IOP PUBLISHING JOURNAL OF PHYSICS G: NUCLEAR AND PARTICLE PHYSICS

J. Phys. G: Nucl. Part. Phys. **34** (2007) 1757–1769 doi:10.1088/0954-3899/34/7/014

A three-parameter model for the neutrino mass matrix

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Received 23 February 2007
Published 6 June 2007
Online at stacks.iop.org/JPhysG/34/1757

Abstract

Using the type-II seesaw mechanism with three Higgs doublets ϕ_α ($\alpha = e, \mu, \tau$) and four Higgs triplets, we build a model for lepton mixing based on a 384-element horizontal symmetry group, generated by the permutation group S_3 and by six $\mathbb{Z}_2$ transformations. The charged-lepton mass matrix is diagonal; the symmetries of the model would require all the three masses m_α to be equal, but different vacuum expectation values of ϕ_α allow m_α to split. The number of parameters in the Majorana neutrino mass matrix $\mathcal{M}_\nu$ depends on two options: full breaking of the permutation group S_3, or leaving a μ–τ interchange symmetry intact, and hard or spontaneous violation of CP. We discuss in detail the case with the minimal number of three parameters, wherein $\mathcal{M}_\nu$ is real, symmetric under μ–τ interchange and has equal diagonal elements. In that case, CP is conserved in lepton mixing, atmospheric neutrino mixing is maximal and $\theta_{13} = 0$; moreover, the type of neutrino mass spectrum and the absolute neutrino mass scale are sensitive functions of the solar mixing angle.

1. Introduction

There are two puzzles associated with neutrinos: why are their masses so much smaller than those of the charged fermions and why does the lepton mixing matrix feature large mixing angles—for reviews see [1]—in contrast to the quark mixing matrix. It is possible that both puzzles are solved through the same mechanism. In this paper, we envisage the type-II seesaw mechanism as a possible solution[3]. We use horizontal symmetries to enforce certain features of lepton mixing, in particular maximal atmospheric neutrino mixing and $\theta_{13} = 0$. In order to achieve this, we enlarge the scalar sector of the Standard Model by adding to it four Higgs triplets and by using altogether three Higgs doublets. Our model has a permutation group S_3 together with six cyclic symmetries $\mathbb{Z}_2$, which commute with each other but not with S_3; the

[3] In our model we do not allow for a type-I seesaw mechanism; we assume right-handed neutrino singlets not to exist.

result is a large discrete symmetry group with 384 elements. This setting allows us to obtain four different neutrino mass matrices, depending on the assumed breaking of the horizontal symmetries and of the symmetry CP. Amazingly, by breaking the horizontal symmetries softly by terms of dimension 2, while leaving a residual μ–τ interchange symmetry to be broken at low energy in the charged-lepton sector, we arrive at a *viable* neutrino mass matrix with *only three real parameters*.

In section 2 we make a general discussion of the type-II seesaw mechanism with an arbitrary number of Higgs doublets and triplets. Our model, with its multiplets, symmetries and Lagrangian, is explained in section 3. In section 4, we investigate in detail the most predictive case of a three-parameter neutrino mass matrix. A generalization thereof is considered in section 5. The conclusions are presented in section 6.

2. The type-II seesaw mechanism

We first review the type-II seesaw mechanism [2, 3] for small neutrino masses. We assume the existence—in the electroweak theory—of several Higgs doublets ϕ_α with hypercharge $1/2$, together with several Higgs triplets Δ_i with hypercharge 1. Let the neutral components of ϕ_α have vacuum expectation values (VEVs) v_α and the neutral components of Δ_i have VEVs δ_i. Just because of the hypercharge symmetry, the vacuum potential V_0 must be of the form[4]

$$V_0 = \left(\mu_\phi^2\right)_{\alpha\beta} v_\alpha^* v_\beta + \left(\mu_\Delta^2\right)_{ij} \delta_i^* \delta_j + \left(t_{i\alpha\beta} \delta_i^* v_\alpha v_\beta + \text{c.c.}\right) + \lambda_{\alpha\beta\gamma\delta} v_\alpha^* v_\beta v_\gamma^* v_\delta + \lambda_{ijkl} \delta_i^* \delta_j \delta_k^* \delta_l + \lambda_{\alpha\beta ij} v_\alpha^* v_\beta \delta_i^* \delta_j. \tag{1}$$

The matrices μ_ϕ^2 and μ_Δ^2 are Hermitian and, likewise, the λ coefficients must obey various conditions in order that V_0 should be real. The VEVs of the triplets are determined by

$$0 = \frac{\partial V_0}{\partial \delta_i^*} = \left(\mu_\Delta^2\right)_{ij} \delta_j + t_{i\alpha\beta} v_\alpha v_\beta + 2\lambda_{ijkl} \delta_j \delta_k^* \delta_l + \lambda_{\alpha\beta ij} v_\alpha^* v_\beta \delta_j. \tag{2}$$

In contrast to μ_ϕ^2, we assume the matrix μ_Δ^2 to be positive definite so that, in the absence of the $t_{i\alpha\beta}$ terms, the only solution to equation (2) would be for all δ_i to vanish. The VEVs v_α are of order of the electroweak scale $v \approx 174$ GeV, or smaller. Assuming $t_{i\alpha\beta}$ to be of order M and the eigenvalues of μ_Δ^2 to be of order M^2, where M is a mass scale much larger than v [3], the approximate solution to equation (2) is given by [5]

$$\delta_i \approx -\left(\mu_\Delta^2\right)_{ij}^{-1} t_{j\alpha\beta} v_\alpha v_\beta. \tag{3}$$

From equation (3), δ_i are of order $v^2/M \ll v$. If, furthermore, all the λ coefficients are of order unity or smaller, then the approximate solution (3) will be corrected on its right-hand side only by terms suppressed by a factor $v^2/M^2 \ll 1$.

