)_{ae}\partial_\mu\sigma_e\ . \qquad (11)$$

Define a new set of spinor fields, q_R and q_L, by

$$\psi_R \equiv e^{-i\sigma\cdot\bar R}q_R\,,\qquad \psi_L \equiv e^{-i\sigma\cdot\bar L}q_L\,, \qquad (12)$$

where $\bar R^{(a)}$ and $\bar L^{(a)}$ generate the spinor representations in the Cartesian basis:

$$\bar R^{(a)} \equiv \hat O^{-1}{}_{ae}R^{(e)}\ ,$$
$$\bar L^{(a)} \equiv \hat O^{-1}{}_{ae}L^{(e)}\ ,$$
$$[\bar R^{(a)}, \bar R^{(b)}] = if_{abc}\bar R^{(c)}\ ,$$
$$[\bar L^{(a)}, \bar L^{(b)}] = if_{abc}\bar L^{(c)}\ .$$

The Stückelberg Lagrangian is defined by

$$\mathcal{L}_S(A, q_R, q_L, \phi, \sigma) \equiv \mathcal{L}(\hat{O}(U(\sigma)A + Q(\sigma)\partial\sigma), e^{-i\sigma\cdot\overline{R}}q_R, e^{-i\sigma\cdot\overline{L}}q_L, \phi) \ . \tag{13}$$

Because of the gauge invariance of the purely vectorial piece of $\mathcal{L}$, no second derivatives of σ appear in $\mathcal{L}_S$. The "Stückelberg rules" for constructing diagrams include the propagators

$$\hat{O}_{aa'}\hat{O}_{bb'}\langle T(A_{a'\mu}A_{b'\nu})\rangle \equiv \frac{-ig_{\mu\nu}\delta_{ab}}{k^2-M_a{}^2}\ ,$$
$$M_a\hat{O}_{aa'}M_b\hat{O}_{bb'}\langle T(\sigma_{a'}\sigma_{b'})\rangle \equiv \frac{i\delta_{ab}}{k^2-M_a{}^2}\ . \tag{14}$$

The propagators of q_R, q_L, and ϕ have the usual form. The vertices of the Stückelberg rules are taken from the cubic and higher terms of $\mathcal{L}_S$. In Appendix A it is proved that the Feynman rules of $\mathcal{L}$ and the Stückelberg rules of $\mathcal{L}_S$ lead to the same S matrix in the following sense: The on-mass-shell scattering amplitudes of $W_{a\mu}$ modes, calculated from the $\mathcal{L}$ Feynman rules, are equal to the on-mass-shell scattering amplitudes of $\hat{O}_{aa'}(A_{a'\mu}+\partial_\mu\sigma_{a'})$ modes, calculated from the $\mathcal{L}_S$ Stückelberg rules. In other words, in terms of the rotated fields, $\tilde{A}_{a\mu}\equiv\hat{O}_{aa'}A_{a'\mu}$ and $\tilde{\sigma}_a\equiv M_a\hat{O}_{aa'}\sigma_{a'}$, we have

$$\mathfrak{M}(W_{a\mu}\cdots\psi_{i\alpha}\cdots\phi_k\cdots) = \mathfrak{M}\left(\left(\tilde{A}_{a\mu}+\frac{1}{M_a}\partial_\mu\tilde{\sigma}_a\right)\cdots q_{i\alpha}\cdots\phi_k\cdots\right) \ . \tag{15}$$

Thus, the Feynman rules of Sec. III and the Stückelberg rules of this section are physically equivalent since they describe the same on-mass-shell Green's functions and the same S matrix.

The next step is to translate the statement of tree unitarity into the Stückelberg language. Since $\epsilon\cdot k=0$, we have

$$\mathfrak{M}(\epsilon_a\cdot\tilde{A}_a\cdots\overline{u}_i\,q_i\cdots\phi_k\cdots) = \mathfrak{M}(\epsilon_a\cdot W_a\cdots\overline{u}_i\psi_i\cdots\phi_k\cdots)\ . \tag{16}$$

Equation (2) shows that each external $W_{a\mu}$ line in $\mathfrak{M}(W_{a\mu}\cdots)$ carries a factor of

$$\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{M_a{}^2}\right)\ .$$

Since this factor is eliminated by dotting it with k_a^μ, it follows that

$$\mathfrak{M}\left(k_a^\mu\left(\tilde{A}_{a\mu}+\frac{1}{M_a}\partial_\mu\tilde{\sigma}_a\right)\cdots\epsilon_b\cdot\tilde{A}_b\cdots\overline{u}_i\,q_i\cdots\phi_k\cdots\right)=0\ . \tag{17}$$

Equations (16)–(17) imply that

$$\mathfrak{M}\left(\epsilon_a\cdot\tilde{A}_a-\frac{1}{M_a}k_a\cdot\tilde{A}_a-i\tilde{\sigma}_a\cdots\overline{u}_i\,q_i\cdots\phi_k\cdots\right) = \mathfrak{M}(\epsilon_a\cdot W_a\cdots\overline{u}_i\psi_i\cdots\phi_k\cdots)\ . \tag{18}$$

Let all the vector wave functions ϵ_a be longitudinally polarized; then Eq. (3) gives

$$\epsilon_a^\mu=\frac{k_a^\mu}{M_a}+\xi_a^\mu$$
$$\epsilon_a\cdot\tilde{A}_a-\frac{1}{M_a}k_a\cdot\tilde{A}_a-i\tilde{\sigma}_a=-i(\tilde{\sigma}_a+i\xi_a\cdot\tilde{A}_a)\ ,$$

where ξ_a^μ is some function of k_a^μ such that $\xi_a^\mu \underset{E\to\infty}{\sim} E^{-1}$. Equation (18) can now be rewritten as

$$\mathfrak{M}(-i(\tilde{\sigma}_a+i\xi_a\cdot\tilde{A}_a)\cdots\overline{u}_i\,q_i\cdots\phi_k\cdots) = \mathfrak{M}(\epsilon_a\cdot W_a\cdots\overline{u}_i\psi_i\cdots\phi_k\cdots)\ . \tag{19}$$

Therefore, in the Stückelberg language tree unitarity implies the high-energy bound

$$\mathfrak{M}(\tilde{\sigma}_a+i\xi_a\cdot\tilde{A}_a\cdots\overline{u}_i\,q_i\cdots\phi_k\cdots)\leq E^{4-N}. \tag{20}$$

It is worth discussing the high-energy behavior of the Stückelberg rules. All of the propagators and "wave functions" (ξ_a, $\overline{u}_i$) in Eq. (20) are good. However, $\mathcal{L}_S$ contains many bad vertices associated with the exponential and nonpolynomial functions: $U(\sigma)$, $Q(\sigma)$, $e^{-i\sigma\cdot\overline{R}}$, $e^{-i\sigma\cdot\overline{L}}$, $P_{kl}(\phi)$, $F_{ab}(\phi)$, $G_{ak}(\phi)$, $V(\phi)$, and $H(\phi)$. *A big advantage of the Stückelberg formulation is that all bad behavior is now isolated in the vertices.* Therefore, *if* all the bad vertices can be transformed away by a point transformation of the Stückelberg and physical scalar fields, then the S matrix will be tree-unitary. In Sec. VI we will prove the inverse relation: If the S matrix is tree-unitary, then there exists a special set of field coordinates in terms of which $\mathcal{L}_S$ has no bad vertices. Furthermore, in the language of these special field variables, the physical scalars and longitudinal vectors combine to form the complete basis of a linear representation (of G) which couples in a gauge-invariant way.

B. Stückelberg gauge invariance

At this stage we have no information on the couplings of the physical scalars (ϕ_k) in the original Lagrangian $\mathcal{L}$, and this arbitrariness is reflected in the Stückelberg Lagrangian $\mathcal{L}_S$. At the same time, $\mathcal{L}_S$ depends on the longitudinal modes (σ) in a very systematic way. This is because $\mathcal{L}_S$ was derived by replacing the fields of $\mathcal{L}$ by gauge-transformed expressions, in which σ plays the role of a local gauge parameter. In this subsection we deduce from these facts that $\mathcal{L}_S$ is *automatically* invariant under a group G of local gauge transformations, realized nonlinearly on the σ fields.

First, we review the group property of the usual local gauge transformations. Let $\Lambda_1^{(a)}(x)$ be any real function. The transformation of $A_{a\mu}$ under the element of G, which corresponds to $U(\Lambda_1)$, is defined to be

$$A_{a\mu} \xrightarrow{U(\Lambda_1)} U(\Lambda_1)_{ab} A_{b\mu} + Q(\Lambda_1)_{ab}\, \partial_\mu \Lambda_1^{(b)} . \tag{21}$$

Now, consider the effect of applying $U(\Lambda_2)$ and then $U(\Lambda_1)$:

$$\begin{aligned} A_{a\mu} \to\, & U(\Lambda_1)_{ab}\left[U(\Lambda_2)_{bc} A_{c\mu} + Q(\Lambda_2)_{bc}\, \partial_\mu \Lambda_2^{(c)}\right] \\ & + Q(\Lambda_1)_{ab}\, \partial_\mu \Lambda_1^{(b)} . \end{aligned} \tag{22}$$

The transformations of Eq. (21) obey the group multiplication law of G in the sense that Eq. (22) is the same as

$$A_{a\mu} \xrightarrow{U(\Lambda_3)} U(\Lambda_3)_{ab} A_{b\mu} + Q(\Lambda_3)_{ab}\, \partial_\mu \Lambda_3^{(b)} , \tag{23}$$

where Λ_3 is defined by $U(\Lambda_3) \equiv U(\Lambda_1)\, U(\Lambda_2)$.

Let $\Lambda^{(a)}(x)$ be any real function; then, define $\hat\sigma_a[\sigma(x), \Lambda(x)]$ to be the following function of $\sigma_a(x)$ and $\Lambda^{(a)}(x)$:

$$\begin{aligned} & U(\hat\sigma) \equiv U(\sigma)\, U(\Lambda) , \\ & \hat\sigma_a \equiv \sigma_a + \Lambda_a \quad (a = \text{Abelian index}). \end{aligned} \tag{24a}$$

Take $\hat A_{a\mu}(A, \Lambda, \partial\Lambda)$, $\hat q_R(q_R, \Lambda)$, $\hat q_L(q_L, \Lambda)$ to be defined by

$$\begin{aligned} & \hat A_{a\mu} \equiv U(-\Lambda)_{ab} A_{b\mu} - Q(-\Lambda)_{ab}\, \partial_\mu \Lambda^{(b)} , \\ & \hat q_R \equiv e^{i\Lambda\cdot \overline{R}} q_R , \\ & \hat q_L \equiv e^{i\Lambda\cdot \overline{L}} q_L . \end{aligned} \tag{24b}$$

The equivalence of Eqs. (22) and (23) implies the identity

$$U(\sigma)_{ab} A_{b\mu} + Q(\sigma)_{ab}\, \partial_\mu \sigma_b = U(\hat\sigma)_{ab} \hat A_{b\mu} + Q(\hat\sigma)_{ab}\, \partial_\mu \hat\sigma_b . \tag{25}$$

In a similar way the group properties of the matrices $e^{i\Lambda\cdot\overline{R}}$ and $e^{i\Lambda\cdot\overline{L}}$ lead to the identities

$$\begin{aligned} & e^{-i\sigma\cdot\overline{R}} q_R = e^{-i\hat\sigma\cdot\overline{R}} \hat q_R , \\ & e^{-i\sigma\cdot\overline{L}} q_L = e^{-i\hat\sigma\cdot\overline{L}} \hat q_L . \end{aligned} \tag{26}$$

From Eqs. (13), (25), and (26) we have

$$\begin{aligned} & \mathcal{L}_S(A, q_R, q_L, \phi, \sigma) \\ & \quad = \mathcal{L}\big(\hat O\big(U(\hat\sigma)\hat A + Q(\hat\sigma)\partial\hat\sigma\big), e^{-i\hat\sigma\cdot\overline{R}}\hat q_R, e^{-i\sigma\cdot\overline{L}}\hat q_L, \phi\big) \\ & \quad = \mathcal{L}_S\big(\hat A(A, \Lambda, \partial\Lambda), \hat q_R(q_R, \Lambda), \hat q_L(q_L, \Lambda), \phi, \hat\sigma(\sigma, \Lambda)\big) . \end{aligned} \tag{27}$$

