Large degeneracy of excited hadrons and quark models

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The pattern of a large approximate degeneracy of the excited hadron spectra (larger than the chiral restoration degeneracy) is present in the recent experimental report of Bugg. Here we try to model this degeneracy with state of the art quark models. We review how the Coulomb Gauge chiral invariant and confining Bethe-Salpeter equation simplifies in the case of very excited quark-antiquark mesons, including angular or radial excitations, to a Salpeter equation with an ultrarelativistic kinetic energy with the spin-independent part of the potential. The resulting meson spectrum is solved, and the excited chiral restoration is recovered, for all mesons with $J > 0$. Applying the ultrarelativistic simplification to a linear equal-time potential, linear Regge trajectories are obtained, for both angular and radial excitations. The spectrum is also compared with the semiclassical Bohr-Sommerfeld quantization relation. However, the excited angular and radial spectra do not coincide exactly. We then search, with the classical Bertrand theorem, for central potentials producing always classical closed orbits with the ultrarelativistic kinetic energy. We find that no such potential exists, and this implies that no exact larger degeneracy can be obtained in our equal-time framework, with a single principal quantum number comparable to the nonrelativistic Coulomb or harmonic oscillator potentials. Nevertheless we find it plausible that the large experimental approximate degeneracy will be modeled in the future by quark models beyond the present state of the art.

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I. INTRODUCTION

In the recent report of Bugg [1–4] a large degeneracy emerges from the spectra of the angularly and radially excited resonances produced in $p\bar{p}$ annihilation by the Crystal Ball Collaboration at LEAR in CERN [5]. This degeneracy may be the third remarkable pattern of the excited spectra of hadronic resonances.

A long time ago, Chew and Fautschi remarked the existence of linear Regge trajectories [6] for angularly excited mesons and baryons [7,8]. A similar linear aligning of excited resonances was also reported for radial excitations [9].

Recently, Glozman *et al.* [3,10,11], Jaffe *et al.* [12], and Afonin [13], have been systematically researching the degeneracy of chiral partners in excited resonances, both in models and in lattice QCD. Le Yaouanc *et al.* [14] and Bicudo *et al.* [15,16] developed a model of spontaneous symmetry breaking. Viewed retrospectively, this already included the degeneracy of chiral partners in the limit of high radial or angular excitations [17], both in light-light and in heavy-light hadrons. This earlier work was based on the Bogoliubov transformation and used the Coulomb gauge or the local coordinate gauge QCD truncations. Here the formalism will be extended to include spin-dependent interactions explicitly.

The recent data on highly excited mesons, observed by LEAR at CERN [1,4,5] and of excited baryons observed by the Crystal Barrel Collaboration at ELSA [18], stress the interest of possible patterns in the excited hadronic resonance spectra. While we still have to wait for new experiments focused on excited mesonic resonances to confirm the report of Bugg, say in PANDA, GLUEX, or BESIII, it is important to research theoretical models of the excited hadrons.

Here we address the question, is it possible to build an equal-time quark model, with linear trajectories, with excited chiral symmetry, and, also, with a principal quantum number? We adopt the framework of the Coulomb gauge confinement, of the mass gap equation, and of the equal-time Bethe-Salpeter equation. In Sec. II we review and expand earlier work on chiral symmetry breaking and mesonic bound states, and show in detail the equations for the simplest potential. We also show how for excited states (and for $J > 0$) the Bethe-Salpeter equation simplifies to a Schrödinger-like Salpeter equation, with ultrarelativistic (massless) kinetic energies and a chiral symmetric equal-time potential. In Sec. III we solve the equation with the method of the double diagonalization of the equal-time Hamiltonian and show how linear equal-time potentials and massless quarks produce linear Regge trajectories, both for angular and radial excitations. We also compare them with the Bohr-Sommerfeld semiclassical quantization. We then address in Sec. IV the large degeneracy, where both radial and angular excitations are degenerate. Extending to ultrarelativistic particles the techniques of the classical Bertrand theorem on closed orbits, we verify that no instantaneous 2-body potential may exactly produce the desired large degeneracy. Nevertheless, we present plausible solutions to this modeling problem in the conclusion, Sec. V.

II. QUARK MASS GAP AND BOUND STATES IN EQUAL TIME

We first review earlier work on chiral symmetry breaking with equal-time confining quark-quark potentials, and show the example of the simplest possible model of this class of potentials, which continues to be explored [19]. Importantly, the Hamiltonian of this model can be approximately derived from QCD,

$$H = \int d^3x \Big[\psi^\dagger(x)(m_0\beta - i\vec{\alpha}\cdot\vec{\nabla})\psi(x) + \frac{1}{2}g^2 \int d^4y \bar{\psi}(x)\gamma^\mu \frac{\lambda^a}{2}\psi(x) \times \langle A^a_\mu(x) A^b_\nu(y)\rangle \bar{\psi}(y)\gamma^\nu \frac{\lambda^b}{2}\psi(y) + \cdots \Big] \quad (1)$$

up to the first cumulant order, of two gluons [16,20–22]. In the modified coordinate gauge the cumulant is

$$g^2\langle A^a_\mu(x)A^b_\nu(y)\rangle \simeq -\tfrac{3}{4}\delta_{ab}g_{\mu 0}g_{\nu 0}[K_0^3(\mathbf{x}-\mathbf{y})^2 - U] \quad (2)$$

and this is a simple density-density harmonic effective confining interaction. m_0 is the current mass of the quark. The infrared constant U confines the quarks but the meson spectrum is completely insensitive to it. The important parameter is the potential strength K_0, the only physical scale in the interaction, and all results can be expressed in units of K_0. A reasonable fit of the hadron spectra is achieved with $K_0 \simeq 0.3 \pm 0.05$ GeV.