Under an $SU(2)$ gauge transformation, the left-handed lepton doublets $D_{L\alpha}$ transform as $D_{L\alpha} \to W D_{L\alpha}$ while the Higgs triplets transform as $\Delta_i \to W \Delta_i W^\dagger$, where W is an $SU(2)$ matrix. Therefore, the Higgs triplets have Yukawa couplings of the form $D_{L\alpha}^T C^{-1} \varepsilon \Delta_i D_{L\beta}$, where C is the charge-conjugation matrix in Dirac space and ε is the 2×2 antisymmetric matrix in gauge-$SU(2)$ space. The VEVs δ_i being very small, the above Yukawa couplings generate very small neutrino mass terms $\delta_i \nu_{L\alpha}^T C^{-1} \nu_{L\beta}$, of order v^2/M times a typical Yukawa-coupling constant. The neutrino masses being of order 0.1 eV, M could easily be of order 10^{14} GeV [3], thus fully justifying the approximate solution (3).

[4] We use the summation convention.

3. The model

Our model follows closely, in the symmetries that it utilizes, a previous model of ours [6]. We have three left-handed lepton doublets $D_{L\alpha}$, three right-handed charged-lepton singlets α_R and three Higgs doublets ϕ_α $(\alpha = e, \mu, \tau)$[5]. There are four Higgs triplets, Δ_α and Δ_4.[6] The symmetries of the model consist of a permutation group S_3 acting simultaneously on all indices α, three $\mathbb{Z}_2$ symmetries

$$\mathbf{z}_\alpha^{(1)}: \quad \phi_\alpha \to -\phi_\alpha, \qquad \alpha_R \to -\alpha_R \tag{4}$$

and another three $\mathbb{Z}_2$ symmetries

$$\mathbf{z}_\alpha^{(2)}: \quad D_{L\alpha} \to -D_{L\alpha}, \qquad \alpha_R \to -\alpha_R \qquad \text{and} \qquad \Delta_\beta \to -\Delta_\beta \quad \text{iff} \quad \beta \neq \alpha. \tag{5}$$

Note that Δ_4 is invariant under all these symmetries. In appendix A we make a study of the full symmetry group of our model.

The Yukawa Lagrangian invariant under all these symmetries is

$$\begin{aligned}\mathcal{L}_{\text{Yukawa}} = &-y_0 \bar{D}_{L\alpha}\phi_\alpha\alpha_R + \tfrac{1}{2} y_1 D_{L\alpha}^T C^{-1}\varepsilon\Delta_4 D_{L\alpha} \\ &+ y_2\left(D_{Le}^T C^{-1}\varepsilon\Delta_\mu D_{L\tau} + D_{L\mu}^T C^{-1}\varepsilon\Delta_\tau D_{Le} + D_{L\tau}^T C^{-1}\varepsilon\Delta_e D_{L\mu}\right) + \text{h.c.}\end{aligned} \tag{6}$$

Thus, the charged-lepton mass matrix is automatically diagonal, the charged lepton α having mass $m_\alpha = |y_0 v_\alpha|$. On the other hand, the neutrino mass matrix is

$$\mathcal{M}_\nu = \begin{pmatrix} y_1\delta_4 & y_2\delta_\tau & y_2\delta_\mu \\ y_2\delta_\tau & y_1\delta_4 & y_2\delta_e \\ y_2\delta_\mu & y_2\delta_e & y_1\delta_4 \end{pmatrix}, \tag{7}$$

all its diagonal matrix elements being equal.

Due to the symmetries of our model, the coupling constants $t_{i\alpha\beta}$ of the previous section assume the very simple form

$$t_{i\alpha\beta} = t\delta_{i4}\delta_{\alpha\beta}. \tag{8}$$

Hence, from equation (3),

$$\delta_i = -t v_\alpha v_\alpha \left(\mu_\Delta^2\right)_{i4}^{-1}. \tag{9}$$

Ordering the triplet fields as $(\Delta_e, \Delta_\mu, \Delta_\tau, \Delta_4)$, the symmetries of our model would enforce

$$\mu_\Delta^2 = \text{diag}\left(\mu_1^2, \mu_1^2, \mu_1^2, \mu_2^2\right), \tag{10}$$

which is not satisfactory since it would lead, through equation (9), to $\delta_e = \delta_\mu = \delta_\tau = 0$. We must have $\left(\mu_\Delta^2\right)_{i4}^{-1} \neq 0$ for $i = e, \mu, \tau$. In order to solve this problem, we assume the symmetries of the model to be broken softly, only by terms of dimension 2. Without any residual symmetry, this means that both matrices μ_Δ^2 and μ_ϕ^2 become fully general, while all other couplings remain unchanged.

However, in order to simplify our model and render it more predictive, we may assume that the interchange symmetry $\mu \leftrightarrow \tau$ [8–11], which is a subgroup of our permutation group S_3, is kept unbroken in μ_Δ^2 and μ_ϕ^2. Then,

$$\left(\mu_\Delta^2\right)^{-1} = \begin{pmatrix} a & b & b & c \\ b^* & d & e & f \\ b^* & e & d & f \\ c^* & f^* & f^* & g \end{pmatrix} \tag{11}$$

[5] Constraints on multi-Higgs doublet models from electroweak precision tests are not very stringent; Higgs bosons with large ZZ couplings must have an average mass in the range allowed for the mass of the Standard Model Higgs boson [4].