Therefore, $\mathcal{L}_S$ is invariant under a group (G) of local gauge transformations, which are realized nonlinearly on the σ fields[14]:

$$\begin{aligned} & \sigma_a \xrightarrow{U(\Lambda)} \hat\sigma_a , \\ & A_{a\mu} \xrightarrow{U(\Lambda)} \hat A_{a\mu} , \\ & q_R \xrightarrow{U(\Lambda)} \hat q_R , \\ & q_L \xrightarrow{U(\Lambda)} \hat q_L , \\ & \phi \xrightarrow{U(\Lambda)} \phi . \end{aligned} \tag{28}$$

The differential statement of this "Stückelberg gauge invariance" is

$$\frac{\delta \mathcal{L}_S(\hat A, \hat q_R, \hat q_L, \phi, \hat\sigma)}{\delta \Lambda^{(a)}} = 0 , \tag{29a}$$

$$\frac{\delta \mathcal{L}_S(\hat A, \hat q_R, \hat q_L, \phi, \hat\sigma)}{\delta \partial_\mu \Lambda^{(a)}} = 0 . \tag{29b}$$

The last two equations can be translated into constraints on various terms in $\mathcal{L}_S$. It is convenient to put the physical and Stückelberg scalars into one array π_p [15]:

$$\pi_k \equiv \phi_k , \quad \pi_a \equiv \sigma_a .$$

Then, Eqs. (13) and (4) imply that $\mathcal{L}_S$ has the form

$$\begin{aligned} \mathcal{L}_S(A, q_R, q_L, \pi) \equiv\, & \mathcal{L}_S(A, q_R, q_L, \phi, \sigma) \\ =\, & -\tfrac{1}{4}\left(\partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} - f_{abc} A_{b\mu} A_{c\nu}\right)^2 + \overline{q}_R\, i(\not\partial + i \not A_a \overline{R}^{(a)})\, q_R + \overline{q}_L\, i(\not\partial + i\not A_a \overline{L}^{(a)}) q_L \\ & + \tfrac{1}{2}\partial_\mu \pi_p \partial^\mu \pi_q g_{pq}(\pi) + A_{a\mu}\partial^\mu \pi_p K_p^{(a)}(\pi) + \tfrac{1}{2} A_{a\mu} A_b^\mu S^{(ab)}(\pi) - V(\pi) - \overline{q}_L\, Y(\pi) q_R - \overline{q}_R\, Y(\pi)^\dagger q_L . \end{aligned} \tag{30}$$

Here, $g_{pq}(\pi)$, $K_p^{(a)}(\pi)$, $S^{(ab)}(\pi)$, $V(\pi)$, and $Y(\pi)$ describe complicated exponential and nonpolynomial vertices [containing factors of $U(\sigma)$, $Q(\sigma)$, $P_{kl}(\phi)$, $F_{ab}(\phi)$, $G_{ak}(\phi)$, $V(\phi)$, and $H(\phi)$]. The explicit forms of these functions will not be needed in this paper. Define $\hat\pi_p(\pi, \Lambda)$ to be the gauge-transformed π_p:

$$\hat\pi_k \equiv \pi_k = \phi_k , \quad \hat\pi_a \equiv \hat\sigma_a . \tag{31}$$

Let $\mathcal{J}^{(a)p}(\pi)$ denote the generator of gauge transformations:

$$\mathcal{J}^{(a)p}(\pi) \equiv \left.\frac{\delta \hat\pi_p}{\delta \Lambda^{(a)}}\right|_{\Lambda = \partial\Lambda = 0} \tag{32}$$

In the above and in the following equations, $\delta/\delta\Lambda$ and $\delta/\delta\partial_\mu\Lambda$ are evaluated at $\Lambda = \partial_\mu\Lambda = 0$. The differential forms of Eqs. (24a) and (24b) are

$$
\begin{aligned}
&\frac{\delta(\partial_\mu \hat{\pi}_p)}{\delta\Lambda^{(a)}} = J^{(a)p}_{,q}\,\partial_\mu \pi_q \ ,\\
&\frac{\delta(\partial_\mu \hat{\pi}_p)}{\delta(\partial_\nu\Lambda^{(a)})} = \delta_{\mu\nu} J^{(a)p} \ ,\\
&\frac{\delta \hat{A}_{b\mu}}{\delta\Lambda^{(a)}} = i(t_a)_{bc} A_{c\mu} \ ,\\
&\frac{\delta \hat{A}_{b\mu}}{\delta(\partial_\nu\Lambda^{(a)})} = -\delta_{\mu\nu}\,\delta_{ab} \ ,\\
&\frac{\delta \hat{q}_R}{\delta\Lambda^{(a)}} = i\overline{R}^{(a)} q_R \ ,\\
&\frac{\delta \hat{q}_L}{\delta\Lambda^{(a)}} = i\overline{L}^{(a)} q_L \ .
\end{aligned} \tag{33}
$$

Equations (29a), (30), and (33) yield

$$
\begin{aligned}
&g_{rq} J^{(a)r}_{,p} + \tfrac{1}{2} g_{pq,r} J^{(a)r} + (p \leftrightarrow q) = 0 \ ,\\
&K^{(a)}_{p,q} J^{(b)q} + J^{(b)q}_{,p} K^{(a)}_q + i(t_b)_{ca} K^{(c)}_p = 0 \ ,\\
&\tfrac{1}{2} S^{(ab)}_{,p} J^{(c)p} + i(t_c)_{da} S^{(db)} + (a \leftrightarrow b) = 0 \ ,\\
&V_{,p} J^{(a)p} = 0 \ ,\\
&i\overline{L}^{(a)} Y - iY\overline{R}^{(a)} - Y_{,p} J^{(a)p} = 0 \ .
\end{aligned} \tag{34a}
$$

Equation (29b) gives

$$
\begin{aligned}
&K^{(a)}_p = g_{pq} J^{(a)q} \ ,\\
&S^{(ab)} = J^{(a)p} K^{(b)}_p \ .
\end{aligned} \tag{34b}
$$

These identities are consequences of the invariance of $\mathcal{L}_S$ under the nonlinear local gauge transformations of Eq. (28); this invariance followed automatically from the fact that $\mathcal{L}_S$ was defined by making gauge-type substitutions in $\mathcal{L}$ [see Eq. (13)]. Essentially, Eqs. (34a) and (34b) are expressions of the fact that the σ_a fields are longitudinal modes of vectors and appear in $\mathcal{L}_S$ in a systematic pattern. When the above gauge-invariance identities are combined with the constraints of tree unitarity, it will be possible to prove that $\mathcal{L}_S$ can be put into a *linearly* gauge-invariant form, containing good vertices only.

C. Geometry of field coordinate manifold

In this subsection we introduce a geometric language for describing the transformation of field variables. This step is necessary for the following reasons. So far, the only restrictions on the scalar couplings in $\mathcal{L}_S$ are given by the Stückelberg gauge identities [Eqs. (34a) and (34b)]. This information is expressed in terms of the π (or ϕ, σ) field variables. In Sec. VI it is proved that the tree unitarity of scattering amplitudes with scalar external particles implies the possibility of changing to a new set of field variables, $\pi' = \pi'(\pi)$, in terms of which $\mathcal{L}_S$ has no bad vertices. It should be emphasized that tree unitarity gives the *existence (but not the explicit form)* of the transformation of variables, $\pi \to \pi(\pi')$. The problem is to combine the Stückelberg gauge identities, which constrain the scalar couplings in the π system, with the tree-unitarity constraints, which prohibit bad vertices in the π' system. This must be done without knowing the explicit form of the transformation: $\pi \to \pi(\pi')$. The transfer of the Stückelberg gauge identities from the π system to the π' system can be accomplished if these identities can be expressed in a "generally covariant" form (a form independent of the choice of field variables). In order to construct such "generally covariant" statements, it is convenient to use a geometric notation for handling transformations of field coordinates.

At each point in space-time the scalar field coordinates π_p can be taken to describe a Riemannian manifold.[16] In Eq. (30) define $g_{pq}(\pi)$ to be a generally covariant tensor, $K^{(a)}_p(\pi)$ to be a vector, and $S^{(ab)}(\pi)$, $V(\pi)$, $Y(\pi)$ to be scalars. Now consider the effect of a transformation of field variables $\pi \to \pi(\pi')$. The new Lagrangian, $\mathcal{L}_S(A, q_R, q_L, \pi(\pi'))$, is obtained from the old Lagrangian, $\mathcal{L}_S(A, q_R, q_L, \pi)$, by substituting $\partial_\mu \pi_p \to \partial_\mu \pi'_p$ and by replacing each covariant object with its transformed representative:

$$
\begin{aligned}
&g_{pq}(\pi) \to g'_{pq}(\pi') = \frac{\delta\pi_r}{\delta\pi'_p}\frac{\delta\pi_s}{\delta\pi'_q}\, g_{rs}(\pi(\pi')) \ ,\\
&K^{(a)}_p(\pi) \to K^{(a)\prime}_p(\pi') = \frac{\delta\pi_q}{\delta\pi'_p} K^{(a)}_q(\pi(\pi')) \ ,\\
&S^{(ab)}(\pi) \to S^{(ab)\prime}(\pi') = S^{(ab)}(\pi(\pi')) \ ,\\
&V(\pi) \to V'(\pi') = V(\pi(\pi')) \ ,\\
&Y(\pi) \to Y'(\pi') = Y(\pi(\pi')) \ .
\end{aligned}
$$

Thus, in the geometric language[17] a change of field variables is the same as a change of coordinates on the field manifold. Generally covariant (coordinate-independent) statements about $\mathcal{L}_S$ express properties of $\mathcal{L}_S$ which are independent of the choice of scalar field variables. The symmetric tensor g_{pq} can be taken to define a metric on the field coordinate manifold. Equations (30), (13), and (4) show that $g_{pq}(0)$ has the following positive-definite form:

$$
\begin{aligned}
&g_{kl}(0) = \delta_{kl} \ ,\\
&g_{ka}(0) = g_{ak}(0) = 0 \ ,\\
&g_{ab}(0) = \hat{O}_{ca}\hat{O}_{cb} M_c^{\,2} \ .
\end{aligned} \tag{35}
$$

Therefore, $g_{pq}(\pi)$ is positive-definite in a neighborhood of the origin. In the same neighborhood of the origin it is possible to define an inverse matrix

$g^{pq}(\pi)$ such that $g^{pr}g_{qr}=\delta_{pq}$. Then, the affine connection Γ^r_{pq}, and the curvature tensor R^p_{qrs}, have the usual forms

$$\begin{aligned}\Gamma^r_{pq} &\equiv \tfrac{1}{2}g^{rs}(g_{qs,p}+g_{sp,q}-g_{pq,s})\ ,\\ R^p_{qrs} &= -\Gamma^p_{qr,s}+\Gamma^p_{qs,r}+\Gamma^p_{tr}\Gamma^t_{qs}-\Gamma^p_{ts}\Gamma^t_{qr}\ .\end{aligned} \tag{36}$$

Covariant differentiation with respect to field coordinates is denoted by a subscript preceded by a semicolon; for example, if $A_p(\pi)$ is a covariant vector, then

$$A_{p;q}\equiv A_{p,q}-\Gamma^r_{pq}A_r.$$

With these definitions it is possible to put the Stückelberg gauge-invariance identities [Eqs. (34a) and (34b)] into a generally covariant form:

$$K^{(a)}_{p;q}+K^{(a)}_{q;p}=0\ , \tag{37a}$$

$$K^{(a)}_{p;q}K^{(b)q}-K^{(b)}_{p;q}K^{(a)q}+f_{abc}K^{(c)}_p=0\ , \tag{37b}$$

$$S^{(ab)}=K^{(a)p}K^{(b)}_p\ , \tag{37c}$$

$$V_{;p}K^{(a)p}=0\ , \tag{37d}$$

$$i\overline{L}^{(a)}Y-iY\overline{R}^{(a)}-Y_{;p}K^{(a)p}=0\ . \tag{37e}$$

Since the above expressions are generally covariant, they constrain $\mathfrak{L}_S$ when $\mathfrak{L}_S$ is written in terms of *any* field variables. In other words, the "Stückelberg gauge invariance" of $\mathfrak{L}_S$, which was derived in the π coordinate system, persists for any choice of scalar field variables. This is not too surprising for the following reason. The original gauge-invariance identities [Eqs. (34a) and (34b)] express the invariance of $\mathfrak{L}_S$ under gauge transformations of A, q_R, q_L, and π: e.g., $\pi_p \xrightarrow{U(\Lambda)} \hat{\pi}_p(\pi,\Lambda)$. Suppose that some new scalar variable, $\pi'=f(\pi)$, is used to rewrite $\mathfrak{L}_S$. In terms of the new π' variable, $\mathfrak{L}_S$ should be invariant under the corresponding gauge transformations of A, q_R, q_L, and π'; e.g.,

$$\pi'\to f\{\hat{\pi}[f^{-1}(\pi'),\Lambda]\}\ .$$

In other words, the gauge invariance of $\mathfrak{L}_S$ in the π system guarantees the invariance of $\mathfrak{L}_S$ under the corresponding realization of the gauge group in the π' system. Thus, the Stückelberg gauge invariance is a coordinate-independent property of $\mathfrak{L}_S$, and the gauge-invariance identities should be expressible in a generally covariant (coordinate-independent) way.