The relativistic invariant Dirac-Feynman propagators [14], can be decomposed in the quark and antiquark Bethe-Goldstone propagators [23], used in the formalism of nonrelativistic quark models,

$$\begin{aligned} S_{\rm Dirac}(k_0,\vec{k}) &= \frac{i}{\not{k} - m + i\epsilon} \\ &= \frac{i}{k_0 - E(k) + i\epsilon}\sum_s u_s u_s^\dagger \beta - \frac{i}{-k_0 - E(k) + i\epsilon}\sum_s v_s v_s^\dagger \beta, \\ u_s(\mathbf{k}) &= \left[\sqrt{\frac{1+S}{2}} + \sqrt{\frac{1-S}{2}}\hat{k}\cdot\vec{\sigma}\gamma_5\right]u_s(0), \\ v_s(\mathbf{k}) &= \left[\sqrt{\frac{1+S}{2}} - \sqrt{\frac{1-S}{2}}\hat{k}\cdot\vec{\sigma}\gamma_5\right]v_s(0) \\ &= -i\sigma_2\gamma_5 u_s^*(\mathbf{k}), \end{aligned} \quad (3)$$

where $S = \sin(\varphi) = \frac{m_c}{\sqrt{k^2+m_c^2}}$, $C = \cos(\varphi) = \frac{k}{\sqrt{k^2+m_c^2}}$ and φ is a chiral angle. In the noncondensed vacuum, φ is equal to $\arctan\frac{m_0}{k}$. In the physical vacuum, the constituent quark mass $m_c(k)$, or the chiral angle $\varphi(k) = \arctan\frac{m_c(k)}{k}$, is a variational function which is determined by the mass gap equation. We anticipate examples of solutions, for different light current quark masses m_0, depicted in Fig. 1.

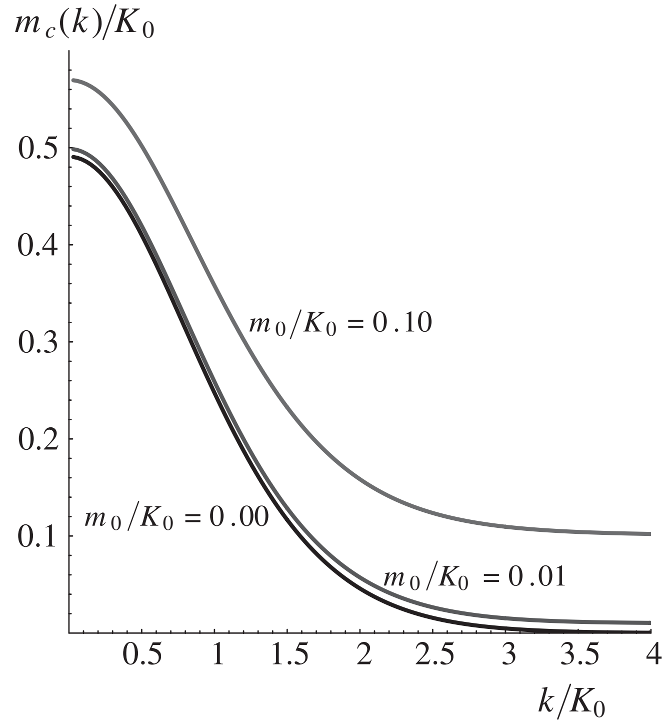

FIG. 1. The constituent quark masses $m_c(k)$, solutions of the mass gap equation, for different current quark masses m_0.

There are three equivalent methods to derive the mass gap equation for the true and stable vacuum, where constituent quarks acquire the constituent mass [19]. One method consists in assuming a quark-antiquark $^3P_0$ condensed vacuum, and in minimizing the vacuum energy density. A second method consists in rotating the quark and antiquark fields with a Bogoliubov-Valatin canonical transformation to diagonalize the terms in the Hamiltonian with two quark or antiquark second quantized fields. A third method consists in solving the Schwinger-Dyson equations for the propagators. Any of these methods lead to the same mass gap equation and quark dispersion relation. Here we replace the propagator of Eq. (3) in the Schwinger-Dyson equation,