[6] The scalar content of our model resembles that of the A_4 model of [7]. However, in that model, three gauge triplets are used instead of our Δ_4.

(a, d, e and g are real), so that $\delta_e = -tcv_\alpha v_\alpha$, $\delta_\mu = \delta_\tau = -tfv_\alpha v_\alpha$ and the neutrino mass matrix is μ–τ symmetric. This immediately leads to the predictions $\theta_{23} = \pi/4$ and $\theta_{13} = 0$. The μ–τ interchange symmetry is supposed to be spontaneously broken through the VEVs of the Higgs doublets: $v_\mu \neq v_\tau$. The Higgs potential is rich enough to allow for this outcome; in appendix B we demonstrate this by working out a simplified case. Of course, the spontaneous breaking at the electroweak scale of the μ–τ interchange symmetry will seep, through radiative corrections, into the rest of the theory, so that at loop level the matrix $\left(\mu_\Delta^2\right)^{-1}$ will not any more be of the form in equation (11), and then $\delta_\mu \neq \delta_\tau$. But, both because this is a loop effect and because it is a correction of order of the ratio of the electroweak scale to the much larger mass terms in μ_Δ^2, we may expect $\delta_\mu - \delta_\tau$ to remain negligible.

In a further simplification of our model, we may also assume the CP violation to be spontaneous: the matrix $\left(\mu_\Delta^2\right)^{-1}$ is then real, but the VEVs v_α display non-trivial relative phases. Then $\delta_e, \delta_\mu, \delta_\tau$ and δ_4 will all have the same phase—the phase of $v_\alpha v_\alpha$. This phase may be rephased away from $\mathcal{M}_\nu$, so that the neutrino mass matrix becomes real. Thus, spontaneous CP breaking in our model yields the remarkable outcome that even though there is CP violation, it remains absent from the mass matrices and from lepton mixing[7].

One thus obtains the following four possibilities.

(i) The general case, in which CP violation is hard and μ–τ symmetry is allowed to be broken in μ_Δ^2. Then,

$$\mathcal{M}_\nu = \begin{pmatrix} m & p\,\mathrm{e}^{\mathrm{i}\psi} & q\,\mathrm{e}^{\mathrm{i}\chi} \\ p\,\mathrm{e}^{\mathrm{i}\psi} & m & r\,\mathrm{e}^{\mathrm{i}\rho} \\ q\,\mathrm{e}^{\mathrm{i}\chi} & r\,\mathrm{e}^{\mathrm{i}\rho} & m \end{pmatrix}, \tag{12}$$

with real m, p, q and r. This case should not be very predictive, since it has seven parameters to predict nine observables—three neutrino masses, three lepton mixing angles, one CKM-type phase and two Majorana phases.

(ii) The case in which μ–τ symmetry is allowed to be broken in μ_Δ^2 but CP violation is spontaneous. Then ψ, χ and ρ in (12) vanish. There is no CP violation in lepton mixing. The four parameters m, p, q and r allow one to predict six observables—three neutrino masses and three lepton mixing angles.

(iii) The case in which the μ–τ interchange symmetry is preserved in μ_Δ^2, while the CP violation is allowed to be hard. Then,

$$\mathcal{M}_\nu = \begin{pmatrix} x & y & y \\ y & x & w \\ y & w & x \end{pmatrix}, \tag{13}$$

with complex parameters x, y and w. There are in this case five parameters—three moduli and two phases.

(iv) The most predictive case, in which the CP violation is spontaneous and μ–τ interchange symmetry is preserved down to the electroweak scale. The neutrino mass matrix is the one in equation (13), but with real x, y and w. The neutrino mass matrix has only three parameters.

4. The three-parameter neutrino mass matrix

In this section we concentrate on case (iv) of the previous section, i.e. on the neutrino mass matrix of equation (13) with real x, y and w. The algebra of the diagonalization of a general

[7] This is not an original situation; in the classical Branco model of CP violation [12], spontaneous CP breaking also does not find a way into the quark mixing matrix.

μ–τ symmetric neutrino mass matrix has been worked out in [9], and we only need to adapt it to the simpler case (iv). In the following, the solar mixing angle—which is defined to be in the first quadrant—is denoted by θ, the neutrino masses are $m_{1,2,3}$, the solar mass-squared difference is $\Delta m^2_\odot = m_2^2 - m_1^2 > 0$ and the atmospheric mass-squared difference is

$$\Delta m^2_{\text{atm}} = \left| m_3^2 - \frac{m_1^2 + m_2^2}{2} \right| = \epsilon \left(m_3^2 - \frac{m_1^2 + m_2^2}{2} \right), \tag{14}$$

where $\epsilon = +1$ indicates a normal neutrino mass ordering and $\epsilon = -1$ an inverted ordering.