Equations (37a) and (37b) can be given a more specialized mathematical meaning. The Stückelberg gauge invariance of $\mathfrak{L}_S$ implies that $g_{pq}(\pi)$ is *form-invariant* under the group of global, nonlinear coordinate transformations: $\pi_p \xrightarrow{U(\Lambda)} \hat{\pi}_p(\pi,\Lambda)$, where $\Lambda^{(a)}$ is constant in space-time. The generator of such a transformation (in this case, $J^{(a)p}=K^{(a)p}$), known as a "Killing vector,"[18] *must* satisfy Eq. (37a) (Killing's equation). Equation (37b) expresses the fact that the $K^{(a)p}$ generate a *group* G of transformations under which the Riemannian manifold is form-invariant.

VI. TREE UNITARITY OF SCALAR AND LONGITUDINAL VECTOR PROCESSES

In Sec. III we noted the result: A set of Feynman rules with good propagators, wave functions, and vertices will automatically lead to a tree-unitary S matrix. In Sec. V A it was observed that the "Stückelberg rules" prescribe good propagators and wave functions; however, the vertices of the Stückelberg rules are very badly behaved since $\mathfrak{L}_S$ contains exponential and nonpolynomial functions of π. It is clear that, *if* there is a change of field variables which transforms away all these bad vertices, then the S matrix will be tree-unitary. In this section the inverse relation is proved: If the S matrix is tree-unitary, then there exists a change of field variables, $\pi\to\pi(\pi')$, which transforms away all bad vertices. In other words, tree unitarity requires the existence of a special π' coordinate system in which g'_{pq} is constant in π', $K^{(a)\prime}_p$ is linear, $S^{(ab)\prime}$ is quadratic, V' is quartic, and Y' is linear; stated in the equivalent geometric language, we shall show that tree unitarity implies the following generally covariant conditions on $\mathfrak{L}_S$:

$$R^p_{qrs}(\pi)=0\ , \tag{38a}$$

$$K^{(a)}_{p;q;r}(\pi)=0\ , \tag{38b}$$

$$S^{(ab)}_{;p;q;r}(\pi)=0\ , \tag{38c}$$

$$V_{;p;q;r;s;t}(\pi)=0\ , \tag{38d}$$

$$Y_{;p;q}(\pi)=0\ , \tag{38e}$$

where π ranges over a neighborhood of the origin.

A. Flatness condition

In order to prove Eq. (38a), it is necessary to discuss the high-energy behavior of individual graphs which are constructed from the Stückelberg rules. Consider any tree diagram contributing to $\mathfrak{M}(\tilde{A}_{a\mu}\cdots\tilde{\sigma}_b\cdots\phi_k\cdots)$. Let the diagram contain $\mathfrak{A}_3$ A^3-type vertices, $\mathfrak{A}_4$ A^4-type vertices, $\mathfrak{G}$ g_{pq}-type vertices, $\mathfrak{K}$ $K^{(a)}_p$-type vertices, $\mathfrak{S}$ $S^{(ab)}$-type vertices, and $\mathfrak{V}$ V-type vertices. In any tree diagram the total number of vertices must be one more than the total number (I) of internal lines; therefore

$$I+1=\mathfrak{A}_3+\mathfrak{A}_4+\mathfrak{G}+\mathfrak{K}+\mathfrak{S}+\mathfrak{V}\ .$$

On the other hand, adding up the energy dependence from propagators and vertex parts [see Eqs. (14) and (30)], we find that *any* graph in $\mathfrak{M}(\tilde{A}\cdots\tilde{\sigma}\cdots\phi\cdots)\leq E^P$, where $P=-2I+\mathfrak{A}_3+2\mathfrak{G}+\mathfrak{K}$.

Eliminating I from the last two formulas gives

$$P=2-\mathcal{A}_3-2\mathcal{A}_4-\mathcal{K}-2\mathcal{S}-2\mathcal{V}\ .$$

This means that $P\leq 1$ for all graphs except those which are constructed with g_{pq}-type vertices only ("pure-g" graphs). For pure-g graphs, $P=2$; at high energy the leading (E^2) piece of a pure-g diagram is a "massless pure-g" graph, which is obtained by setting all internal and external masses equal to zero.

Now consider the tree approximation for $\mathfrak{M}(\tilde{\sigma}_a+i\xi_a\cdot\tilde{A}_a\cdots\phi_k\cdots)$. Since $\xi \underset{E\to\infty}{\sim} E^{-1}$, the preceding discussion shows that any graph for this amplitude diverges no more rapidly than E^2 at high energy. The leading (E^2) piece comes from the massless pure-g diagrams. But tree unitarity, as stated in Eq. (20), requires that $\mathfrak{M}(\tilde{\sigma}_a+i\xi_a\cdot\tilde{A}_a\cdots\phi_k\cdots)$ be bounded by $E^{4-N}\leq E^0$ at high energy. Therefore, the sum of all massless pure-g graphs in this same amplitude must be zero. However, since a pure-g graph cannot have a $\tilde{A}$ external line, it must be that the sum of all massless pure-g diagrams in $\mathfrak{M}(\tilde{\sigma}_a\cdots\phi_k\cdots)$ is also zero. Notice that the sum of all massless pure-g graphs in $\mathfrak{M}(\tilde{\sigma}_a\cdots\phi_k\cdots)$ is the same as the T-matrix element for $\tilde{\sigma}$, ϕ scattering, which would be calculated from the Lagrangian $\mathcal{L}_g\equiv\frac{1}{2}\partial_\mu\pi_p\partial^\mu\pi_q g_{pq}(\pi)$. Hence, *tree unitarity requires that the T matrix of* $\mathcal{L}_g$ *be zero*. In Appendix B it is proved that if the T matrix of $\mathcal{L}_g$ is zero, then $R^p_{qrs}(\pi)=0$ for π in a neighborhood of the origin. Therefore, the tree unitarity of the S matrix of $\mathcal{L}(W,\psi_R,\psi_L,\phi)$ implies the flatness condition, Eq. (38a). This result is easy to understand in the following way. We know that *if* $R^p_{qrs}=0$, then there is a transformation, $\pi\to\pi(\pi')$, which changes $g_{pq}(\pi)$ into $g'_{pq}(\pi')$ = constant and changes $\mathcal{L}_g$ into a free Lagrangian. Thus, *if* $R^p_{qrs}=0$, the T matrix of $\mathcal{L}_g$ must vanish. In Appendix B the inverse statement is proved.

A few remarks should be made at this point. The requirement that there be no E^2 behavior in *all* multiparticle amplitudes led to the condition that the curvature tensor vanish in a neighborhood of the origin. Actually, the vanishing of the curvature near the origin (or, equivalently, the vanishing of the curvature tensor and *all* its higher derivatives *at* the origin) corresponds to an infinite set of tree-unitarity conditions. It can be shown that this correspondence takes the following specific form: The vanishing of the nth ordinary derivative of the curvature tensor at the origin is the tree-unitarity condition that the $(4+n)$-particle scattering amplitude has no E^2 behavior.[19]

With the help of Eq. (38a) it is easy to derive Eqs. (38b) and (38c). Since $K_p^{(a)}$ is a Killing vector [according to Eq. (37a)], we have[18]

$$K^{(a)}_{r;s;q}=-R_{pqrs}K^{(a)p}\ . \tag{39}$$

Then, Eq. (38b) follows from the application of Eq. (38a). Equation (38c) is a consequence of Eq. (38b) and the gauge-invariance identity, Eq. (37c).

B. Quartic character of V and linearity of Y

The conditions on V and Y are most easily derived in a Euclidean coordinate system. Equation (38a) implies the existence of a transformation, $\pi\to\pi(\pi')$, which maps the origin into the origin [$\pi(0)=0$] and which maps $g_{pq}(\pi)\to g'_{pq}(\pi')$ = constant. Since $g_{pq}(0)$ is positive-definite, $g'_{pq}(0)$ must be positive-definite. Therefore, by rotating and scaling the fields, the π' system can always be chosen so that $g'_{pq}(\pi')=\delta_{pq}$ and so that

$$\pi'_k=\phi_k+O(\pi^2),\quad \pi'_a=\tilde{\sigma}_a+O(\pi^2)\ . \tag{40}$$

The "flat" Stückelberg rules for constructing diagrams in the π' system include the propagators

$$\langle T(\pi'_k\,\pi'_l)\rangle=\langle T(\phi_k\,\phi_l)\rangle=\frac{i\delta_{kl}}{k^2-\mu_k{}^2}\ ,$$

$$\langle T(\pi'_a\,\pi'_b)\rangle=\frac{i\delta_{ab}}{k^2-M_a{}^2}\ .$$

The $\tilde{A}$, q_R, and q_L propagators are the same as before. The vertices of the flat Stückelberg rules are taken from the cubic and higher terms in $\mathcal{L}_S(A,q_R,q_L,\pi(\pi'))$. The first equivalence theorem of Appendix A shows that the "flat" Stückelberg rules of this section and the "curved" Stückelberg rules of Sec. VA lead to the same on-mass-shell Green's functions in the tree approximation:

$$\begin{aligned}\mathfrak{M}(\tilde{A}_{a\mu}\cdots\pi'_b\cdots\pi'_k\cdots q_{i\alpha}\cdots)\\ =\mathfrak{M}(\tilde{A}_{a\mu}\cdots\tilde{\sigma}_b\cdots\phi_k\cdots q_{i\alpha}\cdots)\ .\end{aligned} \tag{41}$$

Heuristically, this equivalence relation follows from the fact that the quadratic terms in Eq. (40) have no single-particle poles in the tree approximation and do not contribute to the on-mass-shell Green's functions. The tree-unitarity constraint of Eq. (20) implies that amplitudes constructed according to the "flat" Stückelberg rules obey the high-energy bound

$$\mathfrak{M}(\pi'_a+i\xi_a\cdot\tilde{A}_a\cdots\pi'_k\cdots\bar{u}_iq_i\cdots)\leq E^{4-N}\ . \tag{42}$$

Notice that all propagators and wave functions in Eq. (42) are good. Because of Eqs. (38b) and (38c), $K_p^{(a)\prime}$ is linear in π', and $S^{(ab)\prime}$ is quadratic. Therefore, all vertices in $\mathcal{L}_S(A,q_R,q_L,\pi(\pi'))$ are good, except for terms in $V'(\pi')$ which are $O(\pi'^5)$ and higher and terms in $Y'(\pi')$ which are $O(\pi'^2)$ and higher.