$$\begin{aligned} 0 &= u_s^\dagger(k)\Big\{k\hat{k}\cdot\vec{\alpha} + m_0\beta - \int\frac{dw'}{2\pi}\frac{d^3k'}{(2\pi)^3}iV(k-k') \times \sum_{s'}\Big[\frac{u(k')_{s'}u^\dagger(k')_{s'}}{w' - E(k') + i\epsilon} - \frac{v(k')_{s'}v^\dagger(k')_{s'}}{-w' - E(k') + i\epsilon}\Big]\Big\}v_{s''}(k) \\ E(k) &= u_s^\dagger(k)\Big\{k\hat{k}\cdot\vec{\alpha} + m_0\beta - \int\frac{dw'}{2\pi}\frac{d^3k'}{(2\pi)^3}iV(k-k') \times \sum_{s'}\Big[\frac{u(k')_{s'}u^\dagger(k')_{s'}}{w' - E(k') + i\epsilon} - \frac{v(k')_{s'}v^\dagger(k')_{s'}}{-w' - E(k') + i\epsilon}\Big]\Big\}u_s(k), \end{aligned} \quad (4)$$

where, with the simple density-density harmonic interaction [14], the integral of the potential is a Laplacian and the mass gap equation and the quark energy are finally,

$$\Delta\varphi(k) = 2kS(k) - 2m_0C(k) - \frac{2S(k)C(k)}{k^2}$$
$$E(k) = kC(k) + m_0S(k) - \frac{\varphi'(k)^2}{2} - \frac{C(k)^2}{k^2} + \frac{U}{2}. \tag{5}$$

Numerically, this equation is a nonlinear ordinary differential equation. It can be solved with the Runge-Kutta and shooting method. Examples of solutions for the current quark mass $m_c(k) = k\tan\varphi$, for different current quark masses m_0, are depicted in Fig. 1.

The Salpeter-RPA equations for a meson (a color singlet quark-antiquark bound state) can be derived from the Lippman-Schwinger equations for a quark and an antiquark, or replacing the propagator of Eq. (3) in the Bethe-Salpeter equation. In either way, one gets [23]

$$\phi^+(k,P) = \frac{u^\dagger(k_1)\chi(k,P)v(k_2)}{+M(P) - E(k_1) - E(k_2)},$$
$$\phi^{-t}(k,P) = \frac{v^\dagger(k_1)\chi(k,P)u(k_2)}{-M(P) - E(k_1) - E(k_2)},$$
$$\chi(k,P) = \int \frac{d^3k'}{(2\pi)^3} V(k-k')[u(k_1')\phi^+(k',P)v^\dagger(k_2') + v(k_1')\phi^{-t}(k',P)u^\dagger(k_2')], \tag{6}$$

where $k_1 = k + \frac{P}{2}$, $k_2 = k - \frac{P}{2}$, and P is the total momentum of the meson.

The Salpeter-RPA equations of Bicudo *et al.* [15] and of Llanes-Estrada *et al.* [24] are obtained deriving the equation for the positive energy wave function ϕ^+ and for the negative energy wave function ϕ^-. Solving for χ, one gets the Salpeter equations of Le Yaouanc *et al.* [14]. This results in four potentials $V^{\alpha\beta}$, respectively, coupling $\nu^\alpha = r\phi^\alpha$ to ν^β, in the bound-state Salpeter equation,

$$\begin{cases} (2T + V^{++})\nu^+ + V^{+-}\nu^- = M\nu^+ \\ V^{-+}\nu^+ + (2T + V^{--})\nu^- = -M\nu^-. \end{cases} \tag{7}$$

The relativistic equal-time equations have twice as many coupled equations as the Schrödinger equation. The negative energy component ν^- is smaller than the positive energy component by a factor of the order of $1/M$ in units of $K_0 = 1$. Thus when M is large, and this is the case for most excited mesons, the negative energy components can be neglected and the Salpeter equation simplifies to a Schrödinger equation.

Importantly, the potentials $V^{++} = V^{--}$ and $V^{+-} = V^{-+}$ include the usual spin-tensor potentials [25], produced by the Pauli $\vec{\sigma}$ matrices in the spinors of Eq. (3). They are detailed explicitly in Table I. Because we are interested in highly excited states, where both $\langle r\rangle$ and $\langle k\rangle$ are large, we consider the limit where $\frac{m_c}{k} \to 0$. This implies that the potentials, used in Table I, $\varphi'(k) \to 0$, $\mathcal{C}(k) \to 1$, and $\mathcal{G}(k) = 1 - S(k) \to 1$. Then using the textbook matrix elements of the spin-tensor potentials, the bound-state Salpeter equation decouples into two different equations depending only on $\mathbf{J}$ and not explicitly on $\mathbf{L}$ or $\mathbf{S}$. Without the chiral degeneracy there would be four different resonances for each j, one with $s = 0$ and $j = l$ and three with $s = 1$ and $j = l-1, l, l+1$. With the chiral degeneracy we get only two different equations, one for $j \geq 0$ with

$$\begin{cases} 2T + V^{++} = -\frac{d^2}{dk^2} + 2k - \frac{1}{k^2} + \frac{j(j+1)}{k^2}, \\ V^{+-} = \frac{1}{k^2}, \end{cases} \tag{8}$$

and another for $j \geq 1$ with

$$\begin{cases} 2T + V^{++} = -\frac{d^2}{dk^2} + 2k - \frac{2}{k^2} + \frac{j(j+1)}{k^2}, \\ V^{+-} = \frac{0}{k^2}. \end{cases} \tag{9}$$

Thus states with different l and equal j, i.e. with different parity, are degenerate, and chiral symmetry is restored.

