though complicated numerically, appears to be a more attractive alternative.

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Generalized Renormalizable Gauge Formulation of Spontaneously Broken Gauge Theories

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The spontaneously broken gauge theory is formulated in the generalized renormalizable gauge (R_ξ gauge). A parameter ξ can be adjusted to include existing gauges, U gauge, R gauge, and 't Hooft-Feynman gauge as special cases. Three applications of the R_ξ-gauge formulation are given. First we compute the weak correction to the muon magnetic moment unambiguously in the existing models for leptons. Secondly, we discuss the large-momentum-transfer limit of the Pauli magnetic form factor of the muon. Finally, we discuss the static charge of the neutrino, and show that an appropriate regularization makes it vanish.

I. INTRODUCTION

The possibility of constructing a unified theory of weak and electromagnetic interactions in terms of a spontaneously broken gauge symmetry has attracted a great deal of attention lately, following the works of Weinberg[1] and 't Hooft.[2] In this paper we shall present a formulation of spontaneously broken gauge theories (SBGT) which is particularly suited for practical calculations. In this formulation the gauge condition one adopts is a generalization of the one used by 't Hooft and depends on a parameter ξ which can vary continuously from 0 to ∞. In this gauge, which we shall call generically the R_ξ gauge, the massive-vector-boson propagator is precisely the one invented by Lee and Yang in their discussion[3] of the ξ-limiting process:

$$\begin{aligned}\Delta_{\mu\nu}(p,\xi) &= -i\left[g_{\mu\nu} - \left(1-\frac{1}{\xi}\right)\frac{p_\mu p_\nu}{p^2-M^2/\xi}\right]\frac{1}{p^2-M^2}\\ &= -i\left(g_{\mu\nu}-\frac{1}{M^2}p_\mu p_\nu\right)\frac{1}{p^2-M^2}\\ &\quad -i\frac{1}{M^2}p_\mu p_\nu\frac{1}{p^2-M^2/\xi}. \end{aligned} \tag{1.1}$$

The difference between the R_ξ-gauge formulation of SBGT and the ξ-limiting process applied to the electrodynamics of massive vector bosons is this:

In the former, the negative-metric scalar-boson pole of the vector-boson propagator at $p^2 = M^2/\xi$ is canceled by the pole of the unphysical-scalar-boson propagator

$$\frac{i}{p^2 - M^2/\xi} \tag{1.2}$$

in the S matrix, and the S matrix of the former is independent of the parameter ξ and is unitary, whereas in the latter, one recovers the unitarity of the S matrix only in the limit $\xi \to 0$. The ξ independence of the S matrix[4] in the former is a direct consequence of the non-Abelian gauge invariance of the relevant Lagrangian.

It is worthwhile to note the connection between the R_ξ gauge and other gauges discussed in the literature.

(1) The R Gauge: In the proof of renormalizability of SBGT by Lee and Zinn-Justin,[5] and also in the discussion of Salam and Strathdee,[6] a generalization of the Landau gauge in quantum electrodynamics, the so-called R gauge, was used. The R gauge is obtained from the R_ξ gauge for $\xi = \infty$.

(2) The 't Hooft–Feynman Gauge: This gauge, which was discussed by 't Hooft, is obtained when we set $\xi = 1$. In this gauge the vector-boson propagator is proportional to $g_{\mu\nu}$, and the unphysical-scalar-boson propagator of Eq. (1.2) has a pole at $p^2 = M^2$.

(3) The U Gauge: In this formulation, the unphysical scalar bosons are absent and the vector-boson propagator is the canonical one:

$$\Delta_{\mu\nu}(p) = -i\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{M^2}\right)\frac{1}{p^2 - M^2}. \tag{1.3}$$

In this gauge, the unitarity of the S matrix is manifest since there are no spurious singularities at $p^2 = M^2/\xi$. However, Green's functions are unrenormalizable in this gauge: It is only the S matrix that can be defined in this gauge. The U-gauge is *formally* equivalent to the R_ξ gauge in the limit $\xi \to 0$. The equivalence here is "formal," in the sense that Feynman amplitudes in the two formulations are equal if the limit $\xi \to 0$ is taken before the Feynman integral is performed.

The U-gauge formulation of SBGT deserves some more discussion. Because the quantization of SBGT in this gauge is most straightforward, most of the existing calculations were performed in this formulation, despite the divergence difficulties unique to this gauge. The cancellation of divergences in the S matrix (but not in Green's functions) has been demonstrated by various authors in a number of cases.[7] However, isolation of the finite part of an S-matrix element in this gauge may prove ambiguous. In fact, Jackiw and Weinberg and Bars and Yoshimura[8] have commented on an ambiguity that exists in the calculation of the weak-interaction contribution to the anomalous magnetic moment of the muon. We claim that, based on our own experiences, computation of Feynman amplitudes is enormously simplified in the R_ξ gauge. It is also easier to check the ξ independence of the S matrix (thereby verifying the unitarity of the S matrix) in the R_ξ gauge, than to establish the cancellation of higher-order divergences. When there are ambiguities in fixing the finite part of an S-matrix element in the U gauge, the R_ξ-gauge formulation provides a gauge-invariant (with respect to the non-Abelian gauge group) way of circumventing such difficulties. In fact, we shall resolve the ambiguity in the computation of the anomalous magnetic moment of the muon by evaluating it in the R_ξ gauge. Our study explains also why the ξ-limiting process used by Jackiw and Weinberg and by Bars and Yoshimura yields the correct result.[9]

This paper is organized as follows: In Sec. II we formulate the generalized renormalizable gauge (R_ξ gauge). In Sec. III, we apply the R_ξ gauge to the calculation of weak correction to the magnetic moment of the muon. We will present unambiguous answers for three existing models of Weinberg,[1] of Georgi and Glashow,[10] and of Lee,[11] and Prentki and Zumino.[11] In Sec. IV, we show that the naive calculation of the neutrino static charge gives a nonvanishing result and we discuss how to remedy this situation. In Appendix A, we give details of Sec. III. In Appendix B we point out the reasons for ambiguities present in the U-gauge calculations of the weak correction to the muon magnetic moment. Finally, in Appendix C, we give the Lagrangians and necessary Feynman rules for our calculations.

After the completion of this paper, we received a paper by Yao[11a] in which a formulation similar to ours is discussed in the context of an *Abelian* gauge theory.

II. FORMULATION OF THE R_ξ GAUGE

In this section we shall discuss the formulation of SBGT in a general class of covariant linear gauge conditions. We shall consider, for definiteness, the Georgi-Glashow model based on the O(3) gauge symmetry without fermions. In Appendix C, we will extend our considerations of this section to all three models mentioned in the Introduction, with fermions.

In the absence of fermions, the Georgi-Glashow model consists of a triplet of gauge bosons and a triplet of scalar mesons. The Lagrangian is of the form

$$\mathcal{L} = -\tfrac{1}{4}(\partial_\mu \vec{B}_\nu - \partial_\nu \vec{B}_\mu + g\vec{B}_\mu \times \vec{B}_\nu)^2 + \tfrac{1}{2}[(\partial_\mu + g\vec{B}_\mu \times)\vec{\phi}]^2 - V(\vec{\phi}), \tag{2.1}$$

where $V(q)$ is an isospin-invariant quartic polynomial of the scalar fields $\vec{\phi}$. The potential is assumed to have an absolute minimum at $\vec{\phi} = \vec{v} \neq 0$. We can always choose the isospin z axis to coincide with the direction of $\vec{v}$.

It is convenient to define a unit vector $\hat{\eta}$ along the z axis:

$$\vec{v} = v\hat{\eta}.$$

We also define $\vec{\phi}_t$ and ϕ by

$$\vec{\phi} = \vec{\phi}_t + \hat{\eta}(v + \phi), \qquad \vec{\phi}_t \cdot \hat{\eta} = 0. \tag{2.2}$$

The gauge condition we shall adopt is (see also Appendix B)

$$\partial^\mu \vec{B}_\mu - \frac{1}{\xi} g v \hat{\eta} \times \vec{\phi} = 0, \tag{2.3}$$

where ξ is a non-negative real parameter.

It was shown by 't Hooft[12] that the gauge condition (2.3) may be taken care of by defining the effective action[13]:

$$S[\vec{B}_\mu, \vec{\phi}] = \int d^4x \left[\mathcal{L}(x) - \frac{\xi}{2}\left(\partial^\mu \vec{B}_\mu - \frac{1}{\xi} g \vec{v} \times \vec{\phi}\right)^2 - \frac{1}{2}\left(\frac{1}{\alpha} - \xi\right)(\partial^\mu B^3_\mu)^2 \right] - i\,\mathrm{Tr}\ln[1 + g\mathcal{G}\gamma], \tag{2.4}$$

where $\mathcal{G}$ is defined by

$$\left[(-\partial^2 + i\epsilon)\delta_{ab} - \frac{1}{\xi}(gv)^2(\delta_{ab} - \eta_a\eta_b)\right]\langle b, x|\mathcal{G}|c, y\rangle = \delta_{ac}\delta^4(x-y) \tag{2.5}$$

and γ is defined by

$$\langle a, x|\gamma|b, y\rangle = \left(\epsilon_{abc}\frac{\partial}{\partial x_\mu} B^c_\mu(x) + \frac{1}{\xi} g v [\phi_{ta}(x)\eta_b - \phi(\delta_{ab} - \eta_a\eta_b)]\right)\delta^4(x-y) \tag{2.6}$$

and Tr denotes the trace operation over the space-time variables x, y, as well as over the isospin indices a, b. In Eq. (2.4), ξ and α are in general arbitrary non-negative real numbers.

The effective action S of Eq. (2.4) is to be used in defining the generating functional of connected Green's functions. Thus, if we define

$$\exp(iZ[\vec{J}_\mu, \vec{k}]) = \int [d\vec{B}_\mu] \int [d\vec{\phi}] \exp\left\{ iS[\vec{B}_\mu, \vec{\phi}] + i\int d^4x [\vec{k}(x)\cdot\vec{\phi}(x) - \vec{J}_\mu(x)\cdot\vec{B}^\mu(x)]\right\}, \tag{2.7}$$

functional derivatives of Z at $\vec{J}_\mu = \vec{k} = 0$ give connected Green's functions of the theory. The generating functional Z depends on two parameters α and ξ. Note that the choice $\alpha^{-1} = \xi = \infty$ leads to the R gauge discussed in Refs. 5 and 6.

The effective Lagrangian can be written as

$$\begin{aligned}\mathcal{L} - \frac{\xi}{2}\left(\partial^\mu \vec{B}_\mu - \frac{1}{\xi} g\vec{v}\times\vec{\phi}\right)^2 - \frac{1}{2}\left(\frac{1}{\alpha} - \xi\right)(\partial^\mu B^3_\mu)^2 &= \frac{1}{2}[\partial_\mu\vec{\phi} + g\vec{B}_\mu \times (\vec{\phi}_t \times \phi\hat{\eta})]^2 - \frac{(gv)^2}{2\xi}\vec{\phi}_t^{\,2} \\ &\quad + g^2 v\hat{\eta}\times\vec{B}_\mu \cdot \vec{B}^\mu \times (\vec{\phi}_t + \phi\hat{\eta}) - V(\vec{\phi}) \\ &\quad - \tfrac{1}{4}(\partial_\mu\vec{B}_\nu - \partial_\nu\vec{B}_\mu + g\vec{B}_\mu\times\vec{B}_\nu)^2 + \frac{(gv)^2}{2}(\vec{B}^t_\mu)^2 \\ &\quad - \frac{\xi}{2}(\partial^\mu\vec{B}^t_\mu)^2 - \frac{1}{2\alpha}(\partial^\mu B^3_\mu)^2 .\end{aligned} \tag{2.8}$$

The terms proportional to $\vec{B}_\mu \cdot \partial^\mu \vec{\phi}$ have disappeared from Eq. (2.8). The propagators for various fields are obtained by inverting the matrix of the quadratic form $\mathcal{L}_0$ of the above expression:

$$\mathcal{L}_0 = \frac{1}{2}\left[(\partial_\mu\vec{\phi})^2 - \frac{M^2}{\xi}\vec{\phi}_t^{\,2} - \mu_\phi^{\,2}\phi^2\right] - \tfrac{1}{4}(\partial_\mu\vec{B}_\nu - \partial_\nu\vec{B}_\mu)^2 - \frac{\xi}{2}(\partial^\mu\vec{B}^t_\mu)^2 + \frac{M^2}{2}(\vec{B}^t_\mu)^2 - \frac{1}{2\alpha}(\partial^\mu B^3_\mu)^2, \tag{2.9}$$

where

$$M^2 = gv, \quad \mu_\phi^{\,2} = 2\partial V(v\hat{\eta})/\partial v^2,$$

and they are[14]

$$s^\pm \equiv \frac{1}{\sqrt{2}}(\phi^1 \pm i\phi^2):\ i\frac{1}{k^2 - M^2/\xi + i\epsilon},$$

$$\psi \equiv \phi^3 - v: \quad i\frac{1}{k^2 - \mu_\psi{}^2 + i\epsilon},$$

$$W_\mu \equiv \frac{1}{\sqrt{2}}(B^1_\mu \mp iB^2_\mu): \quad -i\left[g_{\mu\nu} - k_\mu k_\nu \frac{1}{k^2 - M^2/\xi}\left(1 - \frac{1}{\xi}\right)\right]\frac{1}{k^2 - M^2 + i\epsilon}, \tag{2.10}$$

and

$$A_\mu \equiv B^3_\mu: \quad -i\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}(1-\alpha)\right)\frac{1}{k^2 + i\epsilon}.$$

We see that our gauge interpolates between the R gauge ($\xi \to \infty$) and the U gauge ($\xi \to 0$). For $\xi = 1$, we recover the 't Hooft–Feynman gauge, in which the vector-boson propagators are proportional to $g_{\mu\nu}$.