Equation (42) requires

$$\mathfrak{M}_5(\pi'_a + i\xi_a\cdot\tilde{A}_a\cdots\pi'_k\cdots) \leq E^{-1}\,, \tag{43}$$

where $\mathfrak{M}_5$ has *exactly* five boson external lines. Now, we know that any individual diagram, constructed from good propagators, wave functions, and vertices, automatically satisfies the above tree-unitarity condition. Since the left-hand side has exactly five external lines, the only bad vertex which can appear is the five-point contact term from $\pi'^5(\partial^5 V'/\partial^5\pi')_{\pi'=0}$. Therefore, if an individual diagram does not contain this contact term, it is automatically bounded by E^{-1}. The left-hand side of Eq. (43) can be divided into terms which do or do not contain $\tilde{A}$ external lines:

$$\mathfrak{M}_5(\pi'_p\cdots) + \cdots + \mathfrak{M}_5(i\xi_a\cdot\tilde{A}_a\cdots\pi'_p\cdots) + \cdots\,. \tag{44}$$

Any term with an $\tilde{A}$ external line has no diagrammatic contributions which contain the bad five-point contact term. Therefore, all terms but the first term in (44) are automatically bounded by E^{-1}; it follows that the first term must satisfy Eq. (43) separately:

$$\mathfrak{M}_5(\pi'_p\cdots) \leq E^{-1}\,. \tag{45}$$

Every diagram contributing to $\mathfrak{M}_5(\pi'_p\cdots)$ satisfies Eq. (45) except for the single diagram, which is constructed from the five-point contact term. The latter diagram is proportional to $(\partial^5 V'/\partial^5\pi')_{\pi'=0}$ and approaches a constant at high energy. Hence, Eq. (45) requires that $(\partial^5 V'/\partial^5\pi')_{\pi'=0}=0$. We may now proceed by induction to show that the tree unitarity of the n-point amplitude ($n \geq 5$) implies $(\partial^n V'/\partial^n\pi')_{\pi'=0}=0$; i.e., in the explicitly "flat" system $V'(\pi')$ is a quartic polynomial. Therefore, tree unitarity requires that Eq. (38d) is true in any coordinate system.[20]

The linearity of Y can be derived in a similar fashion. We now know that all vertices of the "flat" Stückelberg rules are good except for terms in $Y'(\pi')$ which are of order π'^2 and higher. Tree unitarity, Eq. (42), implies the high-energy bound

$$\mathfrak{M}_4(\pi'_a + i\xi_a\cdot\tilde{A}_a\cdots\pi'_k\cdots\bar{u}_i q_i \bar{q}_j u_j) \leq E^0\,, \tag{46}$$

where $\mathfrak{M}_4$ contains exactly two boson and two spinor external lines. Because the left-hand side has four external lines, the only bad vertex which can appear is the four-point contact term, $\bar{q}(\partial^2 Y'/\partial^2\pi')_{\pi'=0}q\pi'^2$. If an individual diagram does not contain this contact term, it is automatically bounded by E^0. The left-hand side of Eq. (46) can be divided into terms which do or do not contain $\tilde{A}$ external lines:

$$\mathfrak{M}_4(\pi'_p\cdots\bar{u}_i\, q_i\, \bar{q}_j u_j) + \cdots + \mathfrak{M}_4(i\xi_a\cdot\tilde{A}_a\cdots\pi'_p\cdots\bar{u}_i\, q_i\, \bar{q}_j u_j) + \cdots\,. \tag{47}$$

Terms with $\tilde{A}$ external lines have no diagrammatic contributions which contain the bad contact term, and are automatically bounded by E^0; therefore, the first term in Eq. (47) must separately satisfy Eq. (46):

$$\mathfrak{M}_4(\pi'_p\cdots\bar{u}_i\, q_i\, \bar{q}_j u_j) \leq E^0\,. \tag{48}$$

Every diagram, contributing to $\mathfrak{M}_4(\pi'_p\cdots\bar{u}_i\, q_i\, \bar{q}_j u_j)$, is bounded by E^0 except for the single diagram constructed from the four-point contact term. The latter diagram is proportional to $(\partial^2 Y'/\partial^2\pi')_{\pi'=0}$ and diverges as E at high energy. Therefore, the unitarity bound of Eq. (48) requires that $(\partial^2 Y'/\partial^2\pi')_{\pi'=0}=0$. The inductive method shows again that the tree unitarity of amplitudes with n external boson lines ($n \geq 2$) implies that $(\partial^n Y'/\partial^n\pi')_{\pi'=0}=0$. Hence, $Y'(\pi')$ is linear in the π' system, and Eq. (38e) is true in any coordinate system.

Note that Eqs. (38d) and (38e), which are valid for π in a neighborhood of the origin, are equivalent to infinite sets of tree-unitarity constraints. Also observe that it is not surprising that tree-unitarity conditions, being purely S-matrix constraints, can be expressed as field-coordinate-independent statements on $\mathfrak{L}_S$ [like Eqs. (38a)–(38e)].

VII. DERIVATION OF ALL TREE-UNITARY THEORIES

A. Solutions of covariant conditions

We have derived a set of necessary conditions [Eqs. (37) and (38)] which any tree-unitary Lagrangian must satisfy. In this subsection we find all solutions of those conditions; these solutions turn out to be SBGT's (modulo Abelian vector mass terms).

First, Eqs. (37) and (38) are studied in the explicitly flat coordinate system (π'). Equation (38b) means that $K_p^{(a)\prime}(\pi')$ is linear:

$$K_p^{(a)\prime} = iD_{pq}^{(a)}\pi'_q + \lambda_p^{(a)}\,, \tag{49}$$

where $D^{(a)}$ is an imaginary matrix and $\lambda^{(a)}$ is a real vector on the space of all scalar indices. Equation (37a) implies that $D^{(a)}$ is antisymmetric and, therefore, Hermitian. Equation (37b) requires

$$[D^{(a)}, D^{(b)}] = if_{abc}D^{(c)}\,, \tag{50a}$$

$$D^{(a)}\lambda^{(b)} - D^{(b)}\lambda^{(a)} = if_{abc}\lambda^{(c)}\,. \tag{50b}$$

The first condition means that $D^{(a)}$ represents the Lie algebra of G on the space of π'_p fields (combinations of physical scalars and longitudinal vectors). If Eq. (50b) is multiplied by $D^{(a)}$ and then summed over a, the result is

$$\kappa\lambda^{(b)} = D^{(b)}(D^{(a)}\lambda^{(a)})\,, \tag{51}$$

where $\kappa \equiv D^{(a)} D^{(a)}$. The π' coordinates can always be chosen such that each $D^{(a)}$ is a block-diagonal array of irreducible representations ($D^{(a)}_{(i)}$ for the ith block). Decompose $\lambda^{(a)}$ and κ into corresponding blocks ($\lambda^{(a)}_{(i)}$ and $\kappa_{(i)}$). Then, Eqs. (50a), (50b), and (51) are true block by block:

$$[D^{(a)}_{(i)}, D^{(b)}_{(i)}] = if_{abc} D^{(c)}_{(i)} , \tag{52a}$$

$$D^{(a)}_{(i)} \lambda^{(b)}_{(i)} - D^{(b)}_{(i)} \lambda^{(a)}_{(i)} = if_{abc} \lambda^{(c)}_{(i)} , \tag{52b}$$

$$\kappa_{(i)} \lambda^{(b)}_{(i)} = D^{(b)}_{(i)} (D^{(a)}_{(i)} \lambda^{(a)}_{(i)}) . \tag{52c}$$

Equation (52a) implies $[\kappa_{(i)}, D^{(a)}_{(i)}] = 0$. Since the $D^{(a)}_{(i)}$ matrices are irreducible, Schur's lemma shows that $\kappa_{(i)}$ is a constant (depending on the block number i) times the unit matrix. If $\kappa_{(i)} \neq 0$, i is said to denote a "charged" block; if $\kappa_{(i)} = 0$, i denotes a "neutral or redundant" block. On "charged" blocks $\kappa_{(i)}{}^{-1}$ exists, and Eq. (52c) gives

$$\lambda^{(a)}_{(i)} = iD^{(a)}_{(i)} \eta_{(i)} , \tag{53}$$

where $\eta_{(i)} = -i\kappa_{(i)}{}^{-1}(D^{(e)}_{(i)} \lambda^{(e)}_{(i)})$. On "neutral and redundant" blocks each $D^{(a)}_{(i)}$ must be zero, and Eq. (52b) yields

$$(t_a)_{bc} \lambda^{(a)}_{(i)} = 0 . \tag{54}$$

Since the Abelian t_a are zero and the semisimple t_a are linearly independent, we deduce that $\lambda^{(a)}_{(i)} = 0$ for a = semisimple and i = neutral or redundant. Most of the above information is summarized by

$$K^{(a)\prime}_p = \begin{cases} \lambda^{(a)}_p \text{ on neutral or redundant blocks,} \\ iD^{(a)}_{pq}(\pi' + \eta)_q \text{ on charged blocks,} \end{cases} \tag{55}$$

where η_p is the real vector with blocks $\eta_{(i)}$. In the above equation we already know that $\lambda^{(a)}_p$ is zero for a equal to any semisimple vector index. In the following we first find all solutions in which $\lambda^{(a)}_p$ is also zero for all Abelian indices a; then we find all solutions in which $\lambda^{(a)}_p \neq 0$ for some Abelian vector indices.

1. SBGT case

Consider the case $\lambda^{(a)}_p = 0$ for a = Abelian and p = neutral or redundant (N.B.: This condition is automatically true if G is a semisimple group). In this case it can be shown that all solutions of Eqs. (37) and (38) describe SBGT's. Since $D^{(a)}$ vanishes on neutral and redundant blocks, Eq. (55) becomes

$$K^{(a)\prime}_p = iD^{(a)}_{pq}(\pi' + \eta)_q \tag{56}$$

for *all* indices p.

Let $\overline{\pi}_p$ be the translated field, $\overline{\pi}_p \equiv (\pi' + \eta)_p$, and define $\overline{V}(\overline{\pi})$ and $\overline{Y}(\overline{\pi})$ so that $\overline{V}(\overline{\pi}) \equiv V'(\pi')$ and $\overline{Y}(\overline{\pi}) \equiv Y'(\pi')$. Using $g'_{pq} = \delta_{pq}$, $K^{(a)\prime}_p = i(D^{(a)}\overline{\pi})_p$, and $S^{(ab)\prime} = K^{(a)p\prime} K^{(b)\prime}_p$ [Eq. (37c)], $\mathcal{L}_S$ can be rewritten as

$$\mathcal{L}_S(A, q_R, q_L, \pi(\overline{\pi})) = -\tfrac{1}{4}(\partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} - f_{abc} A_{b\mu} A_{c\nu})^2 + \overline{q}_R i(\partial\!\!\!/ + iA\!\!\!/_a \overline{R}^{(a)}) q_R + \overline{q}_L i(\partial\!\!\!/ + iA\!\!\!/_a \overline{L}^{(a)}) q_L$$
$$+ \tfrac{1}{2}[(\partial_\mu + iA_{a\mu} D^{(a)})\overline{\pi}]^2 - \overline{V}(\overline{\pi}) - \overline{q}_L \overline{Y}(\overline{\pi}) q_R - \overline{q}_R \overline{Y}^\dagger(\overline{\pi}) q_L . \tag{57}$$

Here, Eqs. (38d) and (38e) and Eqs. (37d) and (37e) tell us that $\overline{V}(\overline{\pi})$ is a quartic polynomial, $\overline{Y}(\overline{\pi})$ is linear, and that

$$\overline{V}_{,p}\, i(D^{(a)}\overline{\pi})_p = 0 , \tag{58a}$$

$$i\overline{L}^{(a)}\overline{Y} - i\overline{Y}\overline{R}^{(a)} - \overline{Y}_{,p}\, i(D^{(a)}\overline{\pi})_p = 0 . \tag{58b}$$

Observe that $\overline{V}(\overline{\pi})$ has a local minimum at $\overline{\pi}_p = \eta_p$ since $V(\phi)$ has a local minimum at $\phi = 0$.

The Lagrangian now contains only good vertices. Furthermore, Eqs. (57) and (58) guarantee that $\mathcal{L}_S$ is invariant under local gauge transformations which are realized linearly on the scalars:

$$\begin{aligned} A_{a\mu} &\to U(-\Lambda)_{ab} A_{b\mu} - Q(-\Lambda)_{ab}\, \partial_\mu \Lambda_b , \\ q_R &\to e^{i\Lambda\cdot\overline{R}} q_R , \\ q_L &\to e^{i\Lambda\cdot\overline{L}} q_L , \\ \overline{\pi} &\to e^{i\Lambda\cdot D}\overline{\pi} , \end{aligned} \tag{59}$$

where $\Lambda^{(a)}(x)$ is a real function of space-time. The gauge symmetry is spontaneously broken by the vacuum expectation value of the scalar field: $\langle \overline{\pi}_p \rangle_0 = \eta_p$. These properties tell us that $\mathcal{L}_S$ is an SBGT Lagrangian. Therefore, if $\lambda^{(a)}_p = 0$ for Abelian indices a, it has been proved that every solution of Eqs. (37) and (38) describes an SBGT. Note a special case of this result: If the structure constants C_{abc} describe a semisimple group, then the only tree-unitary Lagrangians, which are listed in Eq. (4), are equivalent to SBGT's.