TABLE I. The positive and negative energy spin-independent, spin-spin, spin-orbit, and tensor potentials, computed exactly in the framework of the simple density-density harmonic model of Eq. (2). $\varphi'(k)$, $\mathcal{C}(k)$, and $\mathcal{G}(k) = 1 - S(k)$ are all functions of the constituent quark(antiquark) mass.

	$V^{++} = V^{--}$
Spin-independent	$-\frac{d^2}{dk^2} + \frac{\mathbf{L}^2}{k^2} + \frac{1}{4}(\varphi_q'^2 + \varphi_{\bar q}'^2) + \frac{1}{k^2}(\mathcal{G}_q + \mathcal{G}_{\bar q}) - U$
Spin-spin	$\frac{4}{3k^2}\mathcal{G}_q\mathcal{G}_{\bar q}\mathbf{S}_q \cdot \mathbf{S}_{\bar q}$
Spin-orbit	$\frac{1}{k^2}[(\mathcal{G}_q + \mathcal{G}_{\bar q})(\mathbf{S}_q + \mathbf{S}_{\bar q}) + (\mathcal{G}_q - \mathcal{G}_{\bar q})(\mathbf{S}_q - \mathbf{S}_{\bar q})] \cdot \mathbf{L}$
Tensor	$-\frac{2}{k^2}\mathcal{G}_q\mathcal{G}_{\bar q}[(\mathbf{S}_q \cdot \hat{k})(\mathbf{S}_{\bar q} \cdot \hat{k}) - \frac{1}{3}\mathbf{S}_q \cdot \mathbf{S}_{\bar q}]$
	$V^{+-} = V^{-+}$
Spin-independent	0
Spin-spin	$-\frac{4}{3}[\frac{1}{2}\varphi_q'\varphi_{\bar q}' + \frac{1}{k^2}\mathcal{C}_q\mathcal{C}_{\bar q}]\mathbf{S}_q \cdot \mathbf{S}_{\bar q}$
Spin-orbit	0
Tensor	$[-2\varphi_q'\varphi_{\bar q}' + \frac{2}{k^2}\mathcal{C}_q\mathcal{C}_{\bar q}][(\mathbf{S}_q \cdot \hat{k})(\mathbf{S}_{\bar q} \cdot \hat{k}) - \frac{1}{3}\mathbf{S}_q \cdot \mathbf{S}_{\bar q}]$

TABLE II. Masses of the first angular and radial excitations of the different light-light tachyons and mesons in the chiral limit of a vanishing quark mass m. Each column includes both positive and negative parity degenerate states, except for the pseudoscalar and scalar tachyonic states, which are simply avoided with a sufficiently large constituent quark mass. The meson masses are separated in two different families with the same J because two different Salpeter equations (8) and (9) exist for each J.

n	Pseudoscalar	Scalar	$j=1$	$j=1$	$j=2$	$j=2$	$j=3$	$j=3$
0	$\frac{2\times10^{-1}i}{m^2}$	$\frac{3\times10^{-2}i}{m^2}$	3.71	4.59	6.15	6.45	7.65	7.84
1	$\frac{2\times10^{-3}i}{m^2}$	$\frac{3\times10^{-4}i}{m^2}$	6.49	7.15	8.43	8.69	9.72	9.89
2	$\frac{2\times10^{-5}i}{m^2}$	$\frac{3\times10^{-6}i}{m^2}$	8.76	9.32	10.45	10.68	11.61	11.76
3	$\frac{2\times10^{-7}i}{m^2}$	$\frac{3\times10^{-8}i}{m^2}$	10.77	11.27	12.30	12.51	13.38	13.52
4	$\frac{2\times10^{-9}i}{m^2}$	$\frac{3\times10^{-10}i}{m^2}$	12.61	13.08	14.05	14.25	15.12	15.26

This chiral degeneracy applies to all angular momenta, except for $j=0$. In Table II we show the masses of the different light-light mesonic solutions of Eqs. (8) and (9), including tachyons, corresponding to the limit where $m_c \ll k$.