In the Weinberg model (and also in the model of Lee and of Prentki and Zumino) we have another gauge boson Z_μ. We fix the gauge for this boson by adding the following term to the Lagrangian:

$$\mathcal{L}^{c'} = -\frac{\eta}{2}\left(\partial_\mu Z^\mu + \frac{Gv}{\eta}\chi\right)^2, \qquad G = (g^2 + g'^2)^{1/2} \tag{2.11}$$

where η is a parameter that can vary over the range

$$0 \leq \eta \tag{2.12}$$

and G and χ are the coupling constant of Z_μ and the corresponding unphysical neutral scalar boson, respectively. We must also modify the last term $-i\,\mathrm{Tr}\ln[1 + g\mathcal{G}\gamma]$ in Eq. (2.4) accordingly. The propagators for Z_μ and χ are given by

$$Z_\mu: \quad -i\frac{[g_{\mu\nu} - k_\mu k_\nu(1-\eta)/(M_Z{}^2 - \eta k^2)]}{k^2 - M_Z{}^2 + i\epsilon}, \tag{2.13}$$

$$\chi: \quad i\frac{1}{k^2 - (1/\eta)M_Z{}^2 + i\epsilon}, \tag{2.14}$$

where $M_Z \equiv Gv$.

In an S-matrix element, the pole at $k^2 = M^2/\xi$ of the vector-boson propagator is canceled by the similar pole at M^2/ξ of the $s^\pm$ propagator. Neither of the scalar particles implied by these poles are physical. (In the R-gauge formulation the $s^\pm$ are the would-be Goldstone fields.) In fact, the couplings of $s^\pm$ to other particles can be determined based on the above considerations. As an example, let us determine the coupling of s^- to the $e\nu$ pair. We write the coupling of $W^-_\mu [\equiv (1/\sqrt{2})(B^1_\mu + iB^2_\mu)]$ to the $e\nu$ pair as

$$\mathcal{L}_{We\nu} = g\bar{e}(1-\gamma_5)\nu W^-_\mu.$$

Now consider the T-matrix element for the process

$$e(p) + \nu(q) \to \nu(q') + e(p').$$

To lowest order, the W exchange gives

$$(ig)^2(-i)[\bar{e}(p')\gamma^\mu(1-\gamma_5)\nu(q)][\bar{\nu}(q')\gamma^\lambda(1-\gamma_5)e(p)]\left[\left(g^{\mu\lambda} - \frac{k^\mu k^\lambda}{M^2}\right)\frac{1}{k^2 - M^2} + \frac{k^\mu k^\lambda}{M^2}\frac{1}{k^2 - M^2/\xi}\right],$$

where $k = p' - q = p - q'$, and we have used the vector-boson propagator of Eq. (2.10). The pole term at $k^2 = M^2/\xi$,

$$\left(\frac{igm_e}{M}\right)^2(-i)[\bar{e}(p')(1-\gamma_5)\nu(q)][\bar{\nu}(q')(1+\gamma_5)e(p)]\frac{1}{k^2 - M^2/\xi}, \qquad M^2 = (g\nu)^2$$

must be canceled by the s^- exchange contribution. This requires the $s^- e\nu$ coupling to be

$$\mathcal{L}_{s^- e\nu} = \pm\frac{m_e}{v}\bar{e}(1-\gamma_5)\nu s^-.$$

(The sign ambiguity is superficial, since the sign of v is indeterminate. Once a definite sign convention is made here, all other couplings are uniquely determined.)

We note that the above cancellation is one of the consequences of the following two fundamental relations:

(i) The S matrix is gauge-independent, namely

$$\frac{\partial S}{\partial \xi} \equiv 0.$$

(ii) The propagator $D_{\mu\nu}$ for W_μ and the propagator D for $s^\pm$ satisfy the identity

$$\frac{\partial}{\partial\xi}D_{\mu\nu}(k) \equiv -\frac{k_\mu k_\nu}{M^2}\frac{\partial}{\partial\xi}D(k).$$

III. MAGNETIC FORM FACTOR OF THE MUON

A. Weak Correction to the Magnetic Moment of the Muon

In a unified theory of weak and electromagnetic interactions, one-loop contributions to the anomalous magnetic moment a_μ of the muon are formally of order α, whether they derive from photon exchange or weak-vector-boson exchanges. The anomalous magnetic moment to this order can be written as

$$a_\mu = \frac{\alpha}{2\pi}\left[1+\left(\frac{\mu}{M}\right)^2 f\left(\left(\frac{\mu}{M}\right),\left(\frac{m_{Y^0}}{\mu}\right)\right)\right], \tag{3.1}$$

where μ is the muon mass, M is the W-boson mass, f is a function of the mass ratios (μ/M) and (m_{Y^0}/μ), and m_{Y^0} is the mass of a neutral heavy lepton Y^0, that might exist in such a theory. The second term in Eq. (3.1) is *in magnitude* of order $(\alpha/M^2)\mu^2 \sim G_F\mu^2$ and we shall call it the weak correction to a_μ and denote it by

$$a_\mu^W = \frac{\alpha}{2\pi}\left(\frac{\mu}{M}\right)^2 f\left(\left(\frac{\mu}{M}\right),\left(\frac{m_{Y^0}}{\mu}\right)\right). \tag{3.2}$$

In addition, there are contributions of massive Higgs scalar bosons in such a theory to a_μ. However, they are of order $(\mu/m_\phi)^2$ compared to Eq. (3.2), where m_ϕ is a typical Higgs scalar mass, and since the masses of these scalars are presumably very large, we shall ignore them in the following discussion.

The weak correction to a_μ^W has previously been computed by several authors[8] in the U gauge. In this gauge, the electromagnetic vertex of the muon is quadratically divergent, so that its separation into the electric and anomalous (Pauli) magnetic form factors is ambiguous. As a consequence, one finds that a_μ^W computed in this gauge depends on the way the internal momentum is routed in a diagram, even though it is finite.

In the R_ξ gauge, the electromagnetic vertex of the muon is only logarithmically divergent, and there is no such ambiguity in evaluating a_μ^W. In order to verify the gauge invariance we have evaluated it in three different gauges: $\xi=\infty$, 1, and 0. In this section we will present the results of these calculations. In Appendix A, we will present a general proof that the value of a_μ^W is independent of ξ. In the following we shall refer to the result obtained in the limit $\xi\to 0$ after the Feynman integration as the U-gauge result. For those diagrams involving unphysical scalars in this particular case, the limit $\xi\to 0$ and the integration commute. This explains why the procedure used by Jackiw and Weinberg[8] and by Bars and Yoshimura,[8] of replacing the vector propagator (1.3) by the regularized one, (1.1), yielded the correct result, even though this replacement *per se* is not a gauge-invariant procedure.

Our results are given for three different models. These are the model of Weinberg[9] based on SU(2)×U(1), that of Georgi and Glashow[10] based on O(3), and that of Lee and Prentki and Zumino (LPZ) based on O(3) ×O(2). The diagrams shown in Figs. 1 and 2 contribute to the models given by Weinberg and LPZ, and the diagrams shown in Figs. 1 and 3 contribute to the model of Georgi and Glashow. In these figures s^- and χ are unphysical scalars and Y^0 is a neutral lepton. For the purpose of illustration let us evaluate the diagram shown in Fig. 2b. It gives the contribution

$$-e\frac{\mu^2}{M_Z^2}\frac{(g^2+g'^2)}{4}\int\frac{d^4k}{(2\pi)^4}\frac{\overline{\mu}(p')\gamma_5(\not{p}'-\not{k}+\mu)\gamma_\mu(\not{p}-\not{k}+\mu)\gamma_5\mu(p)}{[(p-k)^2-\mu^2][(p'-k)^2-\mu^2][k^2-M_Z^2/\eta]}, \tag{3.3}$$

where p and p' are the incident and final muon momenta, respectively, and k is the internal momentum of χ. We separate this expression into the charge and anomalous magnetic form factors:

$$-i\epsilon\overline{\mu}(p')\left[F_1(q^2)\gamma_\mu + F_2(q^2)\frac{i}{2\mu}\sigma_{\mu\nu}q^\nu + \text{parity-violating terms}\right]\mu(p), \quad q=p'-p. \tag{3.4}$$

After the k integration $F_2(q^2)$ of Eq. (3.4) is found to be

$$F_2(q^2) = -\left(\frac{g\mu^2}{4M^2}\right)\frac{\mu^2}{8\pi^2}\int_0^1 dx\int_0^{1-x} dy\,\frac{(x+y)^2}{\mu^2(x+y)^2 - q^2xy + (M_Z^2/\eta)(1-x-y)}. \tag{3.5}$$

Integrations over x and y yield the desired result in the gauge characterized by η. For example, to obtain the R-gauge result, we take the limit $\eta\to\infty$. Then

$$a_\mu^w[\text{Fig. 2(b)}]_R=-\frac{G_F\mu^2}{8\pi^2\sqrt{2}}. \tag{3.6}$$

To get from Eq. (3.3) to Eq. (3.6) we have used the relations

$$g^2+g'^2=g^2\frac{M_Z{}^2}{M^2}\quad\text{and}\quad\frac{g^2}{8M^2}=G_F/\sqrt{2}.$$

To obtain U-gauge result, we let $\eta\to 0$ and we find that the diagram does not contribute:

$$a_\mu^w[\text{Fig. 2(b)}]_U=0. \tag{3.7}$$

Finally in a 't Hooft-Feynman gauge, we let $\eta=1$ and we see that

$$a_\mu^w[\text{Fig. 2(b)}]_{\text{'t H-F}}=O(\mu^2/M_Z{}^2). \tag{3.8}$$

Details of all other diagrams can be found in Appendix A. Tables I, II, and III give contributions from Figs. 1, 2, and 3, respectively. For example, the contribution of Fig. 2(b) can be found in Table II, in the column labeled Fig. 2(b). We neglected terms of order μ/M. In Table III, terms of order m_{Y^0}/M, μ/m_{Y^0} were also neglected. It is amusing to see that individual diagrams are quite gauge-dependent, but, as they must, the diagrams always add up to give a gauge-independent result not only to the leading order in $(\mu/M)^2$, $(\mu/m_Y)^2$, but to all orders. (See Appendix A.)