It is worth recapitulating how, in the above argument, the SBGT structure emerged from a combination of gauge-invariance identities [Eq. (37)] and tree-unitarity constraints [Eq. (38)]. In Sec. V C it was noted that the Stückelberg gauge invariance of $\mathcal{L}_S$ persists for any choice of scalar field coordinates. The tree-unitarity equations imply the existence of a special ("flat") coordinate system in which $\mathcal{L}_S$ has no bad vertices. In that special coordinate system the gauge invariance of

$\mathcal{L}_S$ is realized *linearly* [Eq. (59)] on the space ($\overline{\pi}_p$) of physical scalars and longitudinal modes of vectors. For example, the gauge-invariance identities in Eqs. (37a)–(37c) show that the g', K', and S' terms of $\mathcal{L}_S$ take the form of the scalar kinetic energy after a Yang-Mills (minimal) substitution. The other gauge-invariance identities, Eqs. (37d) and (37e), guarantee the invariance of the scalar potential ($\overline{V}$) and the Yukawa term ($\overline{q}\overline{Y}q$) under linear gauge transformations.

2. Possible addition of Abelian vector mass terms

We now consider the remaining possibility: Suppose G is a nonsemisimple group and suppose $\lambda_p^{(a)}$ is nonvanishing for some Abelian vector index a and some neutral or redundant scalar index p. In this case it can be demonstrated that every solution of the covariant conditions describes an SBGT, modified by the addition of Abelian vector mass terms corresponding to invariant Abelian subgroups. Let N_A ($N_A \geq 1$) be the number of Abelian indices, and let the vector indices be labeled so that $a = 1, \ldots, N_A$ denote Abelian indices. Let $N_{\rm NR}$ be the number of neutral and redundant scalar indices, and label the scalars so that $p = 1, \ldots, N_{\rm NR}$ denote neutral and redundant indices. Consider $\lambda_p^{(a)}$ (a = Abelian, p = neutral and redundant) to be a rectangular matrix with $N_{\rm NR}$ rows and N_A columns. It is always possible[21] to transform the neutral and redundant scalar fields by a rotation and to transform the Abelian vector fields by a (different) rotation so that $\lambda_p^{(a)}$ is zero except for diagonal matrix elements; i.e., $\lambda_p^{(a)}$ can be taken to have the "diagonal" form

$$\lambda_p^{(a)} = \begin{bmatrix} M_0^{(1)} & & & & & & \\ & M_0^{(2)} & & & & & \\ & & \ddots & & & & \\ & & & M_0^{(N_0)} & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & \ddots \end{bmatrix}, \qquad (60)$$

where $M_0^{(a)} \neq 0$ and N_0 satisfies $1 \leq N_0 \leq N_A$, $N_0 \leq N_{\rm NR}$. We now distinguish between three types of scalar indices:

$$p \equiv \begin{cases} \text{redundant for } 1 \leq p \leq N_0\,, \\ \text{neutral for } N_0 < p \leq N_{\rm NR}\,, \\ \text{charged for } p > N_{\rm NR}\,. \end{cases}$$

For the subset of Abelian vector indices a such that $1 \leq a \leq N_0$, we have [Eqs. (60) and (55)]

$$\lambda_p^{(a)} = \begin{cases} M_0^{(a)}\delta_{ap}\,, & p = \text{redundant} \\ 0, & p = \text{neutral} \end{cases}$$
$$K_p^{(a)\prime} = \begin{cases} M_0^{(a)}\delta_{ap}\,, & p = \text{redundant} \\ 0, & p = \text{neutral} \\ iD_{pq}^{(a)}(\pi' + \eta)_q\,, & p = \text{charged.} \end{cases} \qquad (61a)$$

For the remaining (Abelian and semisimple) indices $a > N_0$

$$\lambda_p^{(a)} = 0, \quad p = \text{neutral and redundant}$$
$$K_p^{(a)\prime} = \begin{cases} 0, & p = \text{redundant} \\ 0, & p = \text{neutral} \\ iD_{pq}^{(a)}(\pi' + \eta)_q\,, & p = \text{charged}\,. \end{cases} \qquad (61b)$$

It is convenient to use the notation: $\theta_p \equiv \pi_p'$ for p = redundant and $\overline{\pi}_p \equiv (\pi' + \eta)_p$ for p = neutral and charged. Since $D_{pq}^{(a)}$ vanishes on neutral blocks, Eqs. (61a) and (61b) can be rewritten as follows: For $1 \leq a \leq N_0$

$$K_p^{(a)\prime} = \begin{cases} M_0^{(a)}\delta_{ap}\,, & p = \text{redundant} \\ i(D^{(a)}\overline{\pi})_p\,, & p = \text{neutral and charged} \end{cases} \qquad (62a)$$

and for $a > N_0$

$$K_p^{(a)\prime} = \begin{cases} 0, & p = \text{redundant} \\ i(D^{(a)}\overline{\pi})_p\,, & p = \text{neutral and charged}\,. \end{cases} \qquad (62b)$$

Let $\overline{V}(\overline{\pi}, \theta) \equiv V'(\pi')$ and $\overline{Y}(\overline{\pi}, \theta) \equiv Y'(\pi')$. Using $g'_{pq} = \delta_{pq}$ and Eqs. (62a), (62b), and (37c), the Lagrangian of Eq. (30) becomes

$$\mathcal{L}_S(A, q_R, q_L, \pi(\overline{\pi}, \theta)) = -\tfrac{1}{4}(\partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} - f_{abc}A_{b\mu}A_{c\nu})^2 + \overline{q}_R\, i(\not\partial + i\not A_a \overline{R}^{(a)})q_R + \overline{q}_L\, i(\not\partial + i\not A_a \overline{L}^{(a)})q_L$$
$$+ \tfrac{1}{2}[(\partial_\mu + iA_{a\mu}D^{(a)})\overline{\pi}]^2 - \overline{V}(\overline{\pi}, \theta) - \overline{q}_L \overline{Y}(\overline{\pi}, \theta)q_R - \overline{q}_R \overline{Y}^\dagger(\overline{\pi}, \theta)q_L + \sum_{a=1}^{N_0} \tfrac{1}{2}M_0^{(a)2}\left(A_{a\mu} + \frac{1}{M_0^{(a)}}\partial_\mu\theta_a\right)^2. \qquad (63)$$

Equations (38d) and (38e) and (37d) and (37e) show that $\overline{V}(\overline{\pi}, \theta)$ is a quartic polynomial, $\overline{Y}(\overline{\pi}, \theta)$ is linear, and that

$$\overline{V}_{,p}\, i(D^{(a)}\overline{\pi})_p + \frac{\partial \overline{V}}{\partial \theta_a} M_0^{(a)} = 0 \,, \tag{64a}$$

$$i\overline{L}^{(a)}\overline{Y} - i\overline{Y}\overline{R}^{(a)} - \overline{Y}_{,p}\, i(D^{(a)}\overline{\pi})_p - \frac{\partial \overline{Y}}{\partial \theta_a} M_0^{(a)} = 0 \,. \tag{64b}$$

The next step is to prove that $\overline{V}(\overline{\pi}, \theta)$ and $\overline{Y}(\overline{\pi}, \theta)$ are independent of the redundant scalars θ_a. First, let $\tilde{\pi}$ stand for the combination

$$\tilde{\pi} \equiv \left[\exp\left(\frac{-iD^{(1)}\theta_1}{M_0^{(1)}}\right)\right] \overline{\pi} \,.$$

Define the functions $\tilde{V}(\tilde{\pi}, \theta_1, \theta_2, \ldots) \equiv \overline{V}(\overline{\pi}, \theta)$ and

$$\tilde{Y}(\tilde{\pi}, \theta_1, \theta_2, \ldots) \equiv \left[\exp\left(\frac{-i\overline{L}^{(1)}\theta_1}{M_0^{(1)}}\right)\right] \times \overline{Y}(\overline{\pi}, \theta)\left[\exp\left(\frac{i\overline{R}^{(1)}\theta_1}{M_0^{(1)}}\right)\right] .$$

For $a=1$ Eqs. (64a) and (64b) imply

$$\left(\frac{\partial \tilde{V}}{\partial \theta_1}\right)_{\tilde{\pi}\ \text{fixed}} = 0 \,,$$

$$\left(\frac{\partial \tilde{Y}}{\partial \theta_1}\right)_{\tilde{\pi}\ \text{fixed}} = 0 \,.$$

It follows that

$$\overline{V}(\overline{\pi}, \theta) = \tilde{V}\left(\left[\exp\left(\frac{-iD^{(1)}\theta_1}{M_0^{(1)}}\right)\right]\overline{\pi}, 0, \theta_2, \theta_3, \ldots\right) , \tag{65a}$$

$$\overline{Y}(\overline{\pi}, \theta) = \left[\exp\left(\frac{i\overline{L}^{(1)}\theta_1}{M_0^{(1)}}\right)\right] \times \tilde{Y}\left(\left[\exp\left(\frac{-iD^{(1)}\theta_1}{M_0^{(1)}}\right)\right]\overline{\pi}, 0, \theta_2, \ldots\right) \times\left[\exp\left(\frac{-i\overline{R}^{(1)}\theta_1}{M_0^{(1)}}\right)\right] . \tag{65b}$$

Notice that the right-hand sides depend on θ_1 only through the matrix elements of unitary matrices. Since these matrix elements are bounded, $\overline{V}(\overline{\pi}, \theta)$ and $\overline{Y}(\overline{\pi}, \theta)$ are bounded functions of θ_1 for fixed values of $\overline{\pi}$ and θ_a $(a \geq 2)$. However, it has already been demonstrated that $\overline{V}(\overline{\pi}, \theta)$ and $\overline{Y}(\overline{\pi}, \theta)$ are (quartic and linear) polynomials in θ_1 for fixed values of $\overline{\pi}$ and θ_a $(a \geq 2)$. Since the only bounded polynomial is a constant, we conclude that $\overline{V}(\overline{\pi}, \theta)$ and $\overline{Y}(\overline{\pi}, \theta)$ are independent of θ_1. The repeated application of this argument shows that $\overline{V}(\overline{\pi}, \theta)$ and $\overline{Y}(\overline{\pi}, \theta)$ are independent of *all* θ_a: $\overline{V}(\overline{\pi}, \theta) = \overline{V}(\overline{\pi})$ and $\overline{Y}(\overline{\pi}, \theta) = \overline{Y}(\overline{\pi})$.

Thus, the Lagrangian has the form

$$\mathcal{L}_S(A, q_R, q_L, \pi(\overline{\pi}, \theta)) = -\tfrac{1}{4}(\partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} - f_{abc}A_{b\mu}A_{c\nu})^2 + \overline{q}_R\, i(\partial\!\!\!/ + iA\!\!\!/_a \overline{R}^{(a)})q_R + \overline{q}_L\, i(\partial\!\!\!/ + iA\!\!\!/_a \overline{L}^{(a)})q_L + \tfrac{1}{2}[(\partial_\mu + iA_{a\mu}D^{(a)})\overline{\pi}]^2 - \overline{V}(\overline{\pi}) - \overline{q}_L\overline{Y}(\overline{\pi})q_R - \overline{q}_R\overline{Y}^\dagger(\overline{\pi})q_L + \sum_{a=1}^{N_0} \tfrac{1}{2}M_0^{(a)2}\left(A_{a\mu} + \frac{1}{M_0^{(a)}}\partial_\mu\theta_a\right)^2 . \tag{66}$$

Here, $\overline{V}(\overline{\pi})$ is a quartic polynomial, and $\overline{Y}(\overline{\pi})$ is linear; these functions satisfy Eqs. (64a) and (64b):

$$\overline{V}_{,p}\, i(D^{(a)}\overline{\pi})_p = 0 \,, \tag{67a}$$

$$i\overline{L}^{(a)}\overline{Y} - i\overline{Y}\overline{R}^{(a)} - \overline{Y}_{,p}\, i(D^{(a)}\overline{\pi})_p = 0 \,. \tag{67b}$$

Note that $\overline{V}(\overline{\pi})$ has a local minimum at $\overline{\pi} = \eta$ since $V(\phi)$ has a local minimum at $\phi = 0$. Except for the $M_0^{(a)}$ terms, Eqs. (66) and (67) are the same as Eqs. (57) and (58). Therefore, the Lagrangian of Eq. (66) describes an SBGT, modified by the addition of $M_0^{(a)}$ terms; this Lagrangian characterizes all solutions of the covariant conditions, Eqs. (37) and (38), when $\lambda_p^{(a)}$ is nonvanishing. Except for the $M_0^{(a)}$ terms, $\mathcal{L}_S$ is invariant under the local gauge transformations of Eq. (59); the *entire* Lagrangian is invariant under the global version ($\Lambda^{(a)} = \text{constant}$) of the same transformations.