This limit is equivalent to simply allowing the quark mass to vanish, except in the $j=0$ case where technically a finite quark mass is necessary to avoid tachyons. The $j=0$ case is a subtle one, because neglecting the mass m_c is equivalent to considering the chiral limit, and this is equivalent to changing from the physical vacuum to the chiral invariant vacuum which is a false, unstable vacuum. A detailed inspection shows that the different potentials $-\frac{d^2}{dk^2}$, $2k$, $\frac{1}{k^2}$ are bound from below and positive definite in the sense that all their eigenvalues are positive. However $-\frac{1}{k^2}$ is unbound from below. It turns out that for $j=0$ all the solutions of Eq. (8), including all radial excitations, are tachyons [26], relevant for the structure of the chiral invariant false vacuum of QCD [27], corresponding to a different solution of the mass gap equation $m_c = 0$. Even when a very small regularizing quark mass m_c is assumed, constant for simplicity, the tachyons persist. This is confirmed by the numerical solutions of the regularized Salpeter equation, shown in Table II. Technically, in the $j=0$ case, it is necessary to rescale the momentum and mass,

$$k/m_c \to k', \qquad M{m_c}^2 \to M', \tag{10}$$

where any finite solution M' in fact corresponds to a large mass $M = M'/m_c^2$, and where a wave function with a finite k' corresponds to a wave function with small momentum $k = k' m_c$. A long time ago Le Yaouanc *et al.* [14] showed that in the chiral limit the pseudoscalar and the scalar possess tachyonic solutions. Very recently Bicudo showed that this number of tachyons is infinite [26]. Only with a finite m_c quark mass, do the scalar and pseudoscalar mesons have positive masses. But then the excited scalar and pseudoscalar states are not degenerate, in contradistinction with the chiral degeneracy of the excited mesons with $j > 0$.

For excited mesons with $j > 0$, the spectrum in Table II is very well approximated by the solutions of the pair of Schrödinger-like equations,

$$\left[-\frac{d^2}{dk^2} + 2k - \frac{1}{k^2} + \frac{j(j+1)}{k^2}\right]\nu(k) = M\nu(k), \tag{11}$$

$$\left[-\frac{d^2}{dk^2} + 2k - \frac{2}{k^2} + \frac{j(j+1)}{k^2}\right]\nu(k) = M\nu(k), \tag{12}$$

obtained when the negative energy component ν^- are neglected in Eqs. (8) and (9).

This result for a quadratic potential, together with the recent work of Wagenbrunn and Glozman [10] for a linear potential, where the highly excited spectra only depends explicitly on j and not on l, indicates that any chiral invariant potential should also show the same chiral degeneracy in the very excited spectra. Notice it is well known that several different chiral invariant potentials lead to chiral symmetry breaking in the vacuum and in the ground-state hadrons [28], now they all are expected to recover chiral degeneracy of the excited spectra.

III. LINEAR REGGE TRAJECTORIES AND SEMICLASSICAL QUANTIZATION WITH THE LINEAR POTENTIAL

Our simple results of Eqs. (13) and (15) are extended to a linear potential [24,29], to get the linear Regge trajectories. Both from the quark modeling of the heavy quarkonium, and from lattice QCD static potentials, it is suggested that the long range confining quark potential is linear. Notice that Szczepaniak and Swanson [30], in the Coulomb gauge, were able to derive from QCD the linear potential.

Continuing with the limit where $\frac{m}{k} \to 0$, and aiming at large total angular momenta j, we assume for the radial equation, the minimal extension of Eqs. (11) and (12) of Sec. II for a linear potential,

$$(2p + \sigma r)\nu = E\nu, \tag{13}$$

where, in momentum space, the position r is an operator,

$$\hat{r} = \sqrt{-\frac{d^2}{dp^2} + \frac{j(j+1)}{p^2}}, \tag{14}$$

neglecting now the terms $\frac{-1}{k^2}$ or $\frac{-2}{k^2}$. In configuration space, r is the c number and the operator is the momentum p, replaced in Eq. (13) by

$$\hat{p} = \sqrt{-\frac{d^2}{dr^2} + \frac{j(j+1)}{r^2}}. \tag{15}$$

Again, the usual centrifugal barrier for spinless particles is extended for fermions in the chiral restoration limit substituting $\mathbf{L}^2$ by $\mathbf{J}^2$, consistent with the result of Sec. II.

Logically, Eq. (13) together with Eq. (15) is the simplest possible configuration space equation for the linear potential in the limit of chiral symmetry restoration. Therefore this simple model should reproduce the results of Wagenbrunn and Glozman [10], except for a constant energy shift, since in Wagenbrunn and Glozman [10] the $j = 0$ pion is massless, while in Eq. (13) the $j = 0$ ground state clearly has a positive zero point energy.

Numerically, we simply have to solve a Salpeter equation (or Schrödinger equation with ultrarelativistic kinetic energy) except that, in the centrifugal barrier, the spherical angular momentum l is now replaced by the total angular momentum j. Equation (13) is solved with the method of the double diagonalization of the equal-time Hamiltonian. Using finite differences, first we diagonalize the bounded from below $\hat{p}$ operator $-\frac{d^2}{dr^2} + \frac{j(j+1)}{r^2}$. Then we apply the square root, well defined in this diagonal basis. After returning to the original position space basis, we diagonalize the full Hamiltonian. This provides us automatically with the full spectrum including the angularly and radially excited states.

The results are shown in Table III. In Figs. 2 and 3 we graphically demonstrate that the angular excitations and the radial excitations of this simple spectrum are disposed in linear Regge trajectories,

$$j \simeq \alpha_0 + \alpha M^2, \qquad n \simeq \beta_0 + \beta M^2. \tag{16}$$

This agrees qualitatively with the experimental spectrum [1], where the linear Regge trajectories are also present. As anticipated, in the limit of large excitations, our trajectories are also parallel to the ones of Wagenbrunn and Glozman [10].