To obtain the result of Weinberg's model, we add the results of Tables I and II:

$$a_\mu^w=\frac{G_F\mu^2}{8\pi^2\sqrt{2}}\{\tfrac{10}{3}+\tfrac{1}{3}[(3-4\cos^2\theta)^2-5]\}. \tag{3.9}$$

Note that this result is the same as that obtained previously in the U gauge by the ξ-limiting regularization.[8]

The result for the model proposed by LPZ can be obtained from that of Weinberg's model by merely changing the definition for the coupling constants. We obtain

$$a_\mu^w=\frac{G_F\mu^2}{8\pi^2\sqrt{2}}\left(\frac{10}{3}-\frac{4}{3}\frac{u^2+v^2}{u^2}(1+\sin^2\theta\cos^2\theta)\right), \tag{3.10}$$

where θ is related to the physical quantities by

$$\frac{G_F}{\sqrt{2}}=\frac{e^2}{4M^2\sin^2\theta}\quad\text{and}\quad\frac{u^2+v^2}{u^2}=2\frac{M^2}{M_Z{}^2}(1+\tan^2\theta). \tag{3.11}$$

u and v are the vacuum expectation values of Higgs scalars in this model. In Eq. (3.10), a_μ^w can be about the same order of magnitude as that of Eq. (3.9) if $(M^2/M_Z{}^2)\tan^2\theta\sim O(1)$. In that case both of these models predict the weak correction to the muon magnetic moment to be of order 10^{-8}, below the experimental detectability. In any case, the strong-interaction correction a_μ^s has been estimated from the colliding-beam experiment to be of order $(6.5\pm0.5)\times10^{-8}$,[15] so that the latter seems to be bigger than the former.

The Georgi-Glashow model receives contributions from Figs. 1 and 2. The results of the calculation in the special case $m_{Y^0}\ll M$ are given in Table III and Table I. Note that Fig. 3 gives contributions which are much larger than the previous two models by the factor m_{Y^+}/μ. We will thus concentrate our attention on the contributions of order $G_F\mu m_{Y^0}$ in Fig. 3. Evaluating for arbitrary value of m_{Y^0}/M we obtain

$$a_\mu^w=-\frac{G_F(m_{Y^+}+\mu)\mu}{8\sqrt{2}\pi^2\sin^2\theta}\left[1+\frac{6}{(1-y)^3}(\tfrac{1}{2}-2y+\tfrac{3}{2}y^2-y^2\ln y)\right], \tag{3.12}$$

where $y=m_{Y^0}{}^2/M^2$ and $\sin^2\theta=4G_FM^2/\sqrt{2}e^2$. Note also the relation

$$2m_{Y^0}\cos\theta=m_{Y^+}+\mu.$$

In Fig. 4 we have plotted the prediction of Eq. (3.12). Two sets of curves correspond to contours of constant a_μ^w and m_{Y^0} at various values of m_{Y^+} and M. The present status on experimental and theoretical uncertainties in a_μ places the limit[15]

$$2\times10^{-8}\leq a_\mu^w+a_\mu^s\leq6\times10^{-7}, \tag{3.13}$$

where a_μ^s corresponds to the hadronic correction to the muon magnetic moment. Using the value for a_μ^s quoted above, it seems quite safe to guess that

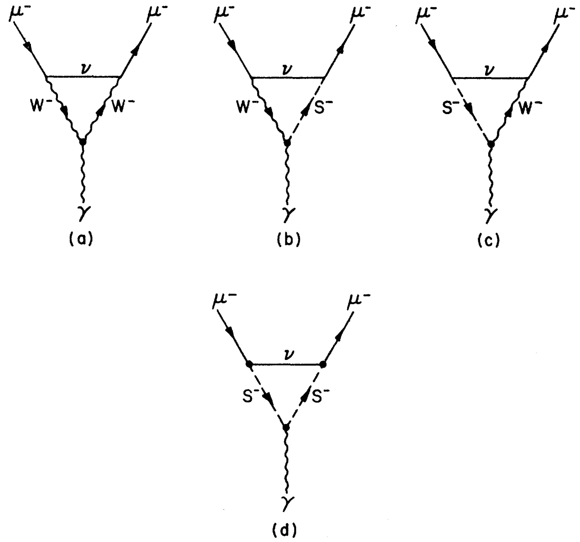

FIG. 1. W-boson and scalar-boson contributions to weak correction to the muon magnetic moment. These give contributions in all three models.

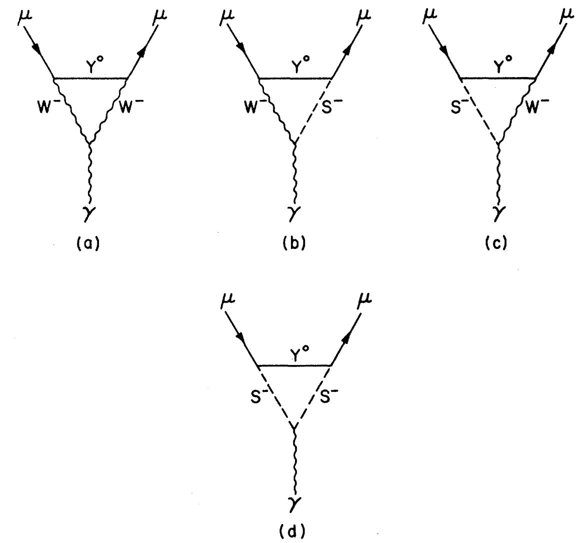

FIG. 3. Y^0-lepton contribution to the weak correction to the muon magnetic moment. These diagrams are for the Georgi-Glashow model.

$$-2\times10^{-7} \leqslant a_\mu^w \leqslant 0\,.$$

If we further demand, for example, that $m_{Y^+} \geqslant 0.5$ GeV, then Fig. 4 readily gives the allowed range of m_{Y^0} and M. The generous lower limit $-2\times10^{-7} = a_\mu^w$ gives $m_{Y^+} \leqslant 5$ GeV. If we use $|a_\mu^w| \leqslant +1.1 \times 10^{-7}$, we get $m_{Y^+} \leqslant 2$ GeV. In these estimates we assume $m_{Y^+} m_\mu / M_\Psi{}^2 \ll 1$, where Ψ is the physical scalar boson. A charged heavy lepton of this mass range can be detected in the near future. A pair of Y^+ and Y^- can be produced in reactions such as

$$\gamma + (Z) \to Y^+ + Y^- + (Z)$$

or

$$e^+ + e^- \to Y^+ + Y^-\,.$$

The detection of coincident $e^-\mu^+$ from the decays

$$Y^- \to \bar\nu_\mu + \nu_e + e^-\,,$$

$$Y^+ \to \bar\nu_\mu + \nu_\mu + \mu^+$$

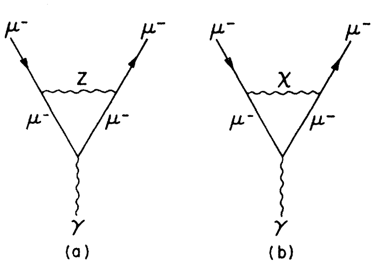

FIG. 2. Z- and χ-boson contribution to the weak correction to the muon magnetic moment. These diagrams are for Weinberg's model. Similar sets of diagrams exist for the LPZ model.

is a signature of the Y^+Y^- pair production.

It is important to recognize that the a_μ^w in Eq. (3.12) does not vanish even in the limit $m_{Y^0} \to \infty$. If one performs a naive U-gauge calculation the first term in Eq. (3.12) is absent and $a_\mu^w \to 0$ for $m_{Y^0} \to \infty$ (i.e., we can make a_μ^w arbitrarily small by letting m_{Y^0} be large).

B. Asymptotic Behavior of $F_2(q^2)$

In this subsection we discuss the behavior of the Pauli magnetic form factor $F_2(q^2)$ as $|q^2| \to \infty$. We caution the reader that $F_2(q^2)$ for $q^2 \neq 0$ is not an on-shell S-matrix element (i.e., not measurable) and is not invariant under non-Abelian gauge group (i.e., depends on the gauge). Figure 5(a) gives a process in which $F_2(q^2)$ is relevant. But Fig. 5(b) is of the same order. We obtain the gauge-invariant answer only when Figs. 5(a) and 5(b) and all other diagrams of the same order are added. $F_2(q^2=0) = a_\mu^w$ is available for experimental measurements only because of the pole due to the photon propagator. Still, the knowledge of the asymptotic behavior of $F_2(q^2)$, though gauge-dependent, is important in the question of renormalizability when diagrams of the type Fig. 5(a) are inserted in more complicated diagrams.

Our conclusion is that $F_2(q^2) \to 0$ as $|q^2| \to \infty$ in all gauges except the U gauge, i.e., for all combinations $\eta \neq 0$, $\xi \neq 0$. This can be easily seen, at least for off-mass-shell muons, as follows: For $\eta \neq 0$, $\xi \neq 0$, the triangle diagram that we consider has the degree of divergence at most zero. Thus due to the

TABLE I. Contributions of diagrams shown in Fig. 1. To obtain the answer these numbers should be multiplied by $G_F\mu^2/8\pi^2\sqrt{2}$.

Gauge \ Diagram	Fig. 1(a)	Fig. 1(b) +1(c)	Fig. 1(d)	Total
U gauge	$\frac{10}{3}$	0	0	$\frac{10}{3}$
't Hooft-Feynman gauge	$\frac{7}{3}$	1	0	$\frac{10}{3}$
R gauge	$\frac{4}{3}$	1	1	$\frac{10}{3}$

kinematical factor, $\sigma_{\mu\nu}q_\nu$, the integral for $F_2(q^2)$ has the degree of divergence −1. Therefore, by Weinberg's theorem,[16] $F_2(q^2) \leq O(1/\sqrt{q^2})$.

We have also done the calculation for the on-mass-shell muon amplitude and verified that $F(q^2) \to 0$ as $|q^2| \to \infty$ in all gauges except the U gauge in the Weinberg model. In order to obtain the result for the U gauge, we let $\xi \to 0$ and then let $q^2 \to -\infty$. The result is (for the Weinberg model)

$$F_2(q^2) \to \frac{G_F\mu^2}{\sqrt{2}8\pi^2}\left(4 - \frac{\mu^2}{M^2}\right)\ln(-q^2) + \text{constant for } q^2 \to -\infty .$$

These results indicate the gauge dependence of the off-shell amplitude. In particular, for the renormalizable gauge (i.e., for $\xi \neq 0$), $F_2(q^2)$ shows a manifestly renormalizable behavior. On the other hand, $F(q^2)$ in the U gauge exhibits a divergent behavior at $q^2 = -\infty$. As remarked in Ref. 17, the logarithmic growth of $F_2(q^2)$ for large q^2 does not necessarily imply any trouble with S-matrix elements for physical processes. When all diagrams of the same order for a physical process are added the bad behavior of $F_2(q^2)$ can be canceled by those of other diagrams.

The dispersion relation for F_2 in the U gauge requires a subtraction (which cannot be determined *a priori*) while its absorptive part may be computed by the standard Landau-Cutkosky rule. On the other hand, F_2 in the R_ξ gauge has an absorptive contribution from unphysical states, while it requires no unknown subtraction.

Note added. After the completion of this paper, we received a preprint of Bardeen *et al.*,[18] in which they evaluate a_μ in the Weinberg model using the n regularization method of 't Hooft and Veltman.[19] Their answer agrees with ours. Quinn and Primack[20] have computed a_μ for the Georgi-Glashow model. We appreciate Professor Quinn's explaining their result to us.

IV. STATIC CHARGE OF THE NEUTRINO

As in the case of Pauli magnetic form factor discussed in Sec. III, the notion of the electric form factor of a (muon) neutrino is an unphysical one in the present theory: The electric form factor, for nonzero momentum transfer, is not an element of the S matrix and, in the present theory, is neither gauge-invariant nor unitary for arbitrary ξ. However, the electric charge, i.e., the value of the electric form factor F_1 at zero momentum transfer, is an element of the S matrix and measurable. It must be zero if due care is exercised in evaluating Feynman integrals.

In the Georgi-Glashow model, there are altogether 10 diagrams contributing to the electric charge

TABLE II. Contributions of diagrams shown in Fig. 2. To obtain the answer these numbers should be multiplied by $G_F\mu^2/8\pi^2\sqrt{2}$.

Gauge \ Diagram	Fig. 2(a)	Fig. 2(b)	Total
U gauge	$\frac{1}{3}[(3-4\cos^2\theta)^2-5]$	0	$\frac{1}{3}[(3-4\cos^2\theta)^2-5]$
't Hooft-Feynman gauge	$\frac{1}{3}[(3-4\cos^2\theta)^2-5]$	0	$\frac{1}{3}[(3-4\cos^2\theta)^2-5]$
R gauge	$\frac{1}{3}[(3-4\cos^2\theta)^2-5]$ +1	−1	$\frac{1}{3}[(3-4\cos^2\theta)^2-5]$

TABLE III. Contributions of diagrams shown in Fig. 3. To obtain the answer these numbers should be multiplied by $G_F m_{Y^+} \mu/2\pi^2\sqrt{2}\sin^2\theta$.