Equations (3.9)–(3.11) and (3.15) of [9] yield, respectively,

$$m_3 = |x - w|, \tag{15}$$

$$m^2_{1,2} = \frac{x^2 + 4y^2 + (x+w)^2 \mp \Delta m^2_\odot}{2}, \tag{16}$$

$$\tan 2\theta = \frac{2\sqrt{2}|y|}{w} \operatorname{sign}(2x + w), \tag{17}$$

$$\Delta m^2_\odot \cos 2\theta = w(2x + w). \tag{18}$$

Experimentally, we know that θ is in the first octant. Hence, $\tan 2\theta > 0$ and equations (17) and (18) both give

$$\operatorname{sign}(2x + w) = \operatorname{sign} w. \tag{19}$$

Again from equations (17) and (18),

$$|y| = \frac{|w| \tan 2\theta}{2\sqrt{2}}, \tag{20}$$

$$x = \frac{\Delta m^2_\odot \cos 2\theta - w^2}{2w}. \tag{21}$$

From equations (14)–(16), we find

$$\epsilon \Delta m^2_{\text{atm}} = \frac{w^2}{2} - 3xw - 2y^2. \tag{22}$$

Inserting equations (20) and (21) into equation (22), we obtain the value of w:

$$w^2 = \frac{4\epsilon \Delta m^2_{\text{atm}} + 6\Delta m^2_\odot \cos 2\theta}{8 - \tan^2 2\theta}. \tag{23}$$

Since $\Delta m^2_{\text{atm}} \gg \Delta m^2_\odot$, the numerator of equation (23) has the sign of ϵ; hence, its denominator must also have the sign of ϵ. This denominator vanishes when $\sin^2\theta = 1/3$, i.e. when θ is just the Harrison–Perkins–Scott (HPS) solar mixing angle [10]. We thus conclude that, in our model,

- if the neutrino mass spectrum is normal, then the solar mixing angle is smaller than its HPS value;
- if the neutrino mass spectrum is inverted, then $\sin^2\theta > 1/3$.

This remarkable result relates the type of neutrino mass spectrum to the value of the solar mixing angle.

From equations (15), (16), (20) and (21),

$$m_{1,2} = \frac{1}{2}\left(\frac{|w|}{\cos 2\theta} \mp \frac{\Delta m^2_\odot \cos 2\theta}{|w|} \right) \quad \text{and} \quad m_3 = \frac{3|w|}{2} - \frac{\Delta m^2_\odot \cos 2\theta}{2|w|}. \tag{24}$$

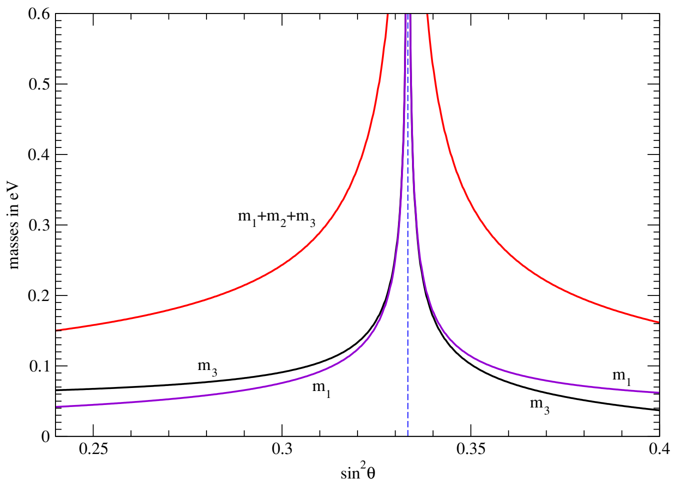

Figure 1. Plot of m_1, m_3 and $\sum_i m_i$ as a function of $\sin^2\theta$ in our three-parameter model. The values of $\Delta m^2_{\mathrm{atm}}$ and $\Delta m^2_\odot$ have been fixed at 2.5×10^{-3} and 7.9×10^{-5}, respectively, in eV^2. The dashed vertical line indicates the singularity at $\sin^2\theta = 1/3$.

(This figure is in colour only in the electronic version)

This gives the overall scale of the neutrino masses. Since $|w|$ diverges when $\tan^2 2\theta \to 8$, we see that in our model the neutrino mass spectrum becomes quasi-degenerate when the solar mixing angle approaches its Harrison–Perkins–Scott value.

One easily sees the reason why our model displays a singularity when $\sin^2\theta = 1/3$. The most general neutrino mass matrix leading to HPS lepton mixing is

$$\mathcal{M}_\nu = \begin{pmatrix} x & y & y \\ y & x+u & y-u \\ y & y-u & x+u \end{pmatrix}. \tag{25}$$

Our $\mathcal{M}_\nu$ in equation (13) has equal diagonal matrix elements. Hence, if it were to accept $\sin^2\theta = 1/3$, it would have to correspond to $u = 0$ in equation (25). But $\mathcal{M}_\nu$ of equation (25) with $u = 0$ leads to two equal neutrino masses; hence it is unrealistic. There is, therefore, a contradiction with experiment in the assumption that our $\mathcal{M}_\nu$ of equation (13) might be compatible with HPS mixing.

Experimentally, $\sin^2\theta$ is close to 1/3; therefore, there is the danger that our neutrino masses are too large and saturate the cosmological bound [13]. As a numerical exercise, we take the 1σ bound on solar mixing from [14]

$$0.27 < \sin^2\theta < 0.32 \Leftrightarrow 3.73 < \tan^2 2\theta < 6.72. \tag{26}$$

The mean value of θ is given by $\sin^2\theta = 0.30$ and $\tan^2 2\theta = 5.25$. Note that the upper 2σ limit $\sin^2\theta = 0.36$ gives $\tan^2 2\theta = 11.76$, which is already significantly larger than 8. Thus, there is experimentally ample room for the neutrino masses to be sufficiently small. This happens because $\tan^2 2\theta$ is a rapidly varying function of θ.

In figure 1 we have plotted m_1, m_3 and $m_1 + m_2 + m_3$ against $\sin^2\theta$ in our model. We have used the best-fit values $\Delta m^2_{\mathrm{atm}} = 2.5 \times 10^{-3}\ \mathrm{eV}^2$ and $\Delta m^2_\odot = 7.9 \times 10^{-5}\ \mathrm{eV}^2$ from [14]; for θ we have used the 3σ bounds $0.24 < \sin^2\theta < 0.40$, from the same source[8].