Interestingly, with the semiclassical Bohr-Sommerfeld quantization relation,

$$\oint p dq \simeq nh, \tag{17}$$

and with the energy $2pc + \sigma r$ of Eq. (13) the linear trajectories can also be derived. The linear trajectories for the angular excitations can be derived from the circular classical orbits,

$$\begin{cases} 2(2\pi\frac{r}{2}p) \simeq lh \\ \sigma = \frac{c}{\frac{r}{2}}p \end{cases} \Rightarrow l \simeq \frac{1}{8\sigma c\hbar}E^2, \tag{18}$$

where we also used the centripetal acceleration. The linear trajectories corresponding to radial excitations can be derived [31] from the linear classical orbits with $\mathbf{L} = 0$,

$$\int_{-E/\sigma}^{E/\sigma} \frac{E - \sigma|r|}{2c} dr \simeq nh \Rightarrow n \simeq \frac{1}{4\pi\sigma c\hbar}E^2. \tag{19}$$

Thus, in our units of $\hbar = c = \sigma = 1$ we get for the Regge slopes, respectively,

$$\alpha = \frac{1}{8}, \qquad \beta = \frac{1}{4\pi}, \tag{20}$$

TABLE III. Masses of the light-light mesons, in dimensionless units of $\sigma = 1$, computed with the ultrarelativistic equal-time chiral degenerate Schrödinger equation (13). The $j = 0$ mesons are distant from the experimental spectrum, but chiral degeneracy is theoretically plausible for the very excited mesons.

j	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$
0	3.16	4.71	5.89	6.87	7.73	8.51	9.21	9.87	10.49
1	4.22	5.46	6.48	7.38	8.17	8.90	9.58	10.21	10.81
2	5.08	6.13	7.05	7.87	8.61	9.30	9.95	10.56	11.13
3	5.81	6.74	7.58	8.34	9.04	9.70	10.31	10.90	11.45
4	6.46	7.31	8.08	8.79	9.45	10.08	10.67	11.23	11.77
5	7.05	7.83	8.55	9.22	9.86	10.45	11.02	11.56	12.08
6	7.60	8.32	9.00	9.64	10.24	10.82	11.36	11.89	12.39
7	8.11	8.79	9.43	10.04	10.62	11.17	11.70	12.21	12.70
8	8.59	9.23	9.84	10.43	10.98	11.51	12.03	12.52	12.99
9	9.04	9.65	10.24	10.80	11.33	11.85	12.34	12.82	13.29
10	9.47	10.06	10.62	11.16	11.68	12.18	12.66	13.12	13.58
11	9.88	10.45	10.99	11.51	12.01	12.49	12.96	13.42	13.86
12	10.28	10.82	11.35	11.85	12.34	12.81	13.26	13.71	14.14
13	10.66	11.19	11.69	12.18	12.65	13.11	13.56	13.99	14.41

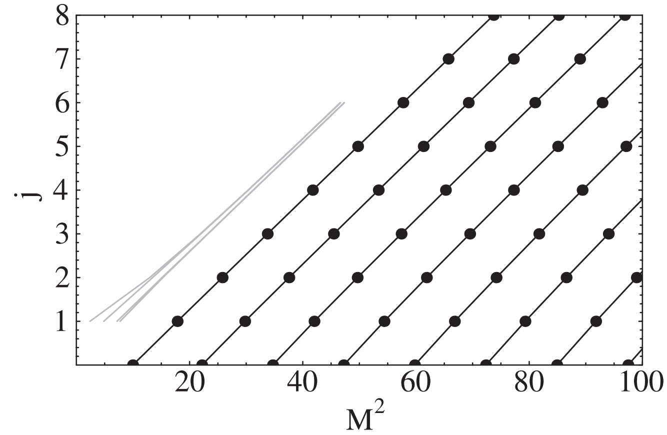

FIG. 2. We show the quasilinear Regge trajectories, of j as a function of M^2. Each line corresponds to a fixed radial n, increasing from left to right. The M are the masses of the light-light mesons, in dimensionless units of $\sigma = 1$, computed with the ultrarelativistic equal-time chiral degenerate Schrödinger equation (13). In gray we also show the trajectories with $n = 0$ and starting at $j = 1$ of Ref. [10].

in excellent agreement with the slopes of Figs. 2 and 3, and with the slopes of the recent Bethe-Salpeter calculation of Wagenbrunn and Glozman [10].