Gauge \ Diagram	Fig. 3(a)	Fig. 3(b) + 3(c)	Fig. 3(d)	Total
U gauge	-1	0	0	-1
't Hooft-Feynman gauge	$-\frac{3}{4}$	$-\frac{1}{4}$	0	-1
R gauge	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{4}$	-1

of the neutrino. In Fig. 6, we show five of them which involve internal muon lines. The other five are similar and involve internal Y^+ lines. We shall evaluate the Feynman integrals in the R gauge ($\xi=\infty$) for convenience. (We have also checked the ξ independence of our results.) The contribution of each of the five diagrams in Fig. 6 is as follows.

$$\text{(a):}\quad F_1^{(a)}(0) = -i\frac{3}{2}\left(\frac{g}{2}\right)^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-M^2)^2} -i\frac{3}{2}\left(\frac{g}{2}\right)^2 \int \frac{d^4k}{(2\pi)^4}\frac{\mu^2}{k^2-\mu^2}\frac{1}{(k^2-M^2)^2},$$

$$\text{(b):}\quad F_1^{(b)}(0) = i\frac{3}{2}\left(\frac{g}{2}\right)^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{\mu^2}{k^2-\mu^2}\frac{1}{k^2-M^2},$$

$$\text{(c):}\quad F_1^{(c)}(0) = -i\frac{1}{2}\left(\frac{f_\mu}{2}\right)^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{1}{k^2-\mu^2}, \qquad (4.1)$$

$$\text{(d):}\quad F_1^{(d)}(0) = -i\frac{3}{2}\left(\frac{g}{2}\right)^2 \int \frac{d^4k}{(2\pi)^4}\frac{\mu^2}{(k^2-M^2)(k^2-\mu^2)^2},$$

$$\text{(e):}\quad F_1^{(e)}(0) = -i\frac{1}{2}\left(\frac{f_\mu}{2}\right)^2 \int \frac{d^4k}{(2\pi)^4}\frac{k^2-2\mu^2}{k^2(k^2-\mu^2)^2},$$

where μ and M are the masses of the muon and the W boson, and f_μ is the coupling constant of the scalar meson to the muon and neutrino:

$$f_\mu = g\mu/M\,.$$

In Eq. (4.1) we have written $F_1^{(a)}(0)$ as a sum of logarithmically divergent and convergent integrals. A simple computation shows that the sum of the second term of $F_1^{(a)}$, $F_1^{(b)}$, and $F_1^{(d)}$ is zero after the integration, so that *if* the sum $F_1^{(c)}+F_1^{(e)}$ vanishes, then the muon contribution to the electric charge of the neutrino is

$$-\frac{3}{2}\left(\frac{g}{2}\right)^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-M^2)^2}$$

and is independent of the muon mass. The Y^+ contribution to the electric charge of the neutrino is exactly opposite to the above-mentioned μ contribution, so the net charge of the neutrino is zero as it should be.

Thus, the matter hinges entirely on whether the sum $F_1^{(c)}+F_1^{(e)}$ is zero. A naive evaluation of these

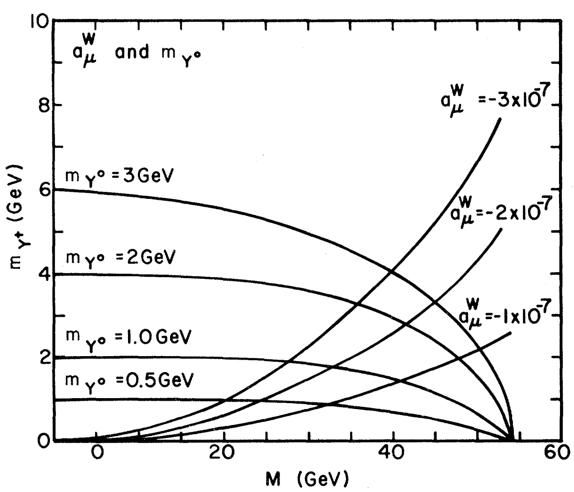

FIG. 4. Predictions of Eq. (3.12). Sets of contours correspond to constant a_μ^w and constant m_{Y^0} on the (m_{Y^+}, M) plane. If we take $-2\times10^{-7} \le a_\mu^w \le 0$, for example, the experimentally allowed region lies below the line labeled -2×10^{-7}. If we further take $m_{Y^+} \ge 0.5$ GeV, the allowed region is bounded from below. The upper bound for m_{Y^+}, in any case, is approximately 5 GeV.

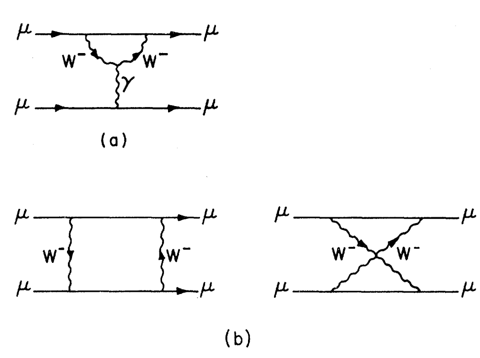

FIG. 5. Example of diagrams contributing to muon elastic scattering.

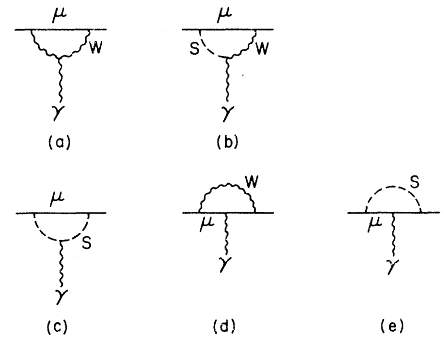

FIG. 6. Diagrams which contribute to the static charge of the neutrino in the Georgi-Glashow model.

two terms gives

$$F_1^{(c)}+F_1^{(e)}=\frac{f_\mu{}^2}{8}I,$$

$$I=i\int\frac{d^4k}{(2\pi)^4}\left[\frac{k^2-2\mu^2}{k^2(k^2-\mu^2)^2}-\frac{1}{k^2(k^2-\mu^2)}\right] \tag{4.2}$$

$$=-\frac{1}{8\pi^2},$$

which is not zero; it is significant that the value of I is independent of the muon mass μ. Note that $F_1^{(c)}+F_1^{(e)}$ cannot be canceled by the similar contribution of Y^+, since the latter is proportional to $(f_{Y^+})^2$, f_{Y^+} being the coupling constant of the scalar meson to the muon-neutrino pair. That is, a naive evaluation of Feynman integrals leads to a nonzero electric charge of the neutrino.

The above paradox has nothing to do with the non-Abelian gauge invariance of the theory or the massless nature of the neutrino. The offending diagrams, Figs. 5(c) and 5(e), are characteristic of a theory in which fermions are coupled to a scalar meson. The sum of the two diagrams shown in Fig. 7 may be written as

$$F_\mu\sim-\frac{\partial}{\partial q^\mu}\int\frac{d^4k}{(2\pi)^4}\gamma_5\left[\frac{1}{(\not q+\not k-m_p)(k^2-m_\pi{}^2)}-\frac{1}{(\not k-m_p)[(k-q)^2-m_\pi{}^2]}\right]\gamma_5, \tag{4.3}$$

and if we can shift the contour of integration $k\to k+q$ in the second term of the integrand, the integral vanishes identically. The integral is, however, linearly divergent, so that the change of the variable of integration is legitimate only after the integral is suitably regularized in a gauge-invariant manner. A simple regularization scheme is to replace the pion propagators in (4.3) by

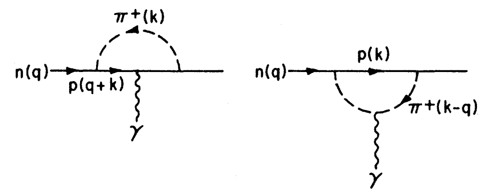

FIG. 7. Diagrams contributing to the neutron static charge.

$$\frac{1}{k^2-m_\pi{}^2}\to\frac{1}{k^2-m_\pi{}^2}-\frac{1}{k^2-\Lambda_0{}^2},$$
$$\frac{1}{(k-q)^2-m_\pi{}^2}\to\frac{1}{(k-q)^2-m_\pi{}^2}-\frac{1}{(k-q)^2-\Lambda_0{}^2}. \tag{4.4}$$

In this case, the naive evaluation of the charge of the neutron gives a result independent of the mass-$m_\pi{}^2$ internal-pion lines so that the regularization implied by Eq. (4.4) yields to a zero neutron charge. The result here is counter to the folklore which says that convergent Feynman integrals need not be regulated: If we perform the differentiation with respect to q prior to integration in Eq. (4.3), as one would to recover the original Feynman integral, then the integral becomes convergent and the conventional wisdom would say that it is not necessary to regulate the integral. What we have learned is that to *keep the charge of a neutral fermion zero, it is necessary to regularize Feynman integrals in a gauge-invariant way, even if the integrals are convergent.*

Let us return now to our problem. We can regularize the scalar-meson line in a gauge-invariant manner as in the σ model: We insert in the Lagrangian the regulator term

$$-\tfrac{1}{2}[(D_\mu\vec\phi')^2-\Lambda_0{}^2\phi'^2]=-\tfrac{1}{2}\{[(\partial_\mu+g\vec B_\mu\times)\vec\phi']^2-\Lambda_0{}^2\vec\phi'^2\}$$

and replace the untranslated scalar fields ϕ by the sum $\phi+\phi'$ in all interaction terms.[21,22] This modification of the Lagrangian is clearly gauge-invariant (with respect to the non-Abelian gauge group), and the integral I in Eq. (4.2) is now regulated to read

$$I_{\rm reg}=i\int\frac{d^4k}{(2\pi)^4}\left[\frac{k^2-2\mu^2}{(k^2-\mu^2)^2}\left(\frac{1}{k^2}-\frac{1}{k^2-\Lambda_0^2}\right)-\frac{k^2}{k^2-\mu^2}\left(\frac{1}{(k^2)^2}-\frac{1}{(k^2-\Lambda_0{}^2)^2}\right)\right],$$

which is zero for all values of $\Lambda_0{}^2$. Thus, a gauge-invariant regularization of the Feynman integral does give the physically correct result $F_1^{(c)}+F_1^{(e)}=0$. We remind the reader that $F_1^{(c)}+F_1^{(e)}$ is non-Abelian gauge-invariant by itself. Thus the regularization procedure stated above is sufficient to remove the neutrino static charge for arbitrary gauge.

V. CONCLUSION

We have given a formulation of the convenient gauge for actual applications of SBGT (R_ξ gauge). Based on this formulation, we have verified the gauge independence of several simple S-matrix elements. This indicates that the ghost-eliminating mechanism is indeed working in examples we have considered. It is important to show that the gauge-independence properties of the S matrix are preserved at every stage of the renormalization program.

In our formulation, the finite part of the S matrix is uniquely determined. Results of our calculation of weak correction to the muon magnetic moment agree with U-gauge calculations with the ξ-limiting regularization procedure. An experimental implication of our results is that the charged heavy lepton in the Georgi-Glashow model is required to be small (of the order of 0.5–5 GeV). It is therefore worthwhile searching for this lepton in the existing accelerators.

A naive calculation of the neutrino static charge in SBGT gives a nonvanishing result. This difficulty, however, also exists in any theory with neutral spinor fields. A prescription to remove the static charge in a manifestly gauge-invariant (non-Abelian) manner was given. To check the self-consistency of SBGT, it is desirable to evaluate other lower-order diagrams and to confirm the absence of any other unexpected "anomalous" behaviors.

APPENDIX A: FORMULAS FOR THE MUON MAGNETIC FORM FACTOR AND PROOF OF THE GAUGE INDEPENDENCE OF THE g_μ FACTOR

In this appendix we give general formulas for the weak corrections to the anomalous magnetic moment of the muon based on our R_ξ gauge in Sec. II. We calculate the neutral-vector-meson contribution in the Weinberg model and the massive-neutral-lepton contribution in the Georgi-Glashow model. The Feynman rules are given in Appendix C. From these two results one can easily derive the magnetic moment for other schemes of lepton interactions.