[8] A different model in which the neutrino mass spectrum is normal or inverted depending on whether $\sin^2\theta$ is smaller or larger than 1/3, and the neutrinos become degenerate in the limit $\sin^2\theta \to 1/3$, has been suggested in [15].

Another important observable is $m_{\beta\beta}$, the effective mass relevant for neutrinoless 2β decay. This is equal to the modulus of the (e, e) matrix element of $\mathcal{M}_\nu$, i.e. in our case, to $|x|$. Thus,

$$m_{\beta\beta} = \frac{|w|}{2} - \frac{\Delta m^2_\odot \cos 2\theta}{2|w|}. \tag{27}$$

Since

$$|w| \approx 2\sqrt{\frac{\Delta m^2_{\mathrm{atm}}}{|8 - \tan^2 2\theta|}} \gg \sqrt{\Delta m^2_\odot \cos 2\theta}, \tag{28}$$

we see that in our model we have the relation

$$m_{\beta\beta} \approx m_3/3. \tag{29}$$

This same relation has recently been obtained in a different model [16].

One may ask oneself whether the neutrino mass matrix displays any characteristic texture in the limit $\tan^2 2\theta \to 8$. A glance at equations (20), (21) and (23) allows one to conclude that, in that limit, all the matrix elements of $\mathcal{M}_\nu$ diverge. Moreover, from equations (21) and (20), respectively, we obtain

$$\frac{x}{w} \to -\frac{1}{2}, \qquad \frac{|y|}{|w|} \to 1 \tag{30}$$

for $\tan^2 2\theta \to 8$, from where the texture of $\mathcal{M}_\nu$ in that limit can be read off.

5. Extension to the complex case

In this section, we investigate what happens when one allows the neutrino mass matrix of equation (13) to have complex matrix elements. Does the intriguing feature of quasi-degenerate neutrinos in the limit of HPS mixing, found in the previous section for the case of real matrix elements, still hold true? We shall see that it does not; indeed, the general neutrino mass matrix (13) does not seem to have much predictive power beyond $U_{e3} = 0$ and maximal atmospheric neutrino mixing.

The symmetric matrix

$$\mathcal{M}_\nu = \begin{pmatrix} x & y & y \\ y & z & w \\ y & w & z \end{pmatrix} \tag{31}$$

is diagonalized in the following way:

$$U^T \mathcal{M}_\nu U = \mathrm{diag}(m_1, m_2, m_3), \tag{32}$$

the matrix U being unitary while m_j $(j = 1, 2, 3)$ are real and non-negative. Due to the special form of $\mathcal{M}_\nu$, wherein $(\mathcal{M}_\nu)_{12} = (\mathcal{M}_\nu)_{13}$ and $(\mathcal{M}_\nu)_{22} = (\mathcal{M}_\nu)_{33}$, U is of the form

$$U = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\varphi}, \mathrm{e}^{\mathrm{i}\vartheta}, \mathrm{e}^{\mathrm{i}\vartheta}) \begin{pmatrix} c & s & 0 \\ -rs & rc & r \\ -rs & rc & -r \end{pmatrix} \mathrm{diag}(\mathrm{e}^{\mathrm{i}\Sigma_1}, \mathrm{e}^{\mathrm{i}\Sigma_2}, \mathrm{e}^{\mathrm{i}\Sigma_3}), \tag{33}$$

where $c = \cos\theta$, $s = \sin\theta$ and $r = 2^{-1/2}$. From equations (31)–(33), we find

$$x = \mathrm{e}^{-2\mathrm{i}\varphi}(c^2 m_1\, \mathrm{e}^{-2\mathrm{i}\Sigma_1} + s^2 m_2\, \mathrm{e}^{-2\mathrm{i}\Sigma_2}), \tag{34}$$

$$z = \frac{\mathrm{e}^{-2\mathrm{i}\vartheta}}{2}(s^2 m_1\, \mathrm{e}^{-2\mathrm{i}\Sigma_1} + c^2 m_2\, \mathrm{e}^{-2\mathrm{i}\Sigma_2} + m_3\, \mathrm{e}^{-2\mathrm{i}\Sigma_3}). \tag{35}$$

We define $\bar{m}_j \equiv m_j\, \mathrm{e}^{-2\mathrm{i}\Sigma_j}$ for $j = 1, 2, 3$. We also define $\chi \equiv 2(\vartheta - \varphi)$. Then, the condition $x = z$, which makes $\mathcal{M}_\nu$ of equation (31) identical with that of equation (13), is equivalent to

$$\bar{m}_1(2c^2\,\mathrm{e}^{\mathrm{i}\chi} - s^2) + \bar{m}_2(2s^2\,\mathrm{e}^{\mathrm{i}\chi} - c^2) - \bar{m}_3 = 0. \tag{36}$$

Thus, the condition $x = z$ is equivalent to the existence of four phases χ and $\Sigma_{1,2,3}$ such that condition (36) is satisfied. This condition states that it is possible to draw a triangle in the complex plane, the sides of that triangle having lengths $\sqrt{A}m_1$, $\sqrt{B}m_2$ and m_3, where

$$A = 4c^4 + s^4 - 4c^2s^2\cos\chi, \qquad B = 4s^4 + c^4 - 4c^2s^2\cos\chi. \tag{37}$$

Therefore, one may eliminate the phases $\Sigma_{1,2,3}$ from condition (36) by writing the sole 'triangle inequality' [17]

$$A^2m_1^4 + B^2m_2^4 + m_3^4 - 2\left(ABm_1^2m_2^2 + Am_1^2m_3^2 + Bm_2^2m_3^2\right) \leqslant 0. \tag{38}$$