It is easy to show that the $M_0^{(a)}$ terms are equivalent to mass terms for the Abelian vector fields $A_{a\mu}$. Let $\Lambda^{(a)}(x) \equiv (1/M_0^{(a)})\theta_a$ for $1 \leq a \leq N_0$ and $\Lambda^{(a)}(x) \equiv 0$ for $a > N_0$. Next, make the substitutions of Eq. (59) in the Lagrangian of Eq. (66). Since $\mathcal{L}_S$ is invariant except for the $M_0^{(a)}$ terms, the net effect of the substitutions is to replace

$$\sum_{a=1}^{N_0} \tfrac{1}{2}M_0^{(a)2}\left(A_{a\mu} + \frac{1}{M_0^{(a)}}\partial_\mu\theta_a\right)^2 \to \sum_{a=1}^{N_0} \tfrac{1}{2}M_0^{(a)2}A_{a\mu}{}^2. \tag{68}$$

The redundant scalars are eliminated completely. In terms of the new variables, $\mathcal{L}_S$ describes an SBGT for a nonsemisimple group, modified by the addition of mass terms for vectors associated with invariant Abelian subgroups. This Lagrangian is the general solution of Eqs. (37) and (38) when $\lambda_p^{(a)}$ is not zero. The SBGT of Eq. (57) and the modified SBGT of Eqs. (66) and (68) characterize *all* possible solutions of the gauge-invariance conditions and covariant tree-unitarity constraints in Eqs. (37) and (38). This forms the basis of the conclusion: If the general Lagrangian of Eq. (4) has a tree-unitary S matrix, then there must be a transformation of field variables which puts the Lagrangian into the form of an SBGT (modulo Abelian vector mass terms).

SBGT models, modified by the addition of Abelian vector mass terms, can be classified as conserved current theories, "hybrid" theories,[9] or combinations thereof. For example, suppose G contains just one invariant Abelian subgroup, and $\mathcal{L}_S$ contains a mass term for the corresponding vector boson. In the conserved current case the Abelian subgroup is *not* spontaneously broken; the corresponding massive Abelian vector couples directly to a conserved source current. The simplest example of this type of theory is massive QED. In the "hybrid" case the Abelian subgroup *is* spontaneously broken. Therefore, the Abelian vectors can mix with other vectors and need not have conserved source currents. A simple example[9] of a hybrid model is Weinberg's [SU(2) $\otimes$ U(1)] theory, modified by the addition of an Abelian vector mass term. The model describes four *massive* vectors, none of which couples to a conserved source current.

B. Full tree unitarity of all solutions

We have shown that if the Lagrangian of Eq. (4) has a tree-unitary S matrix, then there must be a change of variables which puts the Lagrangian into the form of Eqs. (57) or (66). In this subsection the inverse relation is proved: Any Lagrangian given by Eqs. (57) or (66) can be transformed into a Lagrangian which has the form of Eq. (4) and which has a fully tree-unitary[3] S matrix. The first step is to construct the transformation which puts Eqs. (57) or (66) into the form of Eq. (4). Consider the vector-boson mass matrix, which characterizes the terms (of $\mathcal{L}_S$) quadratic in vector fields:

$$M^2{}_{ab} = (iD^{(a)}\eta)_p\,(iD^{(b)}\eta)_p + M_0^{(a)2}\delta_{ab}\ , \tag{69}$$

where p is summed only over neutral and charged scalar indices ($p > N_0$). It is easy to see that M^2 is a real, symmetric, and positive-indefinite matrix. Therefore, there is an orthogonal transformation $\hat{O}_{ab}$, which diagonalizes it: $(\hat{O}M^2\hat{O}^{-1})_{ab} = \delta_{ab}M_a{}^2$. In general, we know that $M_a \geq 0$; at this point it will be assumed that all $M_a > 0$. This assumption is consistent with the original restriction that the Lagrangian in Eq. (4) describes massive vectors only (see Sec. III). Define the real vectors $v_p^{(a)}$ on the space of all (redundant plus neutral plus charged) scalar indices:

$$v_p^{(a)} \equiv \begin{cases} \dfrac{\hat{O}_{ap}M_0^{(p)}}{M_a}\ , & p = \text{redundant} \\[2ex] \dfrac{\hat{O}_{ab}(iD^{(b)}\eta)_p}{M_a}\ , & p = \text{neutral or charged}\ . \end{cases} \tag{70}$$

A direct calculation shows that the $v^{(a)}$ are orthonormal. Define the real vectors $u_p^{(k)}$ so that $\{u^{(k)}, v^{(a)}\}$ is an orthonormal set which is complete over the entire space of all scalar indices. The number of values, taken by the index k, is equal to the number of all scalar indices minus the number of all vector indices. The following invertible transformation serves to define the real fields ϕ_k and σ_a in terms of θ_a and $\overline{\pi}_p$:

$$\begin{aligned} &\theta_p \equiv \phi_k u_p^{(k)} + M_a\hat{O}_{ab}\sigma_b v_p^{(a)}\ , \quad p = \text{redundant} \\ &\overline{\pi}_p \equiv [e^{i\sigma\cdot D}(\eta + u^{(k)}\phi_k)]_p\ , \quad p = \text{neutral or charged.} \end{aligned} \tag{71}$$

Define $W_{a\mu}$, ψ_R, and ψ_L by

$$\begin{aligned} &A_{a\mu} \equiv U(-\sigma)_{ab}\hat{O}^{-1}{}_{bc}W_{c\mu} - Q(-\sigma)_{ab}\partial_\mu\sigma_b\ , \\ &q_R \equiv e^{i\sigma\cdot\overline{R}}\psi_R\ , \\ &q_L \equiv e^{i\sigma\cdot\overline{L}}\psi_L\ . \end{aligned} \tag{72}$$

When $\mathcal{L}_S$ is rewritten in terms of σ, ϕ, W, and ψ, the σ fields disappear. Except for rotations which diagonalize the ϕ and ψ mass matrices, the rewritten Lagrangian takes the form of Eq. (4), with

$$\begin{aligned} &A_{abc} = B_{abcd} = 0\ , \\ &C_{abc} = f_{a'b'c'}\hat{O}_{aa'}\hat{O}_{bb'}\hat{O}_{cc'}\ , \\ &8D_{abcd} = C_{ace}C_{bde} + C_{ade}C_{bce}\ , \\ &R^{(a)} = \hat{O}_{ab}\overline{R}^{(b)}, \\ &L^{(a)} = \hat{O}_{ab}\overline{L}^{(b)}\ , \\ &P_{kl}(\phi) = \delta_{kl}\ , \\ &G_{ak}(\phi) = -\hat{O}_{ab}u_p^{(k)}(iD^{(b)}u^{(l)})_p\,\phi_l\ , \\ &2F_{ab}(\phi) = \delta_{ab}M_a{}^2 + M_b\hat{O}_{ac}v_p^{(b)}(iD^{(c)}u^{(k)})_p\,\phi_k \\ &\qquad + M_a\hat{O}_{bc}v_p^{(a)}(iD^{(c)}u^{(k)})_p\,\phi_k \\ &\qquad + \hat{O}_{ac}\hat{O}_{bd}(iD^{(c)}u^{(k)})_p\,(iD^{(d)}u^{(l)})_p\,\phi_k\phi_l\ , \\ &V(\phi) = \overline{V}(\eta + u^{(k)}\phi_k)\ , \\ &H(\phi) = \overline{Y}(\eta + u^{(k)}\phi_k)\ . \end{aligned} \tag{73}$$

This completes the demonstration that the $\mathcal{L}_S$ in Eqs. (57) or (66) can always be transformed into the $\mathcal{L}(W, \psi_R, \psi_L, \phi)$ in Eq. (4). Notice that the mass dimension of $\mathcal{L}$ is less than or equal to four as anticipated in the comments in Sec. III.

The next step is to prove the full tree unitarity of the Lagrangian specified by Eq. (73). Because of the equivalence theorem in Appendix A, the

S matrix of $\mathcal{L}(W, \psi_R, \psi_L, \phi)$ is given by Eq. (19). The left-hand side is computed by using the Stückelberg rules, derived from the Lagrangians in Eqs. (57) or (66). Since these Lagrangians have mass dimension less than or equal to four, all vertices of the Stückelberg rules are good. In addition, all Stückelberg propagators are good, and the wave functions $(\xi, \bar{u})$ in the left-hand side of Eq. (19) are well-behaved. It follows that the left-hand side of Eq. (19) is also well-behaved $(\lesssim E^{4-N})$ at high energy. Therefore, the Lagrangian of Eq. (73) satisfies the unitarity bound:

$$\mathfrak{M}(\epsilon_a \cdot W_a \cdots \bar{u}_i \psi_i \cdots \phi_k \cdots) \lesssim E^{4-N} .$$

Hence, we have demonstrated that every Lagrangian given by Eqs. (57) and (66) can be transformed into a fully tree-unitary Lagrangian with the form of Eq. (4).

This result, taken together with the result of Sec. VII A, shows that the SBGT of Eq. (57) and the modified SBGT of Eq. (66) constitute the complete set of all fully tree-unitary theories; i.e., the Lagrangian in Eq. (4) is fully tree-unitary *if and only if* it is equivalent to an SBGT, possibly modified by the addition of Abelian vector mass terms.

VIII. DISCUSSION AND CONCLUSIONS

The central conclusion of this paper is that the general Lagrangian of Eq. (4) is tree-unitary if and only if it is equivalent to an SBGT, possibly modified by the addition of mass terms for vectors associated with invariant Abelian subgroups. The method of proof has heavily exploited the freedom to change field variables. It is known that a set of Lagrangians related by transformations of field variables describe the same S matrix. We have shown that it is always possible to find such a transformation which casts any tree-unitary Lagrangian into the "standard form" of a gauge-invariant Lagrangian. In this sense, the high-energy criterion of tree unitarity imposes internal gauge symmetry on heavy-vector-boson Lagrangians. This may be the true meaning of spontaneously broken gauge invariance.

Section II presents a plausibility argument that any perturbatively renormalizable theory must be tree-unitary. If that connection is accepted, it follows that every renormalizable model must be equivalent to an SBGT, conserved-current theory, or hybrid theory of Eqs. (57) or (66). Other authors[1] have shown that the SBGT and conserved-current models are truly renormalizable; it is likely that the hybrid theories share this property. Therefore, it is probable that Eqs. (57) and (66) constitute the "standard forms" for *all* renormalizable interactions. All of these "standard forms" are symmetric under groups of global (and, possibly, local) gauge transformations. This suggests that the appearance of Lie groups of internal symmetries in particle physics is a natural consequence of renormalizability. For example, this may explain the striking fact that weak-interaction currents form a Lie algebra. If the weak interactions are renormalizable and mediated by vector mesons, they would have to be described by a "standard" gauge theory. In such models the fermionic parts of the weak currents (vector-boson source currents) *must* form some Lie algebra. Of course, the criterion of renormalizability alone does not specify *which* Lie group or *which* representations are involved.

The results of this paper suggest several areas of investigation.