1. Neutral Vector Meson in the Weinberg Model

We have two diagrams shown in Figs. 2(a) and 2(b) for the neutral vector- and scalar-meson contribution in the Weinberg model. Figure 2(b) has been discussed in Sec. III [see Eq. (3.5)].

For Fig. 2(a) we have

$$-e\bar{\mu}(p')\left\{\gamma_\beta\left[a\left(\frac{1+\gamma_5}{2}\right)+b\left(\frac{1-\gamma_5}{2}\right)\right](\not{p}'+\mu)\gamma_\mu(\not{p}+\mu)\gamma_\alpha\left[a\left(\frac{1+\gamma_5}{2}\right)+b\left(\frac{1-\gamma_5}{2}\right)\right]\right\}\mu(p)\,\frac{P^{\alpha\beta}(k)}{k^2-M_Z{}^2}\,, \tag{A1}$$

where

$$a=\frac{g'^2}{(g^2+g'^2)^{1/2}}\,,\qquad b=\frac{1}{2}\,\frac{g'^2-g^2}{(g^2+g'^2)^{1/2}}\,, \tag{A2}$$

$$P^{\alpha\beta}(k)=g^{\alpha\beta}-\frac{k^\alpha k^\beta(1-\eta)}{M_Z{}^2-\eta k^2}\,. \tag{A3}$$

The result is given by

$$\begin{aligned}F(q^2)=&-\frac{\mu^2}{8\pi^2}(a^2+b^2)\int dx\,dy\,\frac{(x+y-2)(x+y-1)}{Q^2+M_Z{}^2(1-x-y)}\\&-\frac{4\mu^2}{8\pi^2}ab\int dx\,dy\,\frac{(x+y-1)}{Q^2+M_Z{}^2(1-x-y)}\\&+\frac{\mu^2}{8\pi^2}\,\frac{(a-b)^2}{2M_Z{}^2}\left\{\int dx\,dy[3(x+y)-2]\ln\left[\frac{Q^2+\Lambda(1-x-y)}{Q^2+M_Z{}^2(1-x-y)}\right]\right.\\&\left.-\int dx\,dy[Q^2(x+y)+2q^2xy]\left[\frac{1}{Q^2+M_Z{}^2(1-x-y)}-\frac{1}{Q^2+\Lambda(1-x-y)}\right]\right\},\end{aligned} \tag{A4}$$

where

$$Q^2=\mu^2(x+y)^2-q^2xy\,,\qquad \Lambda=M_Z{}^2/\eta\,.$$

Note the relation

$$\frac{(a-b)^2}{M_Z{}^2}=\frac{1}{4}\,\frac{g^2}{M^2}\,, \tag{A5}$$

with M the mass of the W boson.

Based on Eqs. (3.5), (A4), and (A5) we prove the gauge-independence condition for the *physical* S-matrix element, $(\partial/\partial\eta)F_2(0)\equiv 0$ or equivalently

$$\frac{\partial}{\partial\Lambda}F_2(0)\equiv 0\,. \tag{A6}$$

Equation (A6) demands the following relation:

$$\int_0^1 dt\,t\int_{-1}^1 dz\left\{(3t-2)\frac{(1-t)}{\mu^2t^2+\Lambda(1-t)}-\frac{\mu^2t^3(1-t)}{[\mu^2t^2+\Lambda(1-t)]^2}+\frac{2\mu^2t^2(1-t)}{[\mu^2t^2+\Lambda(1-t)]^2}\right\}\equiv 0\,, \tag{A7}$$

where we changed the Feynman variables to

$$t=(x+y),\qquad z=(x-y)/t\,. \tag{A8}$$

Equation (A7) is indeed satisfied if one notices the relation

$$\frac{\partial}{\partial t}\left[\frac{1-t}{\mu^2t^2+\Lambda(1-t)}\right]=-\mu^2\frac{2t-t^2}{[\mu^2t^2+\Lambda(1-t)]^2}\,. \tag{A9}$$

Therefore the sum of the neutral vector and scalar meson contributions is gauge-independent, and consequently it is free of ghost contributions. Equation (A6) allows us to use any gauge we want to calculate the matrix element. In particular if one takes the limit $\eta\to 0$ in Eq. (A4) one recovers the results based on the ξ-limiting procedure (scalar-meson contribution vanishes in this limit).

In passing we note that the Higgs neutral scalar meson (which is independent of the gauge) in the Weinberg model (see Fig. 7) gives the magnetic form factor

$$-\frac{\mu^2}{8\pi^2}\left(\frac{g^2\mu^2}{4M^2}\right)\int dx\,dy\,\frac{(x+y)(x+y-2)}{\mu^2(x+y)^2-q^2(xy)+M_\psi{}^2(1-x-y)}\,. \tag{A10}$$

This gives a small contribution to $F_2(0)$ for $(\mu^2/M_\psi{}^2)\ll 1$.

2. Neutral Massive Lepton in the Georgi-Glashow Model

The neutral massive lepton in the Georgi-Glashow scheme contains four diagrams; see Figs. 3(a)–3(d). Figure 3(d) gives

$$\frac{e^3}{2M^2}[(\mu-m\cos\theta)^2+(\mu\cos\theta-m)^2]\bar{\mu}(p')\not{k}\mu(p)\frac{(l+l')_\mu}{(k^2-m^2)(l^2-\Lambda)(l'^2-\Lambda)}$$
$$+\frac{e^3}{M^2}(\mu-m\cos\theta)(\mu\cos\theta-m)\bar{\mu}(p')\not{k}\mu(p)\frac{(l+l')_\mu}{(k^2-m^2)(l^2-\Lambda)(l'^2-\Lambda)}\,, \tag{A11}$$

and the result is

$$F(q^2)=-\frac{e^2m\mu}{8\pi^2M^2}[(\mu-m\cos\theta)(\mu\cos\theta-m)]\int dx\,dy\frac{(1-x-y)}{f(\Lambda,\Lambda)}$$
$$-\frac{e^2\mu^2}{16\pi^2M^2}[(\mu-m\cos\theta)^2+(\mu\cos\theta-m)^2]\int dx\,dy\,\frac{(x+y)(1-x-y)}{f(\Lambda,\Lambda)}\,, \tag{A12}$$

where we defined

$$\begin{aligned}&\Lambda=M^2/\xi\,,\\&m=\text{mass of the neutral massive lepton }Y^0\,,\\&f(a,b)=Q^2+x(a-\mu^2)+y(b-\mu^2)+(1-x-y)m^2\,,\\&Q^2=\mu^2(x+y)^2-q^2xy\,.\end{aligned} \tag{A13}$$

Note that $F(q^2)$ in Eq. (A12) vanishes in the U-gauge limit, $\xi\to 0$ (or $\Lambda\to\infty$).

Figures 3(b) and 3(c) give rise to

$$-\frac{e^3}{2}[\mu(1+\cos^2\theta)-2m\cos\theta]\left\{\bar{\mu}(p')\not{k}\gamma_\alpha\mu(p)\frac{P_\mu^\alpha(l)}{(k^2-m^2)(l'^2-\Lambda)(l^2-\Lambda)}+\bar{\mu}(p')\gamma_\beta\not{k}\mu(p)\frac{P_\mu^\beta(l')}{(k^2-m^2)(l^2-\Lambda)(l'^2-\Lambda)}\right\}$$
$$+\frac{e^3}{2}m[m(1+\cos^2\theta)-2\mu\cos\theta]\left\{\bar{\mu}(p')\gamma_\alpha\mu(p)\frac{P_\mu^\alpha(l)}{(k^2-m^2)(l'^2-\Lambda)(l^2-\Lambda)}+\bar{\mu}(p')\gamma_\beta\mu(p)\frac{P_\mu^\beta(l')}{(k^2-m^2)(l^2-\Lambda)(l'^2-\Lambda)}\right\}, \tag{A14}$$

where

$$P_\mu^\alpha(l) = g_\mu^\alpha + \frac{k_\mu k^\alpha(1/\xi - 1)}{l^2 - \Lambda} .$$

The result is given by

$$F(q^2) = \frac{e^2\mu}{16\pi^2 M^2}[\mu(1+\cos^2\theta) - 2m\cos\theta]\left\{2M^2\int dx dy \frac{y}{f(\Lambda, M^2)} + \int dx dy[3(x+y)-2]\ln\left(\frac{f(\Lambda,\Lambda)}{f(\Lambda,M^2)}\right) - \int dx dy(x+y-1)[Q^2 - \mu^2(x+y)]\left[\frac{1}{f(\Lambda,M^2)} - \frac{1}{f(\Lambda,\Lambda)}\right]\right\} \tag{A15}$$

for the first group in Eq. (A14). The second group in Eq. (A14) gives

$$F(q^2) = \frac{1}{16\pi^2}\frac{e^2 m\mu^2}{M^2}[m(1+\cos^2\theta) - 2\mu\cos\theta]\int dx dy(x+y-1)^2\left[\frac{1}{f(\Lambda,M^2)} - \frac{1}{f(\Lambda,\Lambda)}\right] . \tag{A16}$$

Note that $F(q^2)$ in Eqs. (A15) and (A16) vanishes in the U-gauge limit $\Lambda \to \infty$.

Finally Fig. 3(a) gives the expression

$$-e^3 m\cos\theta\bar{\mu}(p')\gamma_\beta\gamma_\alpha\mu(p)\frac{P^{\beta\tau}(l')V_{\tau\mu\sigma}P^{\sigma\alpha}(l)}{(k^2-m^2)(l'^2-M^2)(l^2-M^2)} - \frac{e^3}{2}(1+\cos^2\theta)\bar{\mu}(p')\gamma_\beta \not{k}\gamma_\alpha\mu(p)\frac{P^{\beta\tau}(l')V_{\tau\mu\sigma}P^{\sigma\alpha}(l)}{(k^2-m^2)(l'^2-M^2)(l^2-M^2)} , \tag{A17}$$

where

$$V_{\tau\mu\sigma} \equiv g_{\sigma\tau}(l+l')_\mu - l_\tau g_{\mu\sigma} - l'_\sigma g_{\mu\tau} + g_{\sigma\mu}q_\tau - g_{\tau\mu}q_\sigma \tag{A18}$$

and

$$q = l' - l .$$

The result is rather lengthy. The first group in Eq. (A17) gives rise to

$$\begin{aligned} F(q^2) = -\frac{1}{8\pi^2}\frac{e^2\mu m\cos\theta}{M^2}\Bigg\{ & 3M^2\int dx dy \frac{x+y}{f(M^2,M^2)} \\ & + 3\int dx dy(x-y)\ln\frac{f(\Lambda,M^2)}{f(M^2,M^2)} \\ & + \int dx dy\{(1-x+y)[Q^2 - 2\mu^2(x+y)+\mu^2] + 2q^2 y(x-1)\}\left[\frac{1}{f(M^2,M^2)} - \frac{1}{f(\Lambda,M^2)}\right] \\ & - \frac{1}{2}\frac{q^2}{M^2}\int dx dy[3(x+y)-2]\ln\left(\frac{f(\Lambda,M^2)f(M^2,\Lambda)}{f(M^2,M^2)f(\Lambda,\Lambda)}\right) \\ & + \frac{1}{2}\frac{q^2}{M^2}\int dx dy(x+y-1)[Q^2 - 2\mu^2(x+y)+\mu^2] \\ & \quad \times\left[\frac{1}{f(M^2,M^2)} - \frac{1}{f(M^2,\Lambda)} - \frac{1}{f(\Lambda,M^2)} + \frac{1}{f(\Lambda,\Lambda)}\right]\Bigg\} \end{aligned} \tag{A19}$$

and the second group in Eq. (A17) gives

$$\begin{aligned} F(q^2) = \frac{1}{8\pi^2}\frac{e^2\mu^2}{4M^2}(1+\cos^2\theta)\Bigg\{ & 2M^2\int dx dy \frac{(x+y)[2(x+y)+1]}{f(M^2,M^2)} \\ & - 2\int dx dy[4(x+y)^2 - 9(x+y) + 2 + 6y]\ln\left(\frac{f(\Lambda,M^2)}{f(M^2,M^2)}\right) \\ & + 2\int dx dy\{\mu^2(x+y-1)^2[2y + (x+y)(x+y-1)] - q^2 xy[2y + (1-x-y)(2-x-y)]\} \\ & \quad \times\left[\frac{1}{f(M^2,M^2)} - \frac{1}{f(\Lambda,M^2)}\right] \\ & + 4\int dx dy\ \ q^2(xy-y)\left[\frac{1}{f(M^2,M^2)} - \frac{1}{f(\Lambda,M^2)}\right] \end{aligned}$$

$$+\left(\frac{q^2}{M^2}\right)\int dxdy[4(x+y)^2-9(x+y)+4]\ln\left(\frac{f(\Lambda,M^2)f(M^2,\Lambda)}{f(M^2,M^2)f(\Lambda,\Lambda)}\right)$$
$$-\left(\frac{q^2}{M^2}\right)\int dxdy(x+y-1)[(x+y-2)Q^2+\mu^2(x+y)]$$
$$\times\left[\frac{1}{f(M^2,M^2)}-\frac{1}{f(M^2,\Lambda)}-\frac{1}{f(\Lambda,M^2)}+\frac{1}{f(\Lambda,\Lambda)}\right]\Bigg\}. \tag{A20}$$

We would like to discuss the gauge-independence condition $(\partial/\partial\xi)F(0)\equiv 0$ or

$$\frac{\partial}{\partial\Lambda}F(0)\equiv 0. \tag{A21}$$

From Eqs. (A12), (A15), (A16), (A19), and (A20) we readily recognize that $F(q^2)$ consists of two groups, one of which is proportional to $m\cos\theta$ and the other proportional to $(1+\cos^2\theta)$. These two groups separately satisfy Eq. (A21). The proof of Eq. (A21) can be made as in Eqs. (A7)–(A9) by the repeated use of partial integration. We note that the contributions with a $f(\Lambda,M^2)$ factor and the contribution with a $f(\Lambda,\Lambda)$ factor in Eqs. (A12)–(A20) *separately* satisfy Eq. (A12). We do not write down this lengthy but straightforward proof.