Using

$$m_1^2 = m_3^2 - \epsilon\Delta m^2_{\mathrm{atm}} - \frac{1}{2}\Delta m^2_\odot, \qquad m_2^2 = m_3^2 - \epsilon\Delta m^2_{\mathrm{atm}} + \frac{1}{2}\Delta m^2_\odot, \tag{39}$$

inequality (38) takes the form

$$k_4m_3^4 + 2k_2m_3^2 + k_0 \leqslant 0, \tag{40}$$

where

$$k_4 = 1 - 2(A + B) + (A - B)^2, \tag{41}$$

$$k_2 = [A + B - (A - B)^2]\epsilon\Delta m^2_{\mathrm{atm}} + \tfrac{1}{2}(A - B)(1 - A - B)\Delta m^2_\odot, \tag{42}$$

$$k_0 = \left[(A - B)\epsilon\Delta m^2_{\mathrm{atm}} + \tfrac{1}{2}(A + B)\Delta m^2_\odot\right]^2. \tag{43}$$

Since $k_0 > 0$, inequality (40) does not tolerate $m_3 = 0$; hence, there is a non-trivial lower bound on the neutrino masses. We want to find the numerical value of that bound. Using the values of A and B in equations (37), one finds that

$$k_4 = -16c^2s^2(1 - \cos\chi), \tag{44}$$

$$k_2 = 2(-2 + 13c^2s^2 - 4c^2s^2\cos\chi)\epsilon\Delta m^2_{\mathrm{atm}} + 3(c^2 - s^2)(-2 + 5c^2s^2 + 4c^2s^2\cos\chi)\Delta m^2_\odot, \tag{45}$$

$$k_0 = \left[3(c^2 - s^2)\epsilon\Delta m^2_{\mathrm{atm}} + \tfrac{1}{2}(5 - 10c^2s^2 - 8c^2s^2\cos\chi)\Delta m^2_\odot\right]^2. \tag{46}$$

The case of real x, y and w corresponds to $\cos\chi = +1$. In (and only in) that case, the left-hand side of inequality (40) becomes linear in m_3^2; besides, in that case k_2 vanishes when $c^2s^2 = 2/9$, thereby generating singularities at the points $s^2 = 1/3$ and $s^2 = 2/3$, as we saw in the previous section.

For $\cos\chi \neq +1$, k_4 is negative. Since k_0 is always positive, inequality (40) then yields

$$m_3^2 \geqslant \frac{\sqrt{k_2^2 + |k_4|\,k_0} + k_2}{|k_4|} \equiv L. \tag{47}$$

The task now consists in finding the minimum value of L as a function of $\cos\chi$ (and of $\epsilon = \pm 1$); this minimum value provides the lower bound on m_3^2. It is easy to convince oneself that L always has its minimum when $\cos\chi = -1$, for all experimentally allowed values of s^2, $\Delta m^2_{\mathrm{atm}}$ and $\Delta m^2_\odot$. Computing L as a function of s^2 for fixed $\cos\chi = -1$, $\Delta m^2_{\mathrm{atm}} = 2.5 \times 10^{-3}$ eV and $\Delta m^2_\odot = 7.9 \times 10^{-5}$ eV, we conclude the following.

- When the neutrino mass spectrum is normal, i.e. when $\epsilon = +1$, the minimum value of the lowest neutrino mass, m_1, hardly varies with s^2. One has $m_1 > 1.679 \times 10^{-2}$ eV for $s^2 = 0.24$ and $m_1 > 1.665 \times 10^{-2}$ eV for $s^2 = 0.40$.
- When the neutrino mass spectrum is inverted, i.e. when $\epsilon = -1$, the minimum value of the lowest neutrino mass, m_3, varies strongly as a function of s^2. One has $m_3 > 2.9 \times 10^{-2}$ eV for $s^2 = 0.24$ and $m_3 > 9 \times 10^{-3}$ eV for $s^2 = 0.40$.

Thus, the mass matrix of equation (13) with complex x, y and w is not very predictive: it only allows one to derive a rather mild lower bound on the neutrino masses. There is also no prediction for the effective mass $m_{\beta\beta}$, except for the rather trivial bounds

$$\left|m_1^2 c^2 - m_2^2 s^2\right| \leqslant m_{\beta\beta} \leqslant \left|m_1^2 c^2 + m_2^2 s^2\right|. \tag{48}$$

6. Conclusions

In this paper we have constructed an extension of the Standard Model with three Higgs doublets ϕ_α and four scalar gauge triplets Δ_α and Δ_4. The scalar triplets generate a type-II seesaw mechanism, thus explaining the smallness of the neutrino masses. We have employed a large horizontal symmetry group G, generated by the permutation group S_3 of the indices α and by six cyclic groups $\mathbb{Z}_2$. After spontaneous symmetry breaking, the charged-lepton mass matrix is diagonal; the different VEVs of ϕ_α allow for different charged-lepton masses m_α $(\alpha = e, \mu, \tau)$. In order to obtain a realistic neutrino mass matrix $\mathcal{M}_\nu$, we additionally allow for soft breaking of G, through terms of dimension 2 in the scalar potential. A crucial feature of our model is the equality among the diagonal entries of $\mathcal{M}_\nu$—this is one of the reasons for the predictiveness of the model.