The important point we want to stress is that we find quantitative discrepancies with the experiment, although our theoretical results show linear Regge trajectories and comply the chiral degeneracy. In particular, experimentally, the radial and angular slopes defined in Eq. (16) should be almost identical [1],

$$\alpha_{\mathrm{exp}} = 0.877\ \mathrm{GeV}^{-2}, \qquad \beta_{\mathrm{exp}} = 0.855\ \mathrm{GeV}^{-2}, \quad (21)$$

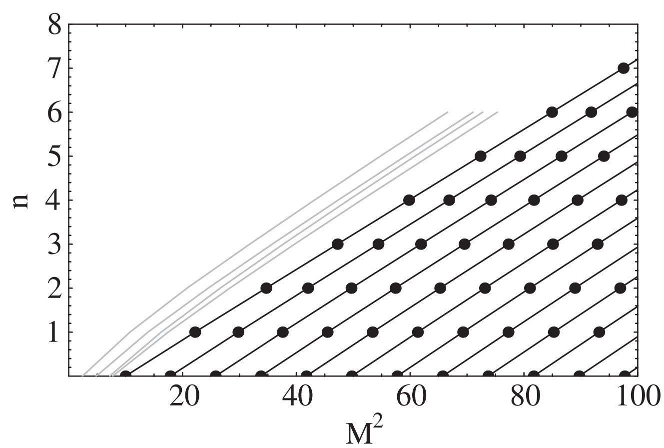

FIG. 3. We show we show the quasilinear Regge trajectories, of n as a function of M^2. Each line corresponds to a fixed angular j, increasing from left to right. The M are the masses of the light-light mesons, in dimensionless units of $\sigma = 1$, computed with the ultrarelativistic equal-time chiral degenerate Schrödinger equation (13). In gray we also show the trajectories with $j = 1$ of Ref. [10].

while our theoretical slopes, depicted in Figs. 2 and 3 and semiclassically computed in Eq. (20), differ by $\pi/2$. Moreover, the meson masses in Table III do not exactly comply with the large degeneracy emerging in the observations of Bugg. Identical slopes would be necessary to reproduce this large experimental degeneracy.

IV. USING CLASSICAL CLOSED ORBITS TO SEARCH FOR A POTENTIAL WITH A PRINCIPAL QUANTUM NUMBER

To better model the large degeneracy, we enlarge our class of timelike central potentials, searching for the best possible potential of this class.

Actually the model of Sec. III, with a linear potential only, is oversimplified. To fit the correct positive intercepts α_0 and β_0 of Eq. (16), it is standard in quark models to include in the potential a constant negative energy shift and a negative short range Coulomb potential. Moreover the result of Sec. II indicates that any chiral invariant potential should comply with the chiral degeneracy in the very excited spectra.

To try to solve the large degeneracy problem, it is then natural to extend our simple linear potential σr of Eq. (13) to a wider class of hadronic potentials. We ask for a spectrum with a principal quantum number, similar to the nonrelativistic spectra of the Coulomb potential or of the harmonic oscillator potential, where the principal quantum numbers are, respectively, $n + l + 1$ and $2n + l + \frac{3}{2}$. The difference here is that our kinetic energy is the ultrarelativistic one $T = 2pc$ whereas in the nonrelativistic case $T = \frac{p^2}{2\mu}$.

Notice that in the nonrelativistic case, classical closed orbits coincide with a quantum principal number. We can also address this problem searching for classical closed orbits with an ultrarelativistic kinetic energy. When all the classical orbits are closed then the Hermann-Jacobi-Laplace-Runge-Lenz vector is conserved, because this vector does not precess. In the Hamilton-Jacobi formalism, this vector commutes with the Hamiltonian. The same commutation then also occurs in the quantum Schrödinger formalism, the formalism we are using now. Then a larger symmetry group, including the angular momentum and the Hermann-Jacobi-Laplace-Runge-Lenz vector exists. Finally this implies that a principal quantum number exists.

Thus, rather than solving the ultrarelativistic Schrödinger equation for all the possible different central potentials (there are infinite possible different potentials), we prefer to extend the Bertrand theorem techniques to the search of classical closed orbits with the ultrarelativistic kinetic energy. For simplicity we consider a kinetic energy $T = pc$ and a general potential $V(r)$, used for a single particle in a central potential, comparable to our two-body problem in the center of mass frame.

Let us consider classical planar trajectories for an ultra-relativistic quark with the speed of light, and with momentum,

$$\mathbf{p} = \frac{p}{c}\mathbf{v} = \frac{p}{c}\dot{r}\hat{e}_r + \frac{p}{c}r\dot{\theta}\hat{e}_\theta, \tag{22}$$

subject to a central force,

$$\mathbf{F} = -\frac{d}{dr}V\hat{e}_r, \tag{23}$$

where the notation is obvious. In the plane we have two constants, the angular momentum L and the energy E,

$$\mathbf{L} = \frac{p}{c}r^2\dot{\theta}\hat{e}_\perp, \qquad E = pc + V(r). \tag{24}$$

Then Newton's law produces the equation for the radius as a function of time,

$$\left(\frac{E-V}{c^2}\dot{r}\right)^{\cdot} - \frac{L^2}{\frac{E-V}{c^2}r^3} = -\frac{d}{dr}V. \tag{25}$$

To study the condition to get closed orbits it is convenient to replace the variable time t by the polar coordinate θ and the function r by its inverse $u = 1/r$. Then the equation simplifies to

$$\begin{aligned}&\frac{d^2}{d\theta^2}u + u = J(u),\\ &J(u) = -\frac{E - V(1/u)}{c^2L^2}\frac{d}{du}V(1/u).\end{aligned} \tag{26}$$

To get the nonrelativistic case one only has to replace in the right-hand side of Eq. (26) the factor $\frac{E-V(1/u)}{c^2} \to m$. Thus the ultrarelativistic equation has two independent constants E and cL while the classical case has only one constant m/L.