Equation (A21) ensures the absence of the ghost contribution to the anomalous magnetic moment. Equation (A21) also allows us to use the most convenient gauge when we calculate numerical values. We also note that in the limit $\xi\to 0$ we recover the result based on the ξ-limiting process.

The neutrino contribution is obtained from the above result by setting $m=0$. The coupling constant should be adjusted according to the specific model one uses.

3. Large-q^2 Behavior of $F(q^2)$

From the above general results for $F(q^2)$ it is easy to see that all the contributions to $F(q^2)$ from scalar mesons vanish in the U-gauge limit, $\xi=0$ and $\eta=0$. They also vanish at $q^2=-\infty$ (i.e., large spacelike momentum transfer). It is also not difficult to see that the Z contribution in Eq. (A4) also vanishes at $q^2=-\infty$ independently of the value of η.

In the following we discuss the large-q^2 behavior of the W contribution in Eqs. (A19) and (A20). Those equations show that all the contributions to $F(q^2)$ vanish at $q^2=-\infty$ for $\xi=1$ (i.e., 't Hooft – Feynman gauge). However the gauge independence of the off-shell amplitude cannot be proved. We expect that $F(q^2)$ at $q^2=-\infty$ may depend on the gauge one chooses. We show this explicitly in the case of the neutrino contribution in the Weinberg model. We thus put $m=0$ in Eqs. (A19) and (A20).

We first note the following relations:

$$-q^2\int dxdy\,\frac{y}{Q^2+x(\Lambda-\mu^2)+y(M^2-\mu^2)}\underset{q^2\to-\infty}{\sim}\ln(-q^2)+\int_0^1 dt\,\ln\,\frac{t}{\mu^2t+M^2-\mu^2} \tag{A22}$$

and

$$-q^2\int dxdy\,\rho(x,y)\ln f(M^2,\Lambda)\underset{q^2\to-\infty}{\sim}-q^2\int_0^1 dt\,\rho(t)t\ln t$$
$$+(\Lambda+M^2-2\mu^2)\int_0^1 dt\,\rho(t)\ln(-q^2t)+\tfrac{1}{2}(\Lambda-M^2)\int_0^1 dt\,\rho(t)\ln\left[\frac{\mu^2t+M^2-\mu^2}{\mu^2t+\Lambda-\mu^2}\right]$$
$$-\tfrac{1}{2}\int_0^1 dt\,\rho(t)\left[\mu^2t-\mu^2+\frac{\Lambda+M^2}{2}\right]\ln[(\mu^2t+M^2-\mu^2)(\mu^2t+\Lambda-\mu^2)]$$
$$+(\Lambda+M^2-2\mu^2)\int_0^1 dt\,\rho(t), \tag{A23}$$

where

$$\rho(x,y)\equiv 4(x+y)^2-9(x+y)+4$$

and

$$\rho(t)\equiv 4t^2-9t+4.$$

Using these two relations in Eq. (A20) we can readily show that

$$F(q^2) \to 0 \quad \text{for} \quad q^2 \to -\infty \text{ and } \xi \neq 0\,, \tag{A24}$$

$$F(q^2) \sim \frac{1}{8\pi^2}\,\frac{g\mu^2}{8M^2}\left(4 - \frac{\mu^2}{M^2}\right)\ln(-q^2) + \text{cost} \quad \text{for } q^2 \to -\infty \text{ after } \xi = 0\,. \tag{A25}$$

Equations (A24) and (A25) indicate the gauge independence of the off-shell amplitude.

We also point out an interesting large-q^2 behavior obtained if one uses (incorrect) regularization schemes other than the ξ-limiting procedure in the U gauge. The last two terms in Eq. (A20) show that the linear divergence in q^2 at large q^2 could exist. For the gauge-invariant calculation this linear divergence cancels. But if one uses other regularization schemes such as the "proper-time" (see Appendix B) or the "Pauli-Villars" regularization with a massive neutrino in the U gauge, this linear divergence indeed survives.

APPENDIX B: AMBIGUITIES IN THE U GAUGE

In this appendix we briefly review the ambiguity that Jackiw and Weinberg and also Bars and Yoshimura[8] encountered in their calculation of the muon g factor.

The logarithmic term in the parametric integral for $F(0)$ in the U-gauge limit (i.e., $\xi \to 0$) causes an ambiguity: Equation (A20) in Appendix A contains the logarithmic term

$$F(0) = -\frac{1}{8\pi^2}\,\frac{g^2\mu^2}{4M^2}\int dx dy [4(x+y)^2 - 9(x+y) + 2 + 6y]\,\ln\left[\frac{x}{\mu^2(x+y)^2 + (x+y)(M^2-\mu^2)}\right] + \text{other terms}\,. \tag{B1}$$

This is the correct answer. On the other hand if one regulates the neutrino propagator we get

$$F(0) = -\frac{1}{8\pi^2}\,\frac{g^2\mu^2}{4M^2}\int dx dy [4(x+y)^2 - 9(x+y) + 2 + 6y]\,\ln\left[\frac{1-x-y}{\mu^2(x+y)^2 + (x+y)(M^2-\mu^2)}\right] + \text{other terms}\,. \tag{B2}$$

If one first exponentiates the Feynman amplitude (a sort of "proper time") and performs a loop integral, the following result is obtained:

$$F(0) = -\frac{1}{8\pi^2}\,\frac{g^2\mu^2}{4M^2}\int dx dy [4(x+y)^2 - 9(x+y) + 2 + 6y]\,\ln\left[\frac{1}{\mu^2(x+y)^2 + (x+y)(M^2-\mu^2)}\right] + \text{other terms}\,. \tag{B3}$$

All these expressions give rise to different answers. This kind of ambiguity is absent in the general R-gauge calculation. Naive U-gauge calculations are plagued by this kind of ambiguity. Some existing proofs of cancellations of divergences in higher-order diagrams are based on the exponential parametrization of propagators that leads to (B3). While such a method is acceptable in establishing the absence of divergences, it will not provide a reliable finite part.

APPENDIX C: LAGRANGIANS AND FEYNMAN RULES

In this appendix we present Lagrangians for the existing models of leptons mentioned in the Introduction. We also write down necessary Feynman rules for our calculations in Secs. III and IV. In the R_ξ gauge discussed in Sec. II, we have unphysical scalars in the Lagrangian. The Feynman rules are therefore more complicated than those in the U-gauge limit. The Feynman rules for those unphysical scalars and also the relative signs for various amplitudes can be conveniently checked based on the gauge independence of the T-matrix element as we discussed in Sec. II.

1. Georgi-Glashow Model

This model is based on the group O(3). We have a triplet of leptons and also a singlet of neutral massive lepton. The mass is generated by a triplet of real scalars. A part of this Lagrangian has been given in Sec. II. (We follow the notation of Bjorken and Drell[16].) The total Lagrangian has the following form:

$$\mathcal{L} = \mathcal{L}_{\text{I}} + \mathcal{L}_{\text{II}} + \mathcal{L}_{\text{III}} - V\,,$$

$$\begin{aligned}\mathcal{L}_{\text{I}} \equiv{}& \overline{Y}^+(i\not\partial - m_{Y^+})Y^+ + \overline{Y}^0(i\not\partial - m_{Y^0})Y^0 \\ &+ \bar{\mu}(i\not\partial - m_\mu)\mu + \bar{\nu}_L i\not\partial \nu_L + e\overline{L}\gamma^\mu(\vec{\text{T}}\cdot\vec{\text{B}})_\mu L + e\overline{R}\gamma^\mu(\vec{\text{T}}\cdot\vec{\text{B}})_\mu R - \frac{e(m_{Y^0}\cos\theta - m_\mu)}{M}[\overline{L}(\vec{\text{T}}\cdot\vec{\text{s}})R + \text{H.c.}] \\ &- \frac{e m_{Y^0}\sin\theta}{M}\{[\sin\theta\,\overline{Y}{}^0_L - \cos\theta\,\bar{\nu}_L][s^- Y^+_R + s^+\mu_R + \psi Y^0_R] + \text{H.c.}\}\,,\end{aligned} \tag{C1}$$

where

$$L \equiv \begin{bmatrix} Y_L^+ \\ \cos\theta\, Y_L^0 + \sin\theta\, \nu_L \\ \mu_L \end{bmatrix}, \qquad R \equiv \begin{bmatrix} Y_R^+ \\ Y_R^0 \\ \mu_R \end{bmatrix}, \tag{C2}$$

with

$$\mu_R \equiv \left(\frac{1+\gamma_5}{2}\right)\mu, \quad \mu_L \equiv \left(\frac{1-\gamma_5}{2}\right)\mu, \quad \text{etc.}, \tag{C3}$$

$$\vec{\mathrm{T}}\cdot\vec{\mathrm{B}}_\mu = \begin{bmatrix} A_\mu & -W_\mu^+ & 0 \\ -W_\mu^- & 0 & W_\mu^+ \\ 0 & W_\mu^- & -A_\mu \end{bmatrix}, \qquad \vec{\mathrm{T}}\cdot\vec{\mathrm{s}} = \begin{bmatrix} \psi & -s^+ & 0 \\ -s^- & 0 & s^+ \\ 0 & s^- & -\psi \end{bmatrix}, \tag{C4}$$

where

$$\begin{aligned} &\psi = \text{Higgs scalar}, \\ &s^{\pm} = \text{unphysical scalars}, \\ &G_F/\sqrt{2} = (e^2 \sin^2\theta/4M^2). \end{aligned} \tag{C5}$$

There is a constraint:

$$2m_{Y^0}\cos\theta = m_{Y^+} + m_\mu. \tag{C6}$$

Note that the covariant derivative is given by

$$\nabla_\mu \psi \equiv [\partial_\mu + ig(\vec{\mathrm{T}}\cdot\vec{\mathrm{B}})_\mu]\psi,$$

with $g \equiv -e$.