(1) Hybrid models. Since the hybrid theories are fully tree-unitary, it is likely that they are also renormalizable. However, this should be proved in detail. As suggested in Ref. 9, "hybridization" might be a useful way of regulating the infrared behavior due to massless vectors associated with unbroken Abelian subgroups in ordinary SBGT's.

(2) More general Lagrangians. We are now in the process of generalizing the techniques of this paper in order to investigate the tree unitarity of Lagrangians which are more general than the one in Eq. (4). The question is: Are there *new* tree-unitary Lagrangians which contain higher powers of W, ∂W, and $\partial\phi$? As mentioned in Sec. III, the chance of an affirmative answer is small.

(3) Higher spins. The Einstein theory of pure gravity is not tree-unitary; in fact, all N-point scattering amplitudes diverge as E^2 at high energy. Can a tree-unitary (and, presumably, renormalizable[22]) model of gravitation be constructed by adding scalar, vector, and massive tensor exchanges, which cancel the bad high-energy behavior? Unfortunately, the answer is "no." In order to maintain Newton's law of gravitation between any two particles, the four-particle scattering amplitude must have a single-graviton-exchange pole with nonzero residue. Because of the spin-2 nature of the graviton, these *pole* terms have pieces which behave like s^2/t, thereby violating unitarity at high energy. On the other hand, scalar exchange diagrams, vector exchange diagrams, and "contact" diagrams are unitarily bounded at high energy except for *polynomials* in s and t. Therefore, these contributions cannot cancel the bad high-energy behavior of graviton exchange. Massive tensor exchange contributes a nonunitary pole term with the *same sign* as the graviton pole term; thus, there can be no cancellation between badly behaved contributions from

graviton and massive tensor exchanges. We conclude that a tree-unitary theory of gravitation cannot be constructed from fields with spin less than or equal to two. It is interesting to ask if there are any tree-unitary theories of scalar, vector, and (massive or massless) tensor particles, which are nongravitational (i.e., which do not reproduce Newton's law). An even more ambitious program would be a search for tree-unitary models which describe towers of higher-spin particles.

Finally, it is interesting to speculate about the relevance of the dimensionality of space-time to this work. We have demonstrated that in four-dimensional space-time tree unitarity is just strong enough to impose internal symmetry on heavy-vector-boson interactions. Multiparticle phase space in two- or three-dimensional space-time grows less rapidly with energy. Therefore, the constraints of tree unitarity are weaker and may not impose a group symmetry on vector-particle interactions in two or three dimensions. On the other hand, vector theories in five-dimensional space-time may be so divergent that there are *no* solutions of all tree-unitarity constraints. So, the connection between high-energy behavior and internal symmetry under Lie groups may be characteristic of four-dimensional space-time.

APPENDIX A: EQUIVALENCE THEOREMS

In this appendix it is shown that Lagrangians, related by point transformations of fields, describe the same S matrix in the tree approximation. First, we state the following general equivalence theorem: Two Lagrangians are equivalent if they are related by an invertible transformation which maps vanishing fields. Then, it is proved that two vector-particle Lagrangians, related by a generalized Stückelberg decomposition, lead to the same S matrix in the tree approximation.

Consider any Lagrangian of a real field ϕ with the form

$$\mathcal{L}(\phi) = \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2}\mu^2\phi^2 + O(\phi^3) + O(\partial\phi^3) + \cdots . \tag{A1}$$

Define a set of $\mathcal{L}$ Feynman rules, in which the ϕ propagator has the usual form [Eq. (2)] and in which vertices are taken from cubic and higher terms of $\mathcal{L}$. Let $h(\phi)$ be the following invertible transformation which maps the origin into the origin:

$$h(\phi') = \phi' + O(\phi'^2) . \tag{A2}$$

Define $\mathcal{L}_h(\phi')$ by

$$\mathcal{L}_h(\phi') \equiv \mathcal{L}[h(\phi')] . \tag{A3}$$

Consider the set of $\mathcal{L}_h$ Feynman rules in which the ϕ' propagator has the usual form and vertices are taken from cubic and higher terms in $\mathcal{L}_h$. The equivalence theorem[8] states that the S-matrix elements, constructed in the tree approximation from $\mathcal{L}$ Feynman rules, are equal to the S-matrix elements, constructed in the tree approximation from $\mathcal{L}_h$ Feynman rules; that is

$$\mathfrak{M}[\phi(k_1)\,\phi(k_2)\cdots]_{\text{tree}} = \mathfrak{M}[\phi'(k_1)\,\phi'(k_2)\cdots]_{\text{tree}} . \tag{A4}$$

Notice that only the on-mass-shell Green's functions need to be equivalent; in general, $\mathcal{L}$ and $\mathcal{L}_h$ describe very different off-mass-shell Green's functions. Heuristically, the theorem means that ϕ' and $\phi = h(\phi')$ are equivalent interpolating fields; the quadratic and higher terms in $h(\phi')$ do not contain single-particle poles and, therefore, do not contribute to on-mass-shell Green's functions.[23]

These results can be restated in terms of the functional formalism[24] for calculating Green's functions. Let $f(x)$ be a real external source function. Define $G(f)$ to be the following functional:

$$G(f) \equiv \text{Ex} \int d^4x [\mathcal{L}(\phi) + f(x)\,\phi(x)] . \tag{A5}$$

Here, "Ex" means that the integral is evaluated at $\phi(x) = \phi_E(x, f)$, the extremum value of the field. It can be proved that the Green's functions, constructed in the tree approximation from the $\mathcal{L}$ Feynman rules, are generated by $G(f)$ in the following way:

$$i^{(n-1)}\langle 0 | T^*\{\phi(x)\cdots\phi(y)\} | 0\rangle = \left[\frac{\delta^n G(f)}{\delta f(x)\cdots\delta f(y)}\right]_{f=0} . \tag{A6}$$

Similarly, the Green's functions, constructed from the $\mathcal{L}_h$ Feynman rules, are generated by $G_h(f)$:

$$G_h(f) \equiv \text{Ex} \int d^4x [\mathcal{L}_h(\phi') + f(x)\,\phi'(x)] . \tag{A7}$$

According to the equivalence theorem, $G(f)$ and $G_h(f)$ generate the same on-mass-shell Green's functions.

Next, we prove that two vector field Lagrangians, related by a generalized Stückelberg decomposition, describe the same S matrix in the tree approximation. Let $\mathcal{L}(W)$ be any Lagrangian of the form

$$\mathcal{L}(W) = -\tfrac{1}{4}(\partial_\mu W_\nu - \partial_\nu W_\mu)^2 + \tfrac{1}{2}M^2 W_\mu{}^2 + O(W^3) + \cdots . \tag{A8}$$

Define a set of $\mathcal{L}$ Feynman rules in which the W propagator has the usual form [Eq. (2)] and in which vertices are taken from cubic and higher terms in $\mathcal{L}$. Let $\alpha(\tilde{\sigma})$ and $\beta(\tilde{\sigma})$ be any power series such that

$$\alpha(\tilde{\sigma}) = 1 + O(\tilde{\sigma}), \quad \beta(\tilde{\sigma}) = 1 + O(\tilde{\sigma}) . \tag{A9}$$

Define the Stückelberg Lagrangian $\mathcal{L}_S$ by decomposing each vector field (W) into a "Stückelberg vector" ($\tilde{A}$) and a "Stückelberg scalar" ($\tilde{\sigma}$) as follows:

$$\mathcal{L}_S(\tilde{A}, \tilde{\sigma}) \equiv \mathcal{L}\left(\alpha(\tilde{\sigma})\tilde{A}_\mu + \frac{1}{M}\beta(\tilde{\sigma})\partial_\mu\tilde{\sigma}\right). \tag{A10}$$

Consider the Stückelberg rules in which vertices are taken from cubic and higher terms in $\mathcal{L}_S$ and in which the propagators are

$$\begin{aligned} \langle T(\tilde{\sigma}\tilde{\sigma})\rangle &= \frac{1}{k^2 - M^2} , \\ \langle T(\tilde{A}_\mu \tilde{A}_\nu)\rangle &= \frac{-ig_{\mu\nu}}{k^2 - M^2} . \end{aligned} \tag{A11}$$

The equivalence theorem states that an on-mass-shell scattering amplitude of W's, constructed in the tree approximation from the $\mathcal{L}$ Feynman rules, is equal to the corresponding on-mass-shell scattering amplitude of $[\tilde{A}_\mu + (1/M)\partial_\mu\tilde{\sigma}]$'s, constructed in the tree approximation from the Stückelberg rules; that is

$$\mathfrak{M}(W_\mu \cdots) = \mathfrak{M}([\tilde{A}_\mu + (1/M)\partial_\mu\tilde{\sigma}]\cdots) . \tag{A12}$$

This equation can be verified immediately for the special case[25]: $\alpha(\tilde{\sigma}) = \beta(\tilde{\sigma}) = 1$. Then, it is clear that the vertices of the $[\tilde{A}_\mu + (1/M)\partial_\mu\tilde{\sigma}]$ modes are exactly the same as the W_μ vertices. Since the W_μ and $[\tilde{A}_\mu + (1/M)\partial_\mu\tilde{\sigma}]$ modes also have identical propagators, their scattering amplitudes (as well as their off-shell Green's functions) are equal.

If α and β are more general functions, the proof of Eq. (A12) is more involved. First, notice that the Green's functions, constructed from the $\mathcal{L}$ Feynman rules, are generated by

$$G_F(J_\mu) = \mathrm{Ex}\int d^4x[\mathcal{L}(W) + J_\mu W^\mu] . \tag{A13}$$

Similarly, the Green's functions, constructed from the Stückelberg rules, are generated by

$$\begin{aligned} G_S(J_\mu, f) = \mathrm{Ex}\int d^4x[&\mathcal{L}_S(\tilde{A}; \tilde{\sigma}) - \tfrac{1}{2}(\partial\tilde{A} - M\tilde{\sigma})^2 \\ &+ J_\mu\tilde{A}^\mu + f\tilde{\sigma}] . \end{aligned} \tag{A14}$$

Define $\tilde{A}_\mu(\tilde{B}_\nu, \tilde{\sigma})$ to be a function of $\tilde{\sigma}$ and another vector field $\tilde{B}_\mu$:

$$\alpha(\tilde{\sigma})\tilde{A}_\mu + \frac{1}{M}\beta(\tilde{\sigma})\partial_\mu\tilde{\sigma} \equiv \tilde{B}_\mu + \frac{1}{M}\partial_\mu\tilde{\sigma} . \tag{A15}$$

Note that this transformation has the form

$$\tilde{A}_\mu = \tilde{B}_\mu + O(\tilde{\sigma}^2) + O(\tilde{\sigma}\tilde{B}) + \cdots . \tag{A16}$$

Therefore, the transformation between $\tilde{A}$ and $\tilde{B}$ is invertible and maps vanishing fields into vanishing fields. Now, define $G_I(J_\mu, f)$:

$$\begin{aligned} G_I(J_\mu, f) \equiv \mathrm{Ex}\int d^4x\{&\mathcal{L}_S(\tilde{A}(\tilde{B}, \tilde{\sigma}), \tilde{\sigma}) \\ &- \tfrac{1}{2}[\partial\tilde{A}(\tilde{B}, \tilde{\sigma}) - M\tilde{\sigma}]^2 \\ &+ J_\mu\tilde{B}^\mu + f\tilde{\sigma}\} . \end{aligned} \tag{A17}$$

According to the equivalence theorem stated earlier in this appendix, $G_I(J_\mu, f)$ and $G_S(J_\mu, f)$ generate the same on-mass-shell Green's functions. At the point $f = (-1/M)\partial J$, $G_I(J_\mu, (-1/M)\partial J)$ can be written as

$$G_I\left(J_\mu, \frac{-1}{M}\partial J\right) = \mathrm{Ex}\int d^4x\left[\mathcal{L}\left(\tilde{B}_\mu + \frac{1}{M}\partial_\mu\tilde{\sigma}\right) - \tfrac{1}{2}(\partial\tilde{A} - M\tilde{\sigma})^2 + J_\mu\left(\tilde{B}^\mu + \frac{1}{M}\partial^\mu\tilde{\sigma}\right)\right] . \tag{A18}$$