There are two relevant options: breaking G softly in the mass matrix of the scalar triplets either fully or keeping a $\mu \leftrightarrow \tau$ symmetry intact and having either hard or spontaneous CP breaking. Our model has the interesting property that spontaneous CP violation has no effect on $\mathcal{M}_\nu$, i.e. it does not generate any physical phases in lepton mixing. The most predictive scenario combines the preservation of μ–τ interchange symmetry with spontaneous CP violation, in which case we arrive at a viable neutrino mass matrix which has only three (real) parameters. This neutrino mass matrix leads to the usual predictions of μ–τ symmetric neutrino mass matrices, namely maximal atmospheric mixing and $\theta_{13} = 0$; hence, there is no CP violation in neutrino oscillations. Besides, the CP property mentioned before also prevents Majorana phases in our case.

The solar mixing angle θ is undetermined. Our three-parameter neutrino mass matrix predicts the neutrino masses m_j as functions of the two mass-squared differences and of θ. For $\sin^2\theta < 1/3$, which seems to be preferred by the data, we have a normal spectrum, while for $\sin^2\theta > 1/3$ the neutrino mass spectrum is inverted. When $\sin^2\theta \to 1/3$, all m_j diverge; see figure 1. As for the effective mass $m_{\beta\beta}$ of neutrinoless 2β decay, our three-parameter mass matrix predicts $m_{\beta\beta} \approx m_3/3$.

Acknowledgments

We thank Ernest Ma for suggestions and discussions that contributed significantly to this work. W G is grateful to H Urbantke for discussions on finite groups. The work of L L was supported by the Portuguese *Fundação para a Ciência e a Tecnologia* through the projects POCTI/FNU/44409/2002, POCTI/FP/63415/2005 and U777–Plurianual.

Appendix A. The group structure of our model

In this appendix, we attempt a mathematical description of the full symmetry group of our model and of its irreducible representations (irreps). Clearly, S_3 commutes neither with $\mathbf{z}_\alpha^{(1)}$ of equation (4) nor with $\mathbf{z}_\alpha^{(2)}$ of equation (5); thus the full symmetry group is rather complicated.

Let us define

$$n_1 = \mathrm{diag}(-1, 1, 1), \qquad n_2 = \mathrm{diag}(1, -1, 1), \qquad n_3 = \mathrm{diag}(1, 1, -1), \tag{A.1}$$

$$c_+ = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad c_- = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{A.2}$$

$$t_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad t_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad t_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.3}$$

Then

$$N = \{\mathbb{1}, n_1, n_2, n_3, n_1 n_2, n_2 n_3, n_3 n_1, -\mathbb{1}\} \tag{A.4}$$

forms an Abelian group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Also,

$$\hat{S}_3 = \{\mathbb{1}, c_+, c_-, t_1, t_2, t_3\} \tag{A.5}$$

forms a three-dimensional (reducible) representation of S_3.

Let us call G the symmetry group utilized in this paper. G may be defined to be the group of the 6×6 matrices

$$\begin{pmatrix} ms & 0 \\ 0 & ns \end{pmatrix}, \qquad m, n \in N, \qquad s \in \hat{S}_3. \tag{A.6}$$

This defining *reducible* representation of G may be called **6**. Clearly, G has $8 \times 8 \times 6 = 384$ elements[9]. Calling (m, n, s) the abstract element of G which is represented in **6** by the matrix of equation (A.6), the group multiplication law is

$$(m_1, n_1, s_1)(m_2, n_2, s_2) = \left(m_1 s_1 m_2 s_1^{-1}, n_1 s_1 n_2 s_1^{-1}, s_1 s_2\right). \tag{A.7}$$

From this group multiplication law, it follows that G has eight one-dimensional irreps:

$$\underline{1}^{(p,q,r)}: \qquad (m, n, s) \to (\det m)^p (\det n)^q (\det s)^r \qquad \text{with} \qquad p, q, r \in \{0, 1\}. \tag{A.8}$$

It is obvious from equation (A.6) that the matrices ms give a three-dimensional irrep of G, and similarly with the matrices ns. The matrices mns give one further three-dimensional irrep of G, since

$$\begin{aligned} &m_1 n_1 s_1 m_2 n_2 s_2 = m_3 n_3 s_3 \qquad \text{with} \qquad m_3 = m_1 s_1 m_2 s_1^{-1}, \\ &n_3 = n_1 s_1 n_2 s_1^{-1} \qquad \text{and} \qquad s_3 = s_1 s_2 \end{aligned} \tag{A.9}$$

complies with the multiplication law (A.7). Thus, G has 24 three-dimensional irreps:

$$\underline{3}_1^{(p,q,r)}: \quad (m, n, s) \to (\det m)^p (\det n)^q (\det s)^r ms, \tag{A.10}$$

$$\underline{3}_2^{(p,q,r)}: \quad (m, n, s) \to (\det m)^p (\det n)^q (\det s)^r ns, \tag{A.11}$$

[9] The $8 \times 6 = 48$ matrices ms, where $m \in N$ and $s \in \hat{S}_3$, form the Coxeter group B_3. (We thank E Ma for drawing our attention to Coxeter groups.) We may write $G = N \rtimes B_3 = (N \times N) \rtimes S_3$, the symbol $\rtimes$ denoting a semi-direct product.

$$\underline{3}_3^{(p,q,r)}: \quad (m, n, s) \to (\det m)^p (\det n)^q (\det s)^r mns, \tag{A.12}$$

with $p, q, r \in \{0, 1\}$.