$\mathcal{L}_{\mathrm{II}}$ is given by

$$\mathcal{L}_{\mathrm{II}} \equiv \tfrac{1}{2}|\partial_\mu\psi + ie[W_\mu^- s^+ - W_\mu^+ s^-]|^2 + |\partial_\mu s^+ + iMW_\mu^+ - ieA_\mu s^+ + ie\psi W_\mu^+|^2. \tag{C7}$$

The quadratic term of $\mathcal{L}_{\mathrm{II}}$ is given by

$$\mathcal{L}_{\mathrm{II}}^{\mathrm{quad}} = |\partial_\mu s^+|^2 + \tfrac{1}{2}(\partial_\mu\psi)^2 + M^2|W_\mu^+|^2 - iM[\partial_\mu s^+ W^{-\mu} - \partial_\mu s^- W^{+\mu}]. \tag{C8}$$

This $\mathcal{L}_{\mathrm{II}}^{\mathrm{quad}}$ suggests the following gauge term:

$$\mathcal{L}^c = -\frac{1}{2\alpha}(\partial_\mu A^\mu)^2 - \xi\left|\partial_\mu W^{+\mu} + \frac{iM}{\xi}s^+\right|^2; \tag{C9}$$

see also Sec. II. $\mathcal{L}_{\mathrm{III}}$ is given by

$$\mathcal{L}_{\mathrm{III}} = -\tfrac{1}{2}|\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+ + ie[W_\mu^+ A_\nu - W_\nu^+ A_\mu]|^2 - \tfrac{1}{4}|\partial_\mu A_\nu - \partial_\nu A_\mu - ie[W_\mu^+ W_\nu^- - W_\mu^- W_\nu^+]|^2. \tag{C10}$$

The mass of $W_\mu^\pm$ is given in Eq. (C8).

The potential is given by

$$\begin{aligned} V(\phi) &= \tfrac{1}{2}\mu_0^2|\phi|^2 + \lambda|\phi|^4 \\ &= \tfrac{1}{2}m_\psi^2\psi^2 + \lambda[4v\psi(2s^+s^- + \psi^2) + (2s^+s^- + \psi^2)^2] + [\mu_0^2 + 4v^2\lambda][s^+s^- + v\psi], \end{aligned} \tag{C11}$$

where

$$m_\psi^2 = \mu_0^2 + 12\lambda v^2, \tag{C12}$$

$\mu_0^2 < 0$, and we have the condition $\mu_0^2 + 4v^2\lambda = 0$. The field ϕ is the unshifted field, and it is expressed as

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \\ \phi^- \end{pmatrix} = \begin{pmatrix} s^+ \\ v+\psi \\ s^- \end{pmatrix}, \qquad \langle\phi\rangle_0 = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}. \tag{C13}$$

The gauge-compensating term for Eq. (C9) is given by (see also Sec. II)

$$-i\,\mathrm{Tr}\ln(1 + \mathcal{G}\cdot\gamma), \tag{C14}$$

where

$$\begin{bmatrix} -(\partial^2+M^2/\xi-i\epsilon) & 0 & 0 \\ 0 & -\partial^2+i\epsilon & 0 \\ 0 & 0 & -(\partial^2+M^2/\xi-i\epsilon)\end{bmatrix}\mathcal{G}=\delta^4(x-y) \tag{C15}$$

and

$$\gamma \equiv \begin{bmatrix} ie\partial_\mu A^\mu-(eM/\xi)\psi & -ie\partial_\mu W^{+\mu}+(eM/\xi)s^+ & 0 \\ -ie\partial_\mu W^{-\mu} & 0 & ie\partial_\mu W^{+\mu} \\ 0 & ie\partial_\mu W^{-\mu}+(eM/\xi)s^- & -ie\partial_\mu A^\mu-(eM/\xi)\psi \end{bmatrix}\delta^4(x-y). \tag{C16}$$

The divergence in this matrix stands for an operator, e.g., $\partial_\mu A^\mu \equiv (\partial_\mu A^\mu)+A^\mu\partial_\mu$. The lowest-order contribution from Eq. (C14) is a ψ-vacuum tadpole diagram. For $\xi=\infty$ (Landau gauge), for example, Eq. (C14) becomes

$$-i\,\mathrm{Tr}\ln\left[1+\frac{1}{-\partial^2+i\epsilon}\partial_\mu(ie\vec{\mathrm{T}}\cdot\vec{\mathrm{B}}^\mu)\right]. \tag{C17}$$

2. Weinberg Model

The detailed form of this Lagrangian is found in Refs. 1 and 6. We briefly summarize it in the following:

$$\mathcal{L}\equiv\mathcal{L}_{\mathrm{I}}+\mathcal{L}_{\mathrm{II}}+\mathcal{L}_{\mathrm{III}}-V,$$

$$\mathcal{L}_{\mathrm{I}}=\bar\nu_L i\not\partial\nu_L+\bar\mu(i\not\partial-m_\mu)\mu-e\bar\mu\not A\mu-\frac{g}{\sqrt2}[\bar\nu_L\not W^+\mu+\mathrm{H.c.}]-\frac{G}{2}\bar\nu_L\not Z\nu_L$$
$$+G\bar\mu\not Z\left[\frac{\cos2\theta}{2}\left(\frac{1-\gamma_5}{2}\right)-\sin^2\theta\left(\frac{1+\gamma_5}{2}\right)\right]\mu-\frac{gm_\mu}{2M}[(\sqrt2\bar\mu\nu_L s^-+\mathrm{H.c.})+\bar\mu\mu\psi+\bar\mu i\gamma_5\mu\chi], \tag{C18}$$

where

$s^\pm$ and χ = unphysical scalar, ψ = Higgs scalar,

$$G=(g^2+g'^2)^{1/2},$$

$$\cos\theta=\frac{g}{G}=\frac{M}{M_Z},\qquad e=-\frac{gg'}{G},\qquad \frac{G_F}{\sqrt2}=\frac{g^2}{8M^2}. \tag{C19}$$

For notational convenience we defined the charge e with an extra minus sign.

$\mathcal{L}_{\mathrm{II}}$ is given by

$$\mathcal{L}_{\mathrm{II}}=\left|\partial_\mu s^++iMW_\mu^++\frac{i}{\sqrt2}gW_\mu^+s^0+i\left(-eA_\mu+\frac{G\cos2\theta}{2}Z_\mu\right)\right|^2+\left|\partial_\mu s^0-i\frac{M_Z}{\sqrt2}Z_\mu-i\frac{GZ_\mu}{2}s^0+\frac{i}{\sqrt2}gW_\mu^-s^+\right|^2, \tag{C20}$$

where

$$s^0\equiv\frac{1}{\sqrt2}(\psi+i\chi).$$

The quadratic part of $\mathcal{L}_{\mathrm{II}}$ is given by

$$\mathcal{L}_{\mathrm{II}}^{\mathrm{quad}}=|\partial_\mu s^+|^2+\tfrac12[(\partial_\mu\psi)^2+(\partial_\mu\chi)^2]+M^2|W_\mu^+|^2$$
$$+\tfrac12 M_Z{}^2(Z_\mu)^2-iM[\partial_\mu s^+W^{-\mu}-\partial_\mu s^-W^{+\mu}]-M_Z\partial_\mu\chi Z^\mu. \tag{C21}$$

The gauge term is given as

$$\mathcal{L}^c=-\frac{1}{2\alpha}(\partial_\mu A^\mu)^2-\xi\left|\partial_\mu W^{+\mu}+i\frac{M}{\xi}s^+\right|^2-\frac{\eta}{2}\left(\partial_\mu Z^\mu+\frac{M_Z}{\eta}\chi\right)^2. \tag{C22}$$

The gauge-compensating term will be discussed later.

For $\mathcal{L}_{\mathrm{III}}$,

$$\mathcal{L}_{\mathrm{III}}=-\tfrac12|\partial_\mu W_\nu^+-\partial_\nu W_\mu^++ie[W_\mu^+A_\nu-W_\nu^+A_\mu]-iG\cos^2\theta[W_\mu^+Z_\nu-W_\nu^+Z_\mu]|^2$$
$$-\tfrac14|\partial_\mu A_\nu-\partial_\nu A_\mu-ie[W_\mu^+W_\nu^--W_\nu^+W_\mu^-]|^2-\tfrac14|\partial_\mu Z_\nu-\partial_\nu Z_\mu+iG\cos^2\theta[W_\mu^+W_\nu^--W_\nu^+W_\mu^-]|^2. \tag{C23}$$

The masses for W_μ and Z_μ are given in Eq. (C21);

$$V=\lambda\left[s^+s^-+\left(s^0+\frac{v}{\sqrt{2}}\right)\left(\bar{s}^0+\frac{v}{\sqrt{2}}\right)-\frac{v^2}{2}\right]^2$$
$$=\tfrac{1}{2}m_\psi{}^2\psi^2+\lambda\{(2/\lambda)^{1/2}m_\psi\psi(s^+s^-+s^0\bar{s}^0)+[s^+s^-+s^0\bar{s}^0]^2\}. \quad \text{(C24)}$$

The gauge-compensating term is given by

$$-i\,\mathrm{Tr}\ln[1+\mathcal{G}\cdot\gamma], \quad \text{(C25)}$$

where

$$\begin{bmatrix} -(\partial^2+M^2/\xi-i\epsilon) & 0 & 0 & 0 \\ 0 & -(\partial^2+M^2/\xi-i\epsilon) & 0 & 0 \\ 0 & 0 & -\partial^2+i\epsilon & 0 \\ 0 & 0 & 0 & -(\partial^2+M^2/\eta-i\epsilon) \end{bmatrix}\mathcal{G}=\delta^4(x-y) \quad \text{(C26)}$$

and the matrix γ is defined by

$$\begin{bmatrix}\delta f^+\\ \delta f^-\\ \delta f^A\\ \delta f^Z\end{bmatrix}\equiv[\gamma]\begin{bmatrix}\Omega^+\\ \Omega^-\\ \Omega^A\\ \Omega^Z\end{bmatrix}\equiv[\gamma]\begin{bmatrix}\frac{1}{\sqrt{2}}(\Omega^1-i\Omega^2)\\ \frac{1}{\sqrt{2}}(\Omega^1+i\Omega^2)\\ \frac{g'}{g}\Omega^3+\frac{2g}{g'}\Omega^0\\ \Omega^3-2\Omega^0\end{bmatrix}, \quad \text{(C27)}$$

with

$$\delta f^{\pm}\equiv\mp ig\partial^\mu[(Z_\mu\cos\theta+A_\mu\sin\theta)\Omega^\pm]-\frac{gM}{2\xi}(\psi\pm i\chi)\Omega^\pm$$
$$\mp ie\cos\theta\,\partial^\mu[W_\mu^\pm(\Omega^A+\cot\theta\,\Omega^Z)]+\frac{eM\cos\theta}{2\xi}s^\pm\left(2\Omega^A-\frac{G}{e}\cos2\theta\,\Omega^Z\right),$$
$$\delta f^A\equiv iG\sin\theta\,\partial^\mu(W_\mu^-\Omega^+-W_\mu^+\Omega^-), \quad \text{(C28)}$$
$$\delta f^Z\equiv ig\partial^\mu(W_\mu^-\Omega^+-W_\mu^+\Omega^-)+\frac{GM_Z}{2\eta}(-\psi\Omega^Z+s^+\Omega^-+s^-\Omega^+).$$

Another way to take care of this compensating term is to introduce four auxiliary complex scalar fields[12] ϕ_a, $a=(+,-,A,Z)$. We add the following extra piece to the effective Lagrangian:

$$\tilde{\mathcal{L}}\equiv\phi_a^\psi(\mathcal{G}^{-1}+\gamma)^{a,b}\phi_b. \quad \text{(C29)}$$

The ordinary perturbative treatment of $\tilde{\mathcal{L}}$ with an extra (−) sign for each closed loop of the ficticious scalar particles ϕ_a gives rise to Eq. (C25).