The right-hand side is evaluated at the extremum fields: $\tilde{B}_\mu = \tilde{B}_{E\mu}(x, J)$, $\tilde{\sigma} = \tilde{\sigma}_E(x, J)$. Any variation about these fields produces no change in the action:

$$\begin{aligned} 0 = \delta\int d^4x\bigg[&\mathcal{L}\left(\tilde{B}_\mu + \frac{1}{M}\partial_\mu\tilde{\sigma}\right) - \tfrac{1}{2}(\partial\tilde{A} - M\tilde{\sigma})^2 \\ &+ J_\mu\left(\tilde{B}^\mu + \frac{1}{M}\partial^\mu\tilde{\sigma}\right)\bigg]_{\substack{\tilde{B} = \tilde{B}_E \\ \tilde{\sigma} = \tilde{\sigma}_E}} . \end{aligned} \tag{A19}$$

In particular, consider the variation

$$\delta_\Lambda\tilde{B}_\mu = \partial_\mu\Lambda, \quad \delta_\Lambda\tilde{\sigma} = -M\Lambda ,$$

where $\Lambda(x)$ is any real function. Since $\delta_\Lambda[\tilde{B}_\mu + (1/M)\partial_\mu\tilde{\sigma}] = 0$, Eq. (A19) implies

$$(\partial\tilde{A}_E - M\tilde{\sigma}_E)\,\delta_\Lambda(\partial\tilde{A} - M\tilde{\sigma}) = 0 . \tag{A20}$$

This means that $\partial\tilde{A}_E - M\tilde{\sigma}_E = 0$ and, therefore,

$$\begin{aligned} G_I\left(J_\mu, \frac{-1}{M}\partial J\right) &= \mathrm{Ex}\int d^4x\bigg[\mathcal{L}\left(\tilde{B}_\mu + \frac{1}{M}\partial_\mu\tilde{\sigma}\right) \\ &\qquad + J_\mu\left(\tilde{B}^\mu + \frac{1}{M}\partial^\mu\tilde{\sigma}\right)\bigg] \\ &= G_F(J_\mu) . \end{aligned} \tag{A21}$$

Since $G_S(J_\mu, (-1/M)\partial J)$ and $G_I(J_\mu, (-1/M)\partial J)$ generate the same on-mass-shell Green's functions, it follows that $G_S(J_\mu, (-1/M)\partial J)$ and $G_F(J_\mu)$ generate the same on-mass-shell Green's functions. Notice that

$$\frac{\delta G_S(J_\mu, (-1/M)\partial J)}{\delta J_\mu(x)} = \left[\left(\frac{\delta}{\delta J_\mu(x)} + \frac{1}{M}\frac{\partial}{\partial x_\mu}\frac{\delta}{\delta f(x)}\right) \times G_S(J_\mu, f)\right]_{f = (-1/M)\partial J} .$$

Therefore, we have proved that the Green's function

$$\left[\frac{\delta^n G_F}{\delta J_\mu(x)\cdots\delta J_\nu(y)}\right]_{J=0}$$

is equal on the mass shell to the Green's function

$$\left[\left(\frac{\delta}{\delta J_\mu(x)}+\frac{1}{M}\frac{\partial}{\partial x_\mu}\frac{\delta}{\delta f(x)}\right)\cdots \times\left(\frac{\delta}{\delta J_\nu(y)}+\frac{1}{M}\frac{\partial}{\partial y_\nu}\frac{\delta}{\delta f(y)}\right)G_S(J_\mu,f)\right]_{J=f=0} .$$

This completes the proof of the equivalence relation, Eq. (A12).

APPENDIX B: PROOF OF FLATNESS CONDITION

Consider a Lagrangian of the form

$$\mathcal{L}(\pi)=\tfrac{1}{2}\partial_\mu\pi_p\partial^\mu\pi_q g_{pq}(\pi)$$

involving a number of scalar fields π_p. The functions $g_{pq}(\pi)$ are given by a power series of the form

$$g_{pq}(\pi)=\delta_{pq}+\tfrac{1}{2}\beta_{pqrs}\pi_r\pi_s+\cdots .$$

The absence of a linear term involves no loss of generality since it can always be eliminated by a point transformation. In this appendix we prove the following result, which was used in Sec. VI: If the T matrix of $\mathcal{L}(\pi)$ vanishes, then the metric $g_{pq}(\pi)$ is flat; i.e., $R^p_{qrs}(\pi)=0$ for π in a neighborhood of the origin. In other words, if the T matrix vanishes, then it is always possible to find a point transformation of the fields

$$\pi_p=F_p(\pi'),\quad F_p(0)=0$$

such that $\mathcal{L}(F(\pi'))=\frac{1}{2}\partial_\mu\pi_p'\partial^\mu\pi_p'$, i.e., $\mathcal{L}(\pi)$ is equivalent to a free theory. It is understood here that F_p represents an invertible change of variables so that it can be taken to be a power series of the form

$$F_p(\pi')=\pi_p'+O(\pi'^2) .$$

We begin by calculating the tree approximation for the amplitude $\pi_p\pi_q\to\pi_r\pi_s$. Setting it equal to zero results in the relation

$$\beta_{prqs}+\beta_{sqpr}-\beta_{psrq}-\beta_{rqps}=0$$

which states that the Riemann-Christoffel tensor of the manifold with metric $g_{pq}(\pi)$ vanishes at the origin $\pi_p=0$:

$$R^p_{qrs}(0)=0 .$$

It is possible to proceed now step-by-step and show that the vanishing of the $(n+4)$-particle amplitudes implies the vanishing of the nth derivatives of R^p_{qrs} at the origin. Thus, the vanishing of the T matrix implies that the g_{pq} manifold is flat in a neighborhood of the origin and, in particular, Euclidean since $g_{pq}(0)=\delta_{pq}$.

Instead of looking in detail at all higher-point trees, we prefer to complete the proof by making use of the functional formalism.[24] The tree-generating functional for $\mathcal{L}(\pi)$ is given by

$$G(f)=\underset{\pi}{\text{extremum}}\int d^4x[\mathcal{L}(\pi)+f_p\pi_p] .$$

The functional derivatives of G yield the Green's functions according to

$$(-i)^{n-1}\frac{\delta^n G}{\delta f_{p_1}(x_1)\cdots\delta f_{p_n}(x_n)}\Bigg|_{f=0}=\langle T(\pi_{p_1}(x_1)\cdots\pi_{p_n}(x_n))\rangle_{\text{tree}}$$

The first functional derivative

$$\frac{\delta G}{\delta f_p(x)}=\pi_p(x;f)$$

satisfies the field equations

$$-\partial^\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\pi_p)}+\frac{\partial\mathcal{L}}{\partial\pi_p}+f_p=0 ,$$

which may be written explicitly as

$$\partial^2\pi_p=\Gamma^p_{qr}(\pi)\partial_\mu\pi_q\partial^\mu\pi_r+g^{pq}f_q .$$

Here $g^{pq}\equiv(g^{-1})_{pq}$ is the contravariant metric tensor and Γ^p_{qr} is the affinity of Eq. (36). The field equations and the appropriate boundary conditions lead to the following integral equation from which any Green's function can be obtained by a finite number of iterations:

$$\pi_p(x)=i\int d^4x'\,\Delta_F(x-x')[J_p(\pi(x'))+f_p(x')] .$$

Here Δ_F is the Feynman propagator for a massless scalar field and $J_p\equiv\Gamma^p_{qr}\partial_\mu\pi_q\partial^\mu\pi_r$. In writing the integral equation we have simplified $g^{pq}f_q$ to f_p since terms like $\pi^n f$ with $n>0$ do not contribute to mass-shell amplitudes. Consider now the *new* Lagrangian $\tilde{\mathcal{L}}(\tilde\pi_p)\equiv\mathcal{L}(\tilde\pi_p+\eta_p)$, where η_p is any sufficiently small x-independent, c-number displacement of the scalar fields. The functional $\tilde\pi_p(x,f)$ associated with $\tilde{\mathcal{L}}$ satisfies

$$\tilde\pi_p(x)=i\int d^4x'\,\Delta_F(x-x')[J_p(\tilde\pi(x')+\eta)+f_p(x')] ,$$

which may be written as

$$\tilde\pi_p(x)+\eta_p=i\int d^4x[J_p(\tilde\pi(x')+\eta)+f_p(x')+\eta_p\partial'^2]\times\Delta_F(x-x') .$$

Comparing this with the integral equation for $\pi_p(x,f)$ we obtain

$$\tilde\pi_p(x,f)+\eta_p=\pi_p(x;f+\eta\partial^2) .$$

Thus the T matrices of $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are simply related in the tree approximation: A given n-particle mass-shell amplitude of $\tilde{\mathcal{L}}$ is equal to the corresponding n-particle amplitude of $\mathcal{L}$ plus a sum over all possible insertions in it of external lines carrying zero four-momentum. But this means that all tree amplitudes of $\tilde{\mathcal{L}}$ vanish since those of $\mathcal{L}$ vanish. In particular, the vanishing of the four-point amplitudes for $\tilde{\mathcal{L}}(\tilde{\pi}) = \mathcal{L}(\tilde{\pi}+\eta)$ implies that

$$R^p_{qrs}(\eta) = 0$$

for all sufficiently small η_p. Hence the metric g_{pq} is flat in some neighborhood of the origin.

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[4]J. M. Cornwall, D. N. Levin, and G. Tiktopoulos, Phys. Rev. Lett. 30, 1268 (1973); 31, 572(E) (1973). In this paper tree-unitarity is imposed on the most general interaction of any number of scalar and vector particles, which has mass dimension less than or equal to four. The coupling constant relations, which guarantee the tree-unitarity of four-particle scattering amplitudes, are derived. These conditions show that the purely vector interactions must have a gauge-invariant (Yang-Mills) form; however, the *four*-particle unitarity conditions are *not* strong enough to show that the vector-scalar and purely scalar interactions *must* describe an SBGT. The uniqueness of the SBGT Lagrangian can be demonstrated only if the *four-, five-, and higher-point* unitarity conditions are solved *simultaneously*. For example, in the case of a special particle spectrum (one massless vector, three massive vectors, one scalar) the four- and five-particle unitarity relations are solved explicitly in the reference cited above; the only solutions, which describe interacting vectors, are the SBGT's of Weinberg [SU(2)⊗U(1)], Georgi-Glashow [SU(2)], and Higgs [U(1)].

[5]C. H. Llewellyn Smith, Phys. Lett. 46B, 233 (1973). This paper also derives the four-particle unitarity conditions for a general interaction of vectors and scalars (see Ref. 4). In addition, spinor interactions are included and constrained by four-particle unitarity. Because tree-unitarity is not explored for five- and higher-point functions, the author does not demonstrate that the *only* tree-unitarity models are SBGT's.

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[10]Notation: All conventions having to do with the metric, Dirac matrices, and external wave functions are taken from J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965). Right- and left-handed spinors are defined by $\psi_R \equiv \frac{1}{2}(1+\gamma_5)\psi$, $\psi_L \equiv \frac{1}{2}(1-\gamma_5)\psi$. Greek letters denote Lorentz indices: μ, ν, λ, ρ for vector fields and α, β for spinor fields. Latin letters denote internal indices: a, b, c, d, e for vector fields and "Stückelberg scalar" fields; k, l, m, n for physical fields; i, j for spinor fields.

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[19]The explicit form of $g_{pq}(\pi)$ can be found and used to express the condition $R^p_{qrs}(0)=0$ in terms of the masses and coupling constants of Eq. (4). The resulting relations are the same as the four-particle unitarity conditions, which were calculated directly in Refs. 4 and 5.

[20]The condition $V_{;p;q;r;s;t}(0)=0$ can be calculated explicitly in the π coordinate system. The resulting constraints on the masses and coupling constants of Eq. (4) are the same as the five-particle unitarity conditions derived in Ref. 4.

[21]Any real rectangular matrix can be "diagonalized" by multiplying it from the left and right by appropriate (possibly different) rotations.

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[25]This is the case discussed in Ref. 13.

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Erratum

Erratum: Derivation of gauge invariance from high-energy unitarity bounds on the S matrix [Phys. Rev. D 10, 1145 (1974)]

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The following reference was inadvertently omitted from Ref. 17: R. Finkelstein, Physica (Utr.) 44, 260 (1969).