The three-dimensional representations of G that we employ in our model are

$$\begin{array}{lll} ms & \text{for} & (\phi_e, \phi_\mu, \phi_\tau), \\ mns & \text{for} & (e_R, \mu_R, \tau_R), \\ ns & \text{for} & (D_{Le}, D_{L\mu}, D_{L\tau}), \\ (\det n) ns & \text{for} & (\Delta_e, \Delta_\mu, \Delta_\tau). \end{array} \tag{A.13}$$

Our group G also has two- and six-dimensional irreps, which are not used in our model. Next we include, for completeness, their construction.

The group S_3 has a two-dimensional irrep D_2, generated by

$$t_1 \to \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}, \qquad t_2 \to \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}, \tag{A.14}$$

where $\omega = (-1 + \mathrm{i}\sqrt{3})/2$. Note that $(\det s) D_2(s)$ is isomorphic to $D_2(s)$:

$$(\det s) D_2(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} D_2(s) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.15}$$

Therefore, G has four two-dimensional irreps:

$$\underline{2}^{(p,q)}: \qquad (m, n, s) \to (\det m)^p (\det n)^q D_2(s) \quad \text{with} \quad p, q \in \{0, 1\}. \tag{A.16}$$

The remaining irreps of G are four six-dimensional ones:

$$\underline{6}^{(p,q)}: \qquad (m, n, s) \to (\det m)^p (\det n)^q D_6(m, n, s) \qquad \text{with} \qquad p, q \in \{0, 1\}. \tag{A.17}$$

The irrep $D_6\,(m, n, s)$ is found in the decomposition of the product of the irreps ms and ns. Suppose there is a space $\mathbb{C}^3$ spanned by $e_{1,2,3}$ transforming like ms and another space $\mathbb{C}^3$ spanned by $e'_{1,2,3}$ transforming like ns. Then the space spanned by $e_k \otimes e'_k$ $(k = 1, 2, 3)$ transforms like mns, while $e_j \otimes e'_k$ with $j \neq k$ span a space which transforms like $D_6(m, n, s)$. It can be shown that this representation $D_6(m, n, s)$ of G is irreducible, and also that it is equivalent to $(\det s) D_6(m, n, s)$.

The group G has the interesting property that it has no faithful irreps. It is obvious that the irreps with dimensions 3 and lower are not faithful. The six-dimensional irreps are not faithful either, as we now explain. Defining the elements a and b of G by $a = (-\mathbb{1}, \mathbb{1}, \mathbb{1})$ and $b = (\mathbb{1}, -\mathbb{1}, \mathbb{1})$, then a, b and ab generate the subgroups $\mathbb{Z}_2^{(a)}$, $\mathbb{Z}_2^{(b)}$ and $\mathbb{Z}_2^{(ab)}$ of G, respectively. The isomorphisms

$$\begin{array}{ll} \underline{6}^{(0,0)} \cong G/\mathbb{Z}_2^{(ab)}, & \underline{6}^{(1,0)} \cong G/\mathbb{Z}_2^{(a)}, \\ \underline{6}^{(0,1)} \cong G/\mathbb{Z}_2^{(b)}, & \underline{6}^{(1,1)} \cong G/\left(\mathbb{Z}_2^{(a)} \times \mathbb{Z}_2^{(b)}\right) \end{array} \tag{A.18}$$

are easy to demonstrate. Thus, none of the six-dimensional irreps represents G faithfully.

Appendix B. Spontaneous breaking of the μ–τ symmetry

Let us consider a simplified model with only two VEVs, v_μ and v_τ. We assume the following symmetries:

$$\begin{array}{ll} z_1 : v_\mu \to -v_\mu, & v_\tau \to v_\tau; \\ z_2 : v_\mu \to v_\mu, & v_\tau \to -v_\tau; \\ z_3 : v_\mu \leftrightarrow v_\tau. & \end{array} \tag{B.1}$$

The symmetries $z_{1,2}$ are assumed to be softly broken by terms of dimension 2, while z_3 is assumed to be exactly conserved. For the sake of clarity, we also assume all coefficients to be real. Then

$$V_0 = a(|v_\mu|^2 + |v_\tau|^2) + b(v_\mu^* v_\tau + v_\tau^* v_\mu) + \lambda(|v_\mu|^2 + |v_\tau|^2)^2 + \lambda'|v_\mu|^2|v_\tau|^2. \tag{B.2}$$

Only the b term breaks $z_{1,2}$ softly.

Without loss of generality we take v_μ to be real and positive, writing

$$v_\mu = v\cos\phi, \qquad v_\tau = v\sin\phi\, \mathrm{e}^{\mathrm{i}\alpha}, \tag{B.3}$$

with $v > 0$ and ϕ in the first quadrant. Then we obtain

$$V_0 = av^2 + bv^2 \sin 2\phi \cos\alpha + \lambda v^4 + \frac{\lambda' v^4}{4}\sin^2 2\phi. \tag{B.4}$$

We require that

$$0 = \frac{\partial V_0}{\partial(2\phi)} = (v^2\cos 2\phi)\left(b\cos\alpha + \frac{\lambda' v^2}{2}\sin 2\phi\right). \tag{B.5}$$

The solution $\cos 2\phi = 0$ corresponds to $|v_\mu| = |v_\tau|$ and is undesirable. But there is another solution,

$$\sin 2\phi = -\frac{2b\cos\alpha}{\lambda' v^2}, \tag{B.6}$$

which we adopt. Since the minimization of V_0 in equation (B.4) with respect to α leads to $b\cos\alpha = -|b|$ being negative, we must assume λ' to be positive. If

$$|b| \ll \frac{\lambda' v^2}{2}, \tag{B.7}$$

which corresponds to the soft-breaking term being very small, then $\sin 2\phi \ll 1$ and $|v_\mu| \ll |v_\tau|$ can be realized.

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