3. The Lepton Model of Lee, Prentki, and Zumino

The group structure of this model is close to that of the Weinberg model. We need two sets of scalar triplets to accomodate "quarks" in this model. However only one triplet of complex scalars is sufficient for the lepton model. We present here this simplified lepton model:

$$\mathcal{L}=\mathcal{L}_{\mathrm{I}}+\mathcal{L}_{\mathrm{II}}+\mathcal{L}_{\mathrm{III}}-V,$$
$$\mathcal{L}_{\mathrm{I}}=\overline{M}^+(i\partial\!\!\!/-m_{M^+})M^++\bar{\nu}_Li\partial\!\!\!/\nu_L+\bar{\mu}(i\partial\!\!\!/-m_\mu)\mu$$
$$-\frac{gm_{M^+}}{M}[(\overline{L}\cdot\bar{\mathrm{s}})M_R^++\mathrm{H.c.}]-\frac{gm_\mu}{M}[(\overline{L}\cdot s)\mu_R+\mathrm{H.c.}]-e[\bar{\mu}A\!\!\!/\mu-\overline{M}^+A\!\!\!/M^+]-g[(\bar{\nu}W\!\!\!\!/^+\mu-\overline{M}^+W\!\!\!\!/^+\nu)+\mathrm{H.c.}]$$
$$+G\bar{\mu}Z\!\!\!/\left[\cos^2\theta\left(\frac{1-\gamma_5}{2}\right)-\sin^2\theta\left(\frac{1+\gamma_5}{2}\right)\right]\mu-G\overline{M}^+Z\!\!\!/\left[\cos^2\theta\left(\frac{1-\gamma_5}{2}\right)-\sin^2\theta\left(\frac{1+\gamma_5}{2}\right)\right]M^+, \quad \text{(C30)}$$

where

$$L\equiv\begin{bmatrix}M_L^+\\ \nu_L\\ \mu_L\end{bmatrix},\qquad s\equiv\begin{bmatrix}s^{++}\\ s^{+}\\ s^{0}\end{bmatrix},\qquad \bar{s}\equiv\begin{bmatrix}\bar{s}^{0}\\ s^{-}\\ s^{--}\end{bmatrix},$$
$$G=(g^2+g'^2)^{1/2},\qquad e=(-)\frac{gg'}{G},\qquad \frac{G_F}{\sqrt{2}}=\frac{g^2}{4M^2},\qquad \cos\theta=\frac{g}{G},\qquad M=gu,\qquad M_Z=\sqrt{2}Gu. \tag{C31}$$

For $\mathcal{L}_{II}$,

$$\mathcal{L}_{II}=|\partial_\mu s^{++}-igW_\mu^+s^+ + i(-2eA_\mu+G\cos 2\theta Z_\mu)s^{++}|^2$$
$$+|\partial_\mu s^+ + iMW_\mu^+ + igW_\mu^+ s^0 - igW_\mu^- s^{++} - i[eA_\mu + G\sin^2\theta Z_\mu]s^+|^2 + \left|\partial_\mu s^0 - i\frac{M_Z}{\sqrt{2}}Z_\mu + igW_\mu^- s^+ - iGZ_\mu s^0\right|^2. \tag{C32}$$

The quadratic part of $\mathcal{L}_{II}$ is given by

$$\mathcal{L}_{II}^{\text{quad}}=|\partial_\mu s^{++}|^2+|\partial_\mu s^+|^2+\tfrac{1}{2}[(\partial_\mu\psi)^2+(\partial_\mu\chi)^2]+M^2|W_\mu^+|^2+\tfrac{1}{2}M_Z^2(Z_\mu)^2-iM[\partial^\mu s^+W_\mu^- - \partial^\mu s^- W_\mu^+]-M_Z\partial_\mu\chi\cdot Z^\mu, \tag{C33}$$

where we defined

$$s^0\equiv\frac{1}{\sqrt{2}}(\psi+i\chi),$$
$$s^\pm \text{ and } \chi=\text{unphysical scalars}, \tag{C34}$$
$$s^{\pm\pm} \text{ and } \psi=\text{Higgs scalars}.$$

The gauge term is the same as Eq. (C22) in the Weinberg model. The Yang-Mills Lagrangian is also given by Eq. (C23).

Finally we discuss the potential:

$$V=\mu_0(\vec{\xi}^*\cdot\vec{\xi})+\frac{\nu}{4}(\vec{\xi}^*\times\vec{\xi})^2+\lambda(\vec{\xi}^*\cdot\vec{\xi})^2$$
$$=\tfrac{1}{2}m_\psi^2\psi^2+\tfrac{1}{2}m_{s^{++}}^2|s^{++}|^2+2\sqrt{2}\lambda u\psi(s^0\bar{s}^0+s^+s^-+s^{++}s^{--})+\lambda(s^0\bar{s}^0+s^+s^-+s^{++}s^{--})^2$$
$$-\tfrac{1}{4}\nu[2\sqrt{2}u\psi(s^0\bar{s}^0+s^{++}s^{--})+2us^+(s^0s^- - s^+s^{--})+2us^-(\bar{s}^0s^+ - s^-s^{++})$$
$$+(s^0\bar{s}^0-s^{++}s^{--})^2+2(\bar{s}^0s^+-s^-s^{++})(s^0s^- - s^+s^{--})], \tag{C35}$$

where ξ is the unshifted complex scalar field

$$\xi=\begin{bmatrix}s^{++}\\ s^+\\ u+s^0\end{bmatrix},\qquad \langle\xi\rangle_0=\begin{bmatrix}0\\0\\u\end{bmatrix}, \tag{C36}$$
$$m_\psi^2=-2\mu_0>0,\qquad m_{s^{++}}^2=\nu u^2>0.$$

There is the following constraint:

$$\lambda=\frac{\nu}{4}+\left(-\frac{\mu_0}{2u}\right)=\frac{1}{4u^2}[m_\psi^2+m_{s^{++}}^2]. \tag{C37}$$

We can make m_ψ and $m_{s^{++}}$ arbitrarily large. The gauge-compensating term is also similar to that of the Weinberg model. We do not discuss it here.

4. Feynman Rules

We summarize several Feynman rules we use in Secs. III and IV. Propagators for vector bosons and also for scalars are found in Sec. II. The fermion propagator has the standard form

$$\frac{i}{\not p - m}. \tag{C38}$$

In the following we give Feynman rules for the lepton models due to Georgi and Glashow (GG), Weinberg (W), and Lee, Prentki, and Zumino (LPZ). We write the Feynman rules for the R_ξ gauge of Sec. II.

All of these models give the identical Feynman rules for Figs. 8(a)–8(d). They are

$$8(\mathrm{a}):\quad (-ie)\bar{\mu}\gamma_\alpha\mu\,; \tag{C39}$$

$$8(\mathrm{b}):\quad (-ie)(l'+l)_\mu\,; \tag{C40}$$

$$8(\mathrm{c}):\quad (ie)[(g_{\alpha\beta}(l'+l)_\mu - l_\beta g_{\mu\alpha} - l'_\alpha g_{\mu\beta}) + (g_{\alpha\mu}q_\beta - g_{\beta\mu}q_\alpha)],\ \text{where } q=l'-l\,; \tag{C41}$$

$$8(\mathrm{d}):\quad (-ie)Mg_{\alpha\mu}\,. \tag{C42}$$

Note that we normalized the sign of the charge by Fig. 8(a). The magnetic form factor appears as a coefficient of $(-ie)\bar{\mu}(i\sigma_{\mu\nu}q^\nu/2\mu)\mu$. For other diagrams Feynman rules depend on the model. We just list them below:

$$8(\mathrm{e}):\quad \mathrm{GG}:\ (ie\sin\theta)\bar{\nu}\gamma_\alpha\left(\frac{1-\gamma_5}{2}\right)\mu\,;$$
$$\mathrm{W}:\ \left(-\frac{ig}{\sqrt{2}}\right)\bar{\nu}\gamma_\alpha\left(\frac{1-\gamma_5}{2}\right)\mu\,; \tag{C43}$$
$$\mathrm{LPZ}:\ (-ig)\bar{\nu}\gamma_\alpha\left(\frac{1-\gamma_5}{2}\right)\mu\,;$$

$$8(\mathrm{f}):\quad \mathrm{GG}:\ \left(\frac{ie\sin\theta\, m_\mu}{M}\right)\bar{\nu}\left(\frac{1+\gamma_5}{2}\right)\mu\,;$$
$$\mathrm{W}:\ \left(\frac{-igm_\mu}{\sqrt{2}M}\right)\bar{\nu}\left(\frac{1+\gamma_5}{2}\right)\mu\,; \tag{C44}$$
$$\mathrm{LPZ}:\ \left(\frac{-igm_\mu}{M}\right)\bar{\nu}\left(\frac{1+\gamma_5}{2}\right)\mu\,;$$

$$8(\mathrm{g}):\quad \mathrm{GG}:\ (ie)\bar{Y}^0\left[\cos\theta\left(\frac{1+\gamma_5}{2}\right)+\left(\frac{1-\gamma_5}{2}\right)\right]\gamma_\alpha\mu\,; \tag{C45}$$

$$8(\mathrm{h}):\quad \mathrm{GG}:\ \left(\frac{ie}{M}\right)\bar{Y}^0\left[(m_\mu\cos\theta - m_{Y^0})\left(\frac{1+\gamma_5}{2}\right) + (m_\mu - m_{Y^0}\cos\theta)\left(\frac{1-\gamma_5}{2}\right)\right]\mu\,; \tag{C46}$$

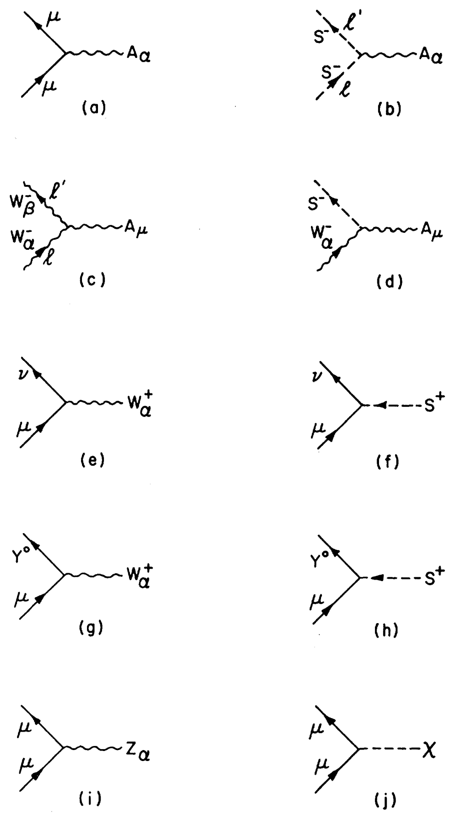

FIG. 8. Several vertex diagrams for lower-order calculations.

$$8(\mathrm{i}):\quad \mathrm{W}:\ iG\bar{\mu}\left[\frac{\cos2\theta}{2}\left(\frac{1+\gamma_5}{2}\right)-\sin^2\theta\left(\frac{1-\gamma_5}{2}\right)\right]\gamma_\alpha\mu\,; \tag{C47}$$
$$\mathrm{LPZ}:\ iG\bar{\mu}\left[\cos^2\theta\left(\frac{1+\gamma_5}{2}\right)-\sin^2\theta\left(\frac{1-\gamma_5}{2}\right)\right]\gamma_\alpha\mu\,;$$

$$8(\mathrm{j}):\quad \mathrm{W}:\ \left(-i\frac{gm_\mu}{2M}\right)\bar{\mu}i\gamma_5\mu\,; \tag{C48}$$
$$\mathrm{LPZ}:\ \left(-i\frac{gm_\mu}{\sqrt{2}M}\right)\bar{\mu}i\gamma_5\mu\,.$$

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Sum Rules and Bounds on Scattering Amplitudes*

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Using sum rules obtained from crossing and analyticity, and unitarity bounds on scattering amplitudes, we show how new relations between low-energy and high-energy scattering can be derived. These relations can provide tests of a wide range of theoretical ideas. As examples, we discuss several inequalities obtained for π-π and π-N scattering. For π-π scattering, a number of relations involving the asymptotic behavior of total cross sections are presented, including bounds limiting the size of violations of the Pomeranchuk theorem. Using finite-energy sum rules for π-N scattering, we derive new types of bounds and show how they can be used to probe such things as the nature of the Pomeranchuk trajectory and the assumption of s-channel helicity conservation. Finally, we introduce inequality constraints between partial-wave amplitudes of different isospin, and indicate how they can be used to explore the nature of exchange degeneracy, absence of exotics, and duality.

I. INTRODUCTION

Certain general principles, namely unitarity, crossing symmetry, and some form of analyticity, severely restrict the allowed behavior of scattering amplitudes. During the past several years, many interesting inequalities have been derived which follow solely from these principles, or from these principles combined with a few pieces of experimental information, or a few additional theoretically plausible assumptions. In this paper, we discuss several bounds at fixed energies, enforcing unitarity through the use of Lagrange inequality multipliers. We then show two ways in which crossing and analyticity can be introduced into the problems by combining our results either with a Froissart-Gribov expression for crossed-channel scattering or with finite-energy sum rules (FESR). Both these approaches yield relations between low- and high-energy scattering. In Sec. II, we describe the fixed-energy bounds problems with which we shall deal, and we introduce a constraint which can be used to explore the nature of duality, absence of exotics, and exchange degeneracy. Using π-π scattering as an example, Sec. III develops the formalism for using the Froissart-Gribov formula with the results of Sec. II, to provide bounds on various quantities. In Sec. IV, the