The theorem of Bertrand can be addressed considering a trajectory u close to the circular trajectory u_0,

$$u = u_0 + \eta, \tag{27}$$

where we fix the angular momentum L and the trajectory is defined by the inverse radius u and derivative at $\theta = 0$ and by the energy E. Defining,

$$\beta^2 = 1 - \frac{d}{du}J|_{u=u_0}, \tag{28}$$

we get, to leading order in η,

$$\frac{d^2\eta}{d\theta^2} + \beta^2\eta = 0, \qquad \eta = h_1\cos\beta(\theta - \theta_0). \tag{29}$$

Thus for very small perturbations to the circular orbit, the condition for a closed orbit is that β is an integer number (if we want the orbit to close right after one turn) or rational (if we want it to close after a finite number of turns). Importantly this implies that β is constant, since it cannot change continuously from one orbit with u_0 to the next one. This restricts the class of possible potentials,

$$\frac{d}{du}J|_{u=u_0} = 1 - \beta^2 = \frac{f^2(\frac{1}{u_0})}{c^2L^2{u_0}^2} - 2 + \frac{u_0}{f_0}\frac{df(\frac{1}{u_0})}{du_0}, \tag{30}$$

and the problem here is that, unlike in the nonrelativistic limit, the equation still depends on the parameter cL, and thus the closed orbits are possible, but there is no potential for which all orbits are closed, since the closing depends on cL.

In the nonrelativistic case it is well known that this problem has two solutions because the first term in the right-hand side is absent. Then the solutions of Eq. (30), producing closed orbits close to the circular one, are the power law potential with $V(r)\alpha r^{\beta^2-2}$. For instance the natural β correspond to the powers -1, 2, 7, 14 ... For more general orbits, we can go up to the third order in the Fourier series for η,

$$\eta = h_0 + h_1\cos\beta\theta + h_2\cos2\beta\theta + h_3\cos3\beta\theta \tag{31}$$

and this already produces Bertrand's theorem, stating that the closed orbit condition is

$$\beta^2(\beta^2 - 1)(\beta^2 - 4) = 0. \tag{32}$$

This includes only the Coulomb and the harmonic oscillator cases, which indeed have all orbits simply closed for a nonrelativistic kinetic energy.

Again, for an ultrarelativistic kinetic energy there is no potential with all orbits closed.

V. CONCLUSION

Here we study possible quark models for the large degeneracy in the experimental meson spectra reported by Bugg [1].

We start with a semirelativistic chiral invariant quark model, with relativistic kinetic energy, with negative energy components, but with an instantaneous potential. For excited states all the different spin-tensor potentials [32–34] merge and we arrive at a Schrödinger-like ultrarelativistic potential quark model with a simple $\mathbf{J}^2$ dependence, leading to the chiral degeneracy of the excited spectra. Using this Schrödinger-like quark model, it is conveniently simple to show that the linear potential, well known for a spectrum with linear Regge trajectories, fails to reproduce the large degeneracy.

The most ambitious approach to the large degeneracy consists in asking for a model with a principal quantum number. Here we apply the equivalent method of searching for classical closed orbits. Unfortunately we find that this large symmetry does not exist, neither for the linear potential nor for any other central potential, in our ultrarelativistic and instantaneous framework.

Another possible approach is the one of an approximate large degeneracy. The linear and radial excitations are so regular in the framework of the ultrarelativistic and equal-time quark model that approximate patterns occur in the spectrum. Notice that in Eq. (20), in Figs. 2 and 3, and in

Table III, an increment of 3 in j is approximately equivalent to an increment of 2 in n. For instance the meson with $j = 5$, $n = 0$ has a similar mass to the one with $j = 2$, $n = 3$. Notice that the $j = 5$ state may have $l = 4$, 5, or 6, while the $j = 2$ state may have $l = 1$, 1, or 3. We get an approximate but relatively large degeneracy of states with different l ranging from 1 to 6. To agree with the experiment, we only need to have an increment in n much closer to the increment in j.

Thus, both for the principal quantum number, and for an approximate degeneracy, a departure from the ultrarelativistic and equal-time quark model is necessary. It is plausible that either retardation, string effects, or coupled channel effects should improve the theoretical model. For instance Morgunov *et al.* [35] already showed that including the angular momentum of the rotating linear confining string changes the slopes of the angular Regge trajectories. Moreover, if one could succeed to include retardation properly in the confining quark model, then one might search again for the possible existence of a principal quantum number. And coupled channels might also affect differently the angular and radial excitations of the spectrum.

In any case the large degeneracy apparent in the data analysis of Bugg, supported by the solid pattern of linear angular and radial Regge trajectories, remains quite plausible from a theoretical perspective. More experimental data on the remarkable patterns of excited resonances are welcome